**Math 4081 HW#1, due WEDNESDAY 1/31/18** **NAME:**

1. Lee 1-5 [SECOND edition] = Lee 1-7 [FIRST edition 1-7]. So either way you will have done it (since hw 1 originally read 1. Lee 1-5, 2. Lee 1-7).

   **ONLY DO THE CASE WHEN $n = 2$.**

   Let $N$ denote the north pole $(0, ..., 0, 1) \in S^n \subset \mathbb{R}^{n+1}$ and let $S$ denote the south pole $(0, ..., 0, -1)$. Define the **stereographic projection** $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, ..., x^{n+1}) = \frac{(x^1, ..., x^n)}{1 - x^{n+1}}$$

   Let $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.
   (a) For any $x \in S^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through $N$ and $x$ intersects the linear subspace where $x^{n+1} = 0$ (Fig. 1.13 in LEE SECOND and FIRST). Similarly, show that $\widetilde{\sigma}(x)$is the point where the line through $S$ and $x$ intersects the same subspace. (For this reason, $z$ is called **stereographic projection from the south pole.**)
   (b) Show that $\sigma$ is bijective, and

$$\sigma^{-l1}(u^1, ..., u^n) = \frac{(2u^1, ..2u^n, |u|^2 - 1)}{|u|^2 + 1}$$

   (c) Compute the transition map $\widetilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(S^n \setminus \{N\}, \sigma)$ and $(S^n \setminus \{S\}, \widetilde{\sigma})$ defines a smooth structure on $S^n$. (The coordinates defined by $\sigma$ or -$\sigma$ are called stereographic coordinates.)
   (d) Show that this smooth structure is the same as the one defined in Example 1.31 from LEE SECOND or Example 1.20 in LEE FIRST.

2. DO ONE OF the following:

A) Lee 1-6 [First Edition] = Lee 1-8 [Second Edition], and the problem I intended!
By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $S^1$ as a subset of the complex plane. An angle function on a subset $U \subset S^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function on an open subset $U \subset S^1$ if and only if $U \neq S^1$. For any such angle function, show that $(U, \theta)$ is a smooth coordinate chart for $S^1$ with its standard smooth structure.

B) Lee 1.6 [SECOND EDITION] (probably more interesting than the first option)
Let $M$ be a nonempty topological manifold of dimension $n \geq 1$. If $M$ has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from $B^n$ to itself, which is a diffeomorphism if and only if $s = 1$.]

3. DO ONE OF the following:

A) Lee 1.7 [First Edition] = Lee 1.9 [Second Edition], AND the problem I intended!

**(Just do the case $n = 1$.) Also check that the projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ is smooth.**

**Complex projective $n$-space**, denoted by $CP^n$, is the set of all 1-dimensional complex-linear subspaces of $C^{n+1}$, with the quotient topology inherited from the natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to CP^n$. Show that $CP^n$ is a compact 2n-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $RP^n$ (done in an earlier example in Chapter 1 in both versions). We use the correspondence

$$(x^i + iy^1, ..., x^{n+1} + iy^{n+1}) \leftrightarrow (x^1, y^1, ...x^{n+1}, y^{n+1})$$

to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{n+2}$

B) WARNING: Lee 1.5 [Second Edition] is **NOT the problem I would do**. Suppose $M$ is a locally Euclidean Hausdorff space. Show that $M$ is second countable if and only if it is paracompact and has countably many connected components. [Hint: assuming $M$ is paracompact, show that each component of $M$ has a locally finite cover by precompact coordinate domains, and extract from this a countable subcover.]

4. Show that $\mathbb{C}P^1$ is diffeomorphic to $S^2$.

5. Consider spherical coordinates on $\mathbb{R}^3$ (not including the line $x = y = 0$) $\rho, \phi, \theta$ defined in terms of the Euclidean coordinates $x, y, z$ by

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

(a) Express $\partial/\partial\rho$, $\partial/\partial\phi$, and $\partial/\partial\theta$ as linear combinations of $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$. (The coefficients in these linear combinations will be functions on $\mathbb{R}^3 \setminus (x = y = 0)$.)

(b) Express $d\rho$, $d\phi$, and $d\theta$ as linear combinations of $dx$, $dy$, and $dz$.

6. Let $V$ and $W$ be finite dimensional vector spaces and let $A : V \to W$ be a linear map. Show that the dual map $A^* : W^* \to V^*$ is given in coordinates as follows. Let $\{e_i\}$ and $\{f_j\}$ be bases for $V$ and $W$, and let $\{e^i\}$ and $\{f^j\}$ be the corresponding dual bases for $V^*$ and $W^*$. If $Ae_i = A_i^j f_j$ then $A^* f^j = A_i^j e^i$.

7. Let $V$ be a finite dimensional vector space and let $\langle \cdot, \cdot \rangle$ be an inner product on $V$. The inner product determines an isomorphism $\phi : V \to V^*$.

   (a) Show that the isomorphism $\phi$ is given in coordinates as follows. Let $\{e_i\}$ be a basis for $V$, let $\{e^i\}$ be the dual basis, and write $g_{ij} = \langle e_i, e_j \rangle$. Then $\phi(e_i) = g_{ij} e^j$.

   (b) The inner product, together with the isomorphism $\phi$, define an inner product on $V^*$. Write this in coordinates as $g^{ij} = \langle e^i, e^j \rangle$. Show that the matrix $(g^{ij})$ is the inverse of the matrix $(g_{ij})$.

8. How difficult was this assignment? How many hours did you spend on it?