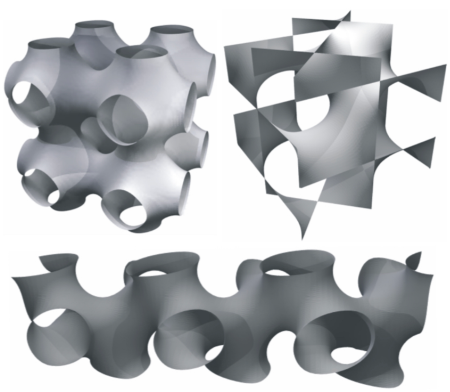
Math 401 HW#4, due Monday 2/7/22 NAME:

For full credit, turn in solutions to 3 out of 4 problems.

1. **Exercise §2 (8)**



(8) A surface is said to be *triply periodic* when it is invariant under three linearly independent translations of $\mathbb{R}^3$. This type of surface plays an important role in crystallography. Prove that the sets $\mathcal{P}$, $\mathcal{D}$, and $\mathcal{G}$, given, respectively, by the equations

$$\cos x + \cos y + \cos z = 0,$$
$$\cos(x-y-z) + \cos(x+y-z) - \cos(x+y+z) = 0,$$
$$\sin(x+y) + \sin(x-y) + \sin(y+z)$$
$$+ \sin(y-z) + \sin(x+z) + \sin(x-z) = 0$$

are examples of triply periodic surfaces. They are called, respectively, *primitive*, *diamond*, and *gyroid*. Show also that, for one of these examples, there are rigid motions of $\mathbb{R}^3$ interchanging the two regions determined by the surface.

Figure 2.8. *Triply periodic surfaces*

Figure 1: Hint: Appeal to trig identities to get linearly independent translations of $\mathbb{R}^3$.

https://math.stackexchange.com/questions/3589185/a-rigid-motion-that-interchanges-the-the-two-regions-determined-by-the-gyroid

2. **Exercise §2 (9)**
   Let $\phi : S \to \mathbb{R}^3$ be a differentiable map defined on a surface that satisfies
   - $(d\phi)_p : T_pS \to \mathbb{R}^3$ is injective for all $p \in S$.
   - $\phi : S \to \phi(S)$ is a homeomorphism.

   Prove that $\phi(S)$ is a surface and that $\phi : S \to \phi(S)$ is a diffeomorphism. If S is compact, show that the second requirement is equivalent to requiring that ϕ be injective.

3. **Exercise §2 (13)**
   Let $f : S_1 \to S_2$ be a differentiable map between surfaces. If $p \in S_1$ and $\{e_1, e_2\}$ is an orthonormal basis of T_pS, we define the **absolute value of the Jacobian of f at p** by
   $$|\mathrm{Jac} f|(p) = |(df)_p(e_1) \wedge (df)_p(e_2)|.$$
   (a) Prove that this definition does not depend on the orthonormal basis.
   (b) Prove that $|\mathrm{Jac} f|(p) \neq 0$ if and only if $(df)_p$ is an isomorphism.

4. **Exercise 2.55**
   Let
   $$S = \{p \in \mathbb{R}^3 \mid |p|^2 - \langle p, a\rangle^2 = r^2\},$$
   with $|a| = 1$ and $r > 0$, be a right cylinder of radius r whose axis is the line passing through the origin with direction a. Prove that
   $$T_pS = \{v \in \mathbb{R}^3 \mid \langle p, v\rangle - \langle p, a\rangle\langle a, v\rangle = 0\}.$$
   Conclude that all the normal lines of S cut the axis perpendicularly.

* Assignment Reflections
How difficult was this assignment? How many hours did you spend on it? Which problems did you like?