Math 401 HW#1, due Monday 1/17/22 **NAME:**

Do not upload solutions to the **In Class Warm Up Problems** to gradescope.
Pick one of 4 or 5 to upload to gradescope

- ⋆ **Multi Warm Up**
  Let $v(t) \in \mathbb{R}^n$ such that $|v(t)| = c$ for all t. Show $v' \cdot v = 0$. Conclude $T(t) \cdot N(t) = 0$.

- ⋆ **Multi Warm Up: Jones Problem 11-1 (in class)**
  Suppose $C \subset \mathbb{R}^2$ is a curve described in polar coordinates by an equation $r = g(\theta)$, where $a \le \theta \le b$. Show that the length of C is
  $$\int_a^b \sqrt{g'(\theta)^2 + g(\theta)^2}\, d\theta.$$
  Remark: This formula is usually written as $\displaystyle\int_a^b \sqrt{\left(\frac{dr}{d\theta}^2 + r^2\right)}\, d\theta.$

1. **Exercise 1.12** (the multi-warm up will be advantageous)
   Consider the logarithmic spiral $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by
   $$\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$$
   with $a > 0$, $b < 0$. Compute the arc length function $S : \mathbb{R} \to \mathbb{R}$ where $S(t) = \int_{t_0}^{t} |\alpha'(\tau)| d\tau$, where t_0 corresponds to an arbitrary choice of $t_0 \in \mathbb{R}$. Reparametrize by arclength (your formula will not be pretty). Describe/sketch the trace of this curve.

- ⋆ **Warm Up (in class)**
  (a) Provide a counterexample to the claim: For every square matrix B, $||Bv|| = |\det B|\ ||v||$.
  (b) Given an orthogonal matrix $A \in \mathrm{O}(n)$, show that A preserves vector norms: $||Av|| = ||v||$. (Recall that an orthogonal matrix A satisfies $AA^T = A^T A = \mathrm{Id}$.)

2. **Exercise 1.8**
   Let $\alpha : I \to \mathbb{R}^3$ be a curve and let $M : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion, e.g. $M = Ax + b$, where $A \in \mathrm{O}(3)$ and $b \in \mathbb{R}^3$ is a fixed vector. Prove that rigid motions preserve the length of curves, namely $L_b^a(\alpha) = L_b^a(M \circ \alpha)$.

3. **Exercise 1.12**
   Let $\phi : J \to I$ be a diffeomorphism and let $\alpha : I \to \mathbb{R}^3$ be a curve. Given $[a, b] \subset J$ with $\phi([a, b]) = [c, d]$ prove that $L_a^b(\alpha \circ \phi) = L_c^d(\alpha)$.

4. **Exercise 1.9, Theory, in lieu of (5)**
   Let $\alpha : I \to \mathbb{R}^3$ be a curve and $[a, b] \subset I$. Prove that
   $$|\alpha(a) - \alpha(b)| \le L_b^a(\alpha).$$
   In other words, straight lines are the shortest curves joining two given points.

5. **Computation, in lieu of (4)**
   For the following curve $\alpha : \mathbb{R} \to \mathbb{R}$, compute $\mathbf{T}, \mathbf{N}, \mathbf{B}$, and $k(t)$.
   $$\alpha(t) = \langle e^t, e^{-t}, t\sqrt{2}\rangle.$$

* **Assignment Reflections**
  How long did this take you? How was the difficulty? Which problems were meaningful?