Math 401 Midterm, due Monday 10/7/19 at 11pm **NAME:**

This is a 5 hour open notes exam. You may use the listed course textbook but are not permitted to use the internet or any other material. You are not permitted to communicate with anyone about this exam except me. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Exercise 1.31 (10 points)
   Let $\alpha : I \to \mathbb{R}^3$ be a curve p.b.a.l. Prove that α is a segment of a straight line if and only if the curvature of α vanishes everywhere.

2. Exercise 2.55 (10 points)
   Let
   $$S = \{p \in \mathbb{R}^3 \mid |p|^2 - \langle p, a\rangle^2 = r^2\},$$
   with $|a| = 1$ and $r > 0$, be a right cylinder of radius r whose axis is the line passing through the origin with direction a. Prove that
   $$T_pS = \{v \in \mathbb{R}^3 \mid \langle p, v\rangle - \langle p, a\rangle\langle a, v\rangle = 0\}.$$
   Conclude that all the normal lines of S cut the axis perpendicularly.

3. Exercise 3.22 (10 points)
   Consider as a surface S the hyperbolic paraboloid given by
   $$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}.$$
   Show that the second fundamental form of S at the point $(0, 0, 0)$ is not a semi-definite bilinear form, i.e. the Gauss curvature of S is negative at this point.

4. Exercise 3.25 (invariance under rigid motions) (10 points)
   Let S be an orientable surface and let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the rigid motion given by $\phi(p) = Ap + b$ where $A \in O(3)$ and $b \in \mathbb{R}^3$. If N is a Gauss map for the surface S, prove that $N' = A \circ N \circ \phi^{-1}$ is a Gauss map for the image surface $S' = \phi(S)$. Conclude that
   $$(dN')_{\phi(p)} = A \circ (dN)_p \circ A^{-1}$$
   and
   $$\sigma'_{\phi(p)}((d\phi)_p(v), (d\phi)_p(w)) = \sigma'_{\phi(p)}(Av, Aw) = \sigma_p(v, w),$$
   for each $p \in S$, $v, w \in T_pS$, where σ and σ' stand, respectively, for the second fundamental forms of S and S'. Finally, find the relationship between the Gauss and mean curvatures of S and S'.