4. **Calculating ds in a different coordinate system**

   Cylindrical polar coordinates are defined by

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}$$

   (a) Confirm that $dx = d\rho\cos\phi - \rho\sin\phi d\phi$.

   (b) Calculate a similar expression for dy.

   (c) Starting from $ds^2 = dx^2 + dy^2 + dz^2$ show that $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$.

   (d) Having warmed up with that calculation, repeat with spherical polar coordinates which are defined by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

   and show that $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$.

   Hint: The spherical result is easier to get starting from the cylindrical result and using $\rho = r\sin\theta$.

4. **Solution: Calculating ds in a different coordinate system**

   (a) This is a simple application of the product rule $dx = d\rho\cos\phi - \rho\sin\phi d\phi$.

   (b) $dy = d\rho\sin\phi + \rho\cos\phi d\phi$.

   (c) Now

$$dx^2 + dy^2 = (d\rho\cos\phi - \rho\sin\phi d\phi)^2 + (d\rho\sin\phi + \rho\cos\phi d\phi)^2$$

   The cross terms cancel so

$$dx^2 + dy^2 = d\rho^2(\cos^2\phi + \sin^2\phi) + \rho^2 d\phi^2(\sin^2\phi + \cos^2\phi) = d\rho^2 + \rho^2 d\phi^2$$

   Adding dz^2 gives the desired result.

   (d) Start with $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$. Now $\rho^2 = r^2\sin^2\theta$ and the product rule applied to $d\rho$ gives

$$d\rho = dr\sin\theta + r\cos\theta d\theta$$

   and also

$$dz = dr\cos\theta - r\sin\theta d\theta$$

   Thus substituting into ds^2 from above gives

$$ds^2 = (dr\sin\theta + r\cos\theta d\theta)^2 + r^2\sin^2\theta d\phi^2 + (dr\cos\theta - r\sin\theta d\theta)^2$$

   Once again the cross-terms cancel leaving

$$ds^2 = dr^2(\sin^2\theta + \cos^2\theta) + r^2(\cos^2\theta + \sin^2\theta)d\theta^2 + r^2\sin^2\theta d\phi^2$$

   which simplifies to the desired result $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$.

5. **Geodesics on the Sphere**

The equation of a sphere in spherical polar coordinates is particularly simple: it is $r = a$, where a is a constant.

(a) Starting with ds in spherical polar coordinates, write down the simplified form of ds when $r = a$ is a constant.

(b) Use this expression for ds to write down an integral that represents the distance between two points connected by a path that lies on the surface of a sphere. Write the integral in the form where ϕ is a function of θ.

(c) Write down a first integral for this integrand.

(d) Show that

$$\phi - \phi_0 = \sin^{-1}[\alpha \cot \theta]$$

satisfies the first integral, where ϕ_0 and α are two independent constants.

(e) The equation of a plane through the origin is $Ax + By + Cz = 0$. Rewrite this equation in spherical polar coordinates. Rearrange the equation to make it look like the solution above and find α and ϕ_0 in terms of A, B and C.

(f) Thus give a simple geometric description and method of finding geodesics on a sphere.

5. **Solution: Geodesics on the Sphere**

(a) If $r = a$ is a constant then

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta\, d\phi^2 .$$

(b) The integral is

$$I = \int ds = a \int_{\theta_A}^{\theta_B} \sqrt{1 + \sin^2\theta\, \phi'^2}\, d\theta .$$

Thus $F(\theta, \phi, \phi') = \sqrt{1 + \sin^2\theta\, \phi'^2}$.

(c) Since $\partial F/\partial \phi = 0$ a first integral is

$$\frac{\sin^2\theta\, \phi'}{\sqrt{1 + \sin^2\theta\, \phi'^2}} = C$$

or

$$\phi' = \pm \left(\frac{C}{\sin\theta \sqrt{\sin^2\theta - C^2}} \right) .$$

(d) Direct differentiation and some algebra, yields the result.

(e) In spherical coordinates this becomes

$$Ar \sin\theta \cos\phi + Br \sin\theta \sin\phi + Cr \cos\theta = 0$$

The r cancels, the $\cos\theta$ can be moved to the other side and both sides divided by $\sin\theta$ to give

$$A \cos\phi + B \sin\phi = -C \cot\theta$$

Trig identities can be used to rewrite the left hand side as

$$\sqrt{A^2 + B^2} \sin(\phi - \phi_0) = -C \cot\theta$$

where $\phi_0 = -\tan^{-1}(A/B)$ and $\alpha = -C/\sqrt{A^2 + B^2}$.

(f) In other words, the curve with the shortest distance lies simultaneously on the surface of a sphere AND on a plane through the origin. The intersection of such a plane and a sphere is called a *great circle*.