Math 4081 HW#4, due Monday 3/5/18 NAME:

1. Lee SECOND 6-10: Suppose $F : N \to M$ is a smooth map that is transverse to an embedded submanifold $X \subset M$, and let $W = F^{-1}(X)$. For each $p \in W$, show that $T_pW = (dF_p)^{-1}(T_{F(p)}X)$. Conclude that if two embedded submanifolds $X, X' \subset M$ intersect transversely, then $T_p(X \cap X') = T_pX \cap T_pX'$ for every $p \in X \cap X'$.

2. (Each part is worth 3 points.) Let M be a smooth manifold and let $f : M \to M$ be a smooth map.

   (a) Define the *diagonal*
   $$\Delta = \{(p,p) \mid p \in M\} \subset M \times M$$
   and the *graph*
   $$\Gamma(f) = \{(p, f(p)) \mid p \in M\} \subset M \times M.$$
   Check that Δ and $\Gamma(f)$ are submanifolds of $M \times M$ which are canonically diffeomorphic to M.

   (b) A *fixed point* of f is a point $p \in M$ with $f(p) = p$. A fixed point p is *nondegenerate* if $1 - df_p : T_pM \to T_pM$ is invertible. Show that all fixed points of f are nondegenerate if and only if $\Gamma(f)$ is transverse to Δ.

   (c) The *Lefschetz sign* of a nondegenerate fixed point p, denoted by $\epsilon(p) \in \{\pm 1\}$, is the sign of the determinant of $1 - df_p$. If all fixed points are nondegenerate, and if there are only finitely many fixed points, define the signed count of fixed points by
   $$\#\operatorname{Fix}(f) = \sum_{f(p)=p} \epsilon(p) \in \mathbb{Z}.$$
   Show that if $\Gamma(f)$ is transverse to Δ, and if M is compact and oriented, then the intersection number[1]
   $$\Gamma(f) \cdot \Delta = \#\operatorname{Fix}(f).$$

   (d) Let A be a 2×2 integer matrix. The map $A : \mathbb{R}^2 \to \mathbb{R}^2$ descends to a map $f_A : T^2 \to T^2$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. If A does not have 1 as an eigenvalue, show that all fixed points of f_A are nondegenerate, and compute $\#\operatorname{Fix}(f)$ in terms of A.

[1]For those of you who know what homology is, this can be used to prove the *Lefschetz fixed point theorem*
$$\#\operatorname{Fix}(f) = \sum_i (-1)^i \operatorname{Tr}(f_* : H_i(M;\mathbb{Q}) \to H_i(M;\mathbb{Q})).$$

3. How difficult was this assignment?