Math 402/500 HW #7, due Friday 4/9/21 NAME:

1. Does there exist a Morse function on $S^2 \times S^2$ that has a minimum, a maximum, one critical point of index 2, and whose other critical points all have indices 1 or 3? If you need help computing the homology of $S^2 \times S^2$ please ask Leo or go to his problem session.

2. Show that every Morse function on a compact odd-dimensional manifold must have an even number of critical points. (Ask Leo to tell you about the Euler Characteristic of a topological space if you haven't seen this before.) If M is a 3-dimensional homology sphere, e.g. $H_*(M) \cong H_*(S^3)$, but with $\pi_1(M) \neq \{\mathrm{id}\}$, show that any Morse function must have at least six critical points. Hint: You may use the fact that $H_1(M)$ is the abelianization of $\pi_1(M)$.

3. Find counterexamples with $X = S^1$ to each of the following statements:

   (a) Suppose (f_t, g_t) is Morse-Smale for all $t \in [0, 1]$, so that there is a canonical identification $\mathrm{Crit}(f_0) \backsimeq \mathrm{Crit}(f_1)$. Then the family $\Gamma = \{(f_t, g_t)\}$ is admissible, and Φ_Γ is given by the canonical identification above.

   (b) $\Phi_{\Gamma_2 * \Gamma_1} = \Phi_{\Gamma_2} \circ \Phi_{\Gamma_1}$

optional Prove the Morse inequalities from Hutchings Theorem 3.1.

optional Use Hutchings Theorem 3.1 to prove the Poincaré-Hopf index theorem: If X is a closed oriented smooth manifold, then $\int_X e(TX) = \chi(X)$, where the left hand side describes the signed number of zeroes of a generic vector field on X. (See Guillemin-Pollack for a bit more about the VF perspective).

optional Use Hutchings Theorem 3.1 to prove Poincaré duality for closed oriented manifolds.

optional Show that the diagram in Hutchings commutes. (exercise 3, page 23)

everyone: How difficult was this assignment? How many hours did you spend on it?