Math 4081 HW#9, due Wednesday 4/25/18 NAME:

1. Lee 11-14 SECOND Consider the following two 1-forms on $\mathbb{R}^3$

$$\begin{aligned} \omega &= -\frac{4z\ dx}{(x^2+1)^2} + \frac{2y\ dy}{y^2+1} + \frac{2x\ dz}{x^2+1} \\ \eta &= -\frac{4xz\ dx}{(x^2+1)^2} + \frac{2y\ dy}{y^2+1} + \frac{2\ dz}{x^2+1} \end{aligned}$$

   (a) Set up and evaluate the line integral of each 1-form along the straight line segment from $(0,0,0)$ to $(1,1,1)$.

   (b) Determine whether either of ω or η is exact.

   (c) For each 1-form that is exact, find a potential function and use it to recompute the line integral.

2. Lee 16-9 SECOND

   Let ω be the $(n-1)$-form on $\mathbb{R}^n \setminus \{0\}$

$$\omega = |x|^{-n} \sum_{i=1}^{n} (-1)^{i-1} x^i \; dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n.$$

   (a) Show that $\iota^*_{S^{n-1}}\omega$ is the Riemannian volume form of S^{n-1} with respect to the round metric and the standard orientation.

   (b) Show that ω is closed but not exact on $\mathbb{R}^n \setminus \{0\}$.

3. Lee 16-18 a, b, c SECOND = Lee 14-12 a, c FIRST
Let (M, g) be an oriented Riemannian n-manifold. This problem outlines an important generalization of the operator

$$* : C^{\infty}(M) \to \Omega^n(M),$$

defined in this chapter.

(a) For each $k = 1, ..., n$, show that g determines a unique inner product on on $\Lambda^k(T_p^* M)$ (denoted by $\langle \cdot, \cdot, \rangle_g$ just like the inner product on Tp M) satisfying

$$\langle \omega^1 \wedge ... \wedge \omega^k, \eta^1 \wedge ... \wedge \eta^k \rangle_g = \det\left(\ \langle (\omega^i)^{\#}, (\eta^j)^{\#} \rangle_g \ \right)$$

whenever $\omega^1, ..., \omega^k, \eta^1, ..., \eta^k$ are covectors at p. [Hint given in Lee 16-18 (a) SECOND = Lee 14-12 (a) FIRST].

(b) Show that the Riemannian volume form dV_g is the unique positively oriented n-form that has unit norm with respect to this inner product.

(c) For each $k = 0, ..., n$ show that there is a unique smooth bundle homomorphism

$$* : \Lambda^k T^* M \to \Lambda^{n-k} T^* M$$

satisfying

$$\omega \wedge * \eta = \langle \omega, \eta \rangle_g dV_g$$

for all smooth k-forms ω, η. (For $k = 0$, interpret the inner product as ordinary multiplication.) This map is called the **Hodge star operator.** [Hint given in Lee 16-18 (c) SECOND = Lee14-12 (b) FIRST]

4. Lee 17-1 SECOND

   Let M be a smooth manifold with or without boundary, and let $\omega \in \Omega^p(M)$, $\eta \in \Omega^q(M)$ be closed forms. Show that the deRham cohomology class of $\omega \wedge \eta$ depends only on the cohomology classes of ω and η, and thus there is a well-defined bilinear map

$$\cup : H^p_{\mathrm{dR}}(M) \times H^q_{\mathrm{dR}}(M) \to H^{p+q}_{\mathrm{dR}}(M),$$

   called the **cup product** given by $[\omega] \cup [\eta] = [\omega \wedge \eta]$.

5. How difficult was this assignment?