Math 402/500 HW #5, due Friday 3/12/21 NAME:

1. Identify $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and consider the Morse function $f : \mathbb{T}^2 \to \mathbb{R}$

$$f(x, y) = \cos(2\pi x) + \sin(2\pi y).$$

   (a) Find the critical points and connecting Morse trajectories (draw a figure similar to Fig. 1)

   (b) Prove that f is a Morse function with a Morse-Smale gradient flow.

   (c) Compute the Morse complex (detail the chain groups, differential, and homology).

   (d) Give an example of a gradient flow on $\mathbb{T}^2$ which is not Morse-Smale. (Try to come up with an explicit function and accompanying figure!)

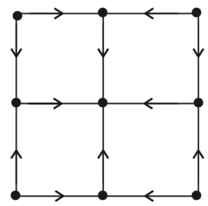

Figure 1: A gradient flow on $\mathbb{T}^2$

2. Consider the function $f : \mathbb{CP}^n \to \mathbb{R}$ given by $f([z_0 : z_1 : ... : z_n]) = \sum_{j=1}^{n} j|z_j|^2$.

   Find the critical points and compute their Morse indices. Use this to compute the homology groups of $\mathbb{CP}^n$. Compare with the means of computing the homology groups of $\mathbb{CP}^n$ via cellular homology. (You may use reddit, mathworld, wikipedia, mathstackexchange, etc to do the latter.)

optional Let $\{\gamma_n\}$ be a sequence of flow lines from p to q, and let $\hat{\gamma} = (\hat{\gamma}_0, \dots, \hat{\gamma}_k)$ be a **k-times broken flow line** from p to q; that is, there exist distinct critical points $r_0, \dots, r_{k+1}$ with $r_0 = p$ and $r_{k+1} = q$ such that $\hat{\gamma}_i$ is a flow line from r_i to r_{i+1} for $i = 0, \dots, k$. Let us say that $\lim_{n\to\infty}[\gamma_n] = [\hat{\gamma}]$ if for each n there exist real numbers $s_{n,0} < s_{n,1} < \cdots < s_{n,k}$ such that $\gamma_n(s_{n,i} + \cdot) \to \hat{\gamma}_i$ in C^∞ on compact sets.

Show that any sequence of flow lines $\{\gamma_n\}$ from p to q has a subsequence which converges to some k-times broken flow line as above for some $k \geq 0$.

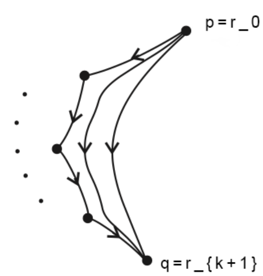

Figure 2: Limit behavior for connecting orbits

If you get stuck, see §3.2 of Audin-Damian or D. Salamon, *Morse theory, the Conley Index, and Floer homology*, Bull. LMS 22 (1990), 113-140. https://people.math.ethz.ch/~salamon/PREPRINTS/morseconley.pdf

everyone: How difficult was this assignment? How many hours did you spend on it?