Math 4081 HW#6, due Wednesday 4/4/18 NAME:

1. Lee 11.7 [SECOND].
   In the following subproblems, M and N are smooth manifolds, $F : M \to N$ is a smooth map, and ω is a covector field on N. Compute $F^*\omega$ in each case.

   (a) $M = N = \mathbb{R}^2$, $F(s,t) = (st, e^t)$,
   $\omega = xdy - ydx$

   (b) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$, $F(\theta, \varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi)$,
   $\omega = z^2 dx$

   (c) $M = \{(s,t) \in \mathbb{R}^2 \mid s^2 + t^2 < 1\}$ and $N = \mathbb{R}^3 \setminus \{0\}$, $F(s,t) = \left(s, t, \sqrt{1 - s^2 - t^2}\right)$,
   $\omega = (1 - x^2 - y^2)dz$

2. Lee 11.11 [SECOND].
   Let M be a smooth manifold, and $C \subset M$ be an embedded submanifold. Let $f \in C^\infty(M)$, and suppose $p \in C$ is a point at which f attains a local maximum or minimum value among points in C. Given a smooth local defining function $\Phi : U \to \mathbb{R}^k$ for C on a neighborhood U of p in M, show that there are real numbers $\lambda_1, ...\lambda_k$ (called Lagrange multipliers) such that

$$df_p = \lambda_1 d\Phi^1|_p + ... + \lambda_k d\Phi^k|_p.$$

3. Check in local coordinates that if α is a 1-form and V and W are vector fields on M, then

$$d\alpha(V, W) = V\alpha(W) - W\alpha(V) - \alpha([V, W]).$$

4. Let $\omega : \mathbb{R}^4 \otimes \mathbb{R}^4 \to \mathbb{R}$ be an alternating bilinear form. Show that there exist linear maps $\alpha, \beta : \mathbb{R}^4 \to \mathbb{R}$ with $\omega = \alpha \wedge \beta$ if and only if $\omega \wedge \omega = 0$. *Hint:* Choose a basis in which ω looks simple.

5. Let M be a smooth manifold with a Riemannian metric $g : TM \otimes TM \to \mathbb{R}$. If $f : M \to \mathbb{R}$ is a smooth function, the *gradient* of f with respect to g is the vector field ∇f defined by

$$df = g(\nabla f, \cdot).$$

(a) In local coordinates $\{x^i\}$, if $g(\partial/\partial x^i, \partial/\partial x^j) = g_{ij}$, explain how to compute ∇f in terms of g_{ij} and $\partial f/\partial x^i$. *Hint:* See HW#1.

(b) Let $f : M \to \mathbb{R}$ and let $p \in M$. Show that if $V \in T_pM$ satisfies $df_p(V) > 0$, then there exists a Riemannian metric g on M with $\nabla f(p) = V$.

6. How difficult was this assignment?