Math 4081, Final Exam **Name (print):** ______________________

Prof. Jo Nelson

Monday 5/7/18

This exam contains 7 pages (including this cover page) and 6 problems.

Put your name on the top of every page, in case the pages become separated.

You may *not* discuss this exam with anyone other than me.

Answer all questions, writing in complete sentences as appropriate. When using lemmas, propositions, or theorems from Lee please give a reference. The following rules apply:

- **Turning in the exam**. You should upload the final to gradescope on **Monday by 10pm 5/7/18**.
- **Allowable materials**. This is an open book exam. In particular, you are allowed to use your notes, wikipedia, the course textbooks, and other textbooks. You can use electronic versions of textbooks. You may NOT google the questions.
- **Office hours**. I will have office hours 12-1pm and 2.30-3.30pm on Monday 4/30/18. You may come to discuss the midterm with me during my office hours. You may also email me questions, though beware that I do not check my email on the weekends.
- Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. (10 points) Let X, Y be vector fields on a smooth manifold M that are pointwise linearly independent and $[X, Y] = 3X - 2Y$. For all $p \in M$, does there exist a submanifold through p with tangent space spanned by $\{X, Y\}$.

2. (10 points) Let (M, g) be a compact Riemannian manifold with boundary, let $\tilde{g}$ denote the induced Riemannian metric on ∂M, and let N be the outward unit normal vector field along ∂M.

   a) Show that the divergence operator satisfies the following product rule for $f \in C^\infty(M)$ and a vector field $X \in \mathfrak{X}(M)$:
$$div(fX) = f \text{ div } X + \langle \text{grad } f, X \rangle_g$$

   b) Prove the following "integration by parts" formula:
$$\int_M \langle \text{grad } f, X \rangle_g dV_g = \int_{\partial M} f \langle X, N \rangle_g dV_{\tilde{g}} - \int_M (f \text{ div } X) dV_g.$$

3. (10 points) Let (M, g) be an oriented Riemannian manifold and X a smooth vector field on M. Show that

$$\begin{aligned} \iota_X dV_g &= *X^\flat \\ \text{div } X &= *d * X^\flat \end{aligned}$$

and, when $\dim M = 3$,

$$\text{curl } X = (*dX^\flat)^\sharp$$

Note that the explanation of the curl operator appears on pages 426-427 of Lee SECOND.

4. (10 points) For these problems, you may find HW # 9, problem 3 helpful, as it concerns the Hodge star operator, which is the homomorphism $* : \Lambda^k T^*M \to \Lambda^{n-k} T^*M$ satisfying

$$\omega \wedge *\eta = \langle \omega, \eta \rangle_g \, dV_g.$$

In c) and d) take $\mathbb{R}^n$ to be a Riemannian manifold equipped with the Euclidean metric (e.g. inner product) and the standard orientation.

a) Show that $* : \Lambda^0 T^*M \to \Lambda^n T^*M$ is given by $*f = f dV_g$

b) Show that $* * \omega = (-1)^{k(n-k)} \omega$ if $\omega \in \Omega^k(M)$.

c) Calculate $*dx^i$ for $i = 1, ..., n$

d) Calculate $*(dx^i \wedge dx^j)$ in the case when $n = 4$.

5. (10 points) A symplectic manifold is a smooth manifold M equipped with a nondegenerate closed 2-form ω. A closed nondegenerate 2-form is said to be a symplectic form.

a) Show that if there exists a symplectic form on a smooth manifold M, then $\dim M = 2n$.

b) Show that the only sphere S^n which admits a symplectic form is S^2. Hint: Use Stokes' theorem and the computation of the de Rham cohomology groups of S^n.

6. (10 points) For each $n \geq 1$, compute the de Rham cohomology groups of $\mathbb{R}^n \setminus \{e_1, -e_1\}$ and for each nonzero cohomology group, give specific differential forms whose cohomology classes form a basis.