Math 401 HW #5, due Monday 2/28/22 NAME:

1. Exercise 3.16
   Consider two diffeomorphic surfaces S_1 and S_2. Show that S_1 is orientable if and only if S_2 is orientable.

2. Exercise 3.17
   If S is an oriented surface, N is the corresponding unit normal field, and $p \in S$, then we say that a basis $\{a, b\}$ of the tangent plane T_pS is *positively oriented* when $\det(a, b, N(p)) > 0$. Otherwise we say that it is *negatively oriented*. If S_1 and S_2 are two oriented surfaces, we say that a local diffeomorphism $f : S_1 \to S_2$ *preserves orientation* if its differential at each point of S_1 takes positively oriented bases on S_1 into positively oriented bases on S_2. We define a function
   $$\mathrm{Jac}f : S_1 \to \mathbb{R}$$
   that is called the *Jacobian* of f - compare with HW #4 - by the equation
   $$(\mathrm{Jac}f)(p) = \det((df)_p(e_1), (df)_p(e_2), N_2(f(p))),$$
   where $\{e_1, e_2\}$ is a positively oriented orthonormal basis of T_pS_1.
   Prove that, if S_1 and S_2 are connected, f preserves orientation if and only if its Jacobian is positive everywhere.

3. Exercise 3.22 (10 points)
   Consider as a surface S the hyperbolic paraboloid given by
   $$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}.$$
   Show that the second fundamental form of S at the point $(0, 0, 0)$ is not a semi-definite bilinear form, i.e. the Gauss curvature of S is negative at this point.

4. Exercise 3.25 (invariance under rigid motions) (10 points)
   Let S be an orientable surface and let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the rigid motion given by $\phi(p) = Ap + b$ where $A \in O(3)$ and $b \in \mathbb{R}^3$. If N is a Gauss map for the surface S, prove that $N' = A \circ N \circ \phi^{-1}$ is a Gauss map for the image surface $S' = \phi(S)$. Conclude that
   $$(dN')_{\phi(p)} = A \circ (dN)_p \circ A^{-1}$$
   and
   $$\sigma'_{\phi(p)}((d\phi)_p(v), (d\phi)_p(w)) = \sigma'_{\phi(p)}(Av, Aw) = \sigma_p(v, w),$$
   for each $p \in S$, $v, w \in T_pS$, where σ and σ' stand, respectively, for the second fundamental forms of S and S'. Finally, find the relationship between the Gauss and mean curvatures of S and S'.

* Assignment Reflections
   How difficult was this assignment? How many hours did you spend on it? Which problems did you find to provide a worthwhile learning experience?