Math 451 HW # 8, due Friday 11/4/22 **NAME:**

1. Lee 11-7 [Second Edition].
   In the following subproblems, M and N are smooth manifolds, $F : M \to N$ is a smooth map, and ω is a covector field on N. Compute $F^*\omega$ in each case.
   (a) $M = N = \mathbb{R}^2,\ F(s,t) = (st, e^t)$,
   $\omega = xdy - ydx$
   (b) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3,\ F(\theta,\varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi)$,
   $\omega = z^2 dx$

2. Define a 1-form α on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ by
$$\alpha = \left(\frac{-y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2}\right)dy.$$
   (a) Calculate $\int_C \alpha$ for any circle C of radius r around the origin.
   (b) Prove that in the half plane $\{x > 0\}$, α is the differential of a function.

3. Lee 11-14 [Second Edition]
   Consider the following two 1-forms on $\mathbb{R}^3$
$$\begin{aligned} \omega &= -\frac{4z\ dx}{(x^2+1)^2} + \frac{2y\ dy}{y^2+1} + \frac{2x\ dz}{x^2+1} \\ \eta &= -\frac{4xz\ dx}{(x^2+1)^2} + \frac{2y\ dy}{y^2+1} + \frac{2\ dz}{x^2+1} \end{aligned}$$
   (a) Determine whether either of ω or η is exact.
   (b) Set up and evaluate the line integral of each 1-form along the straight line segment from $(0,0,0)$ to $(1,1,1)$, OR, for each 1-form that is exact, find a potential function and use it to recompute the line integral.

4. Lee 11-15 [Line Integrals of Vector Fields].
   Let X be a smooth vector field on an open subset $U \subset \mathbb{R}$. Given a piecewise smooth curve segment $\gamma : [a,b] \to U$, define the **line integral of** X **over** γ,
$$\int_\gamma X \cdot ds := \int_a^b X_{\gamma(t)} \cdot \gamma'(t)dt,$$
   where the dot on the right-hand side denotes the Euclidean dot product between tangent vectors at $\gamma(t)$, identified with elements of $\mathbb{R}^n$. A **conservative vector field** is one whose line integral around every piecewise smooth closed curve is zero.
   (a) Show that X is conservative if and only if there exists a smooth function $f \in C^\infty(U)$ such that $X = \operatorname{grad} f$. *Hint: consider the covector field* ω *defined by* $\omega_x(v) = X_x \cdot v$.
   (b) Suppose $n = 3$. Show that if X is conservative, then curl $X = 0$, where
$$\text{curl } X = \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right)\frac{\partial}{\partial x^3}.$$
   (c) Show that if $U \subset \mathbb{R}^3$ is star-shaped, then X is conservative on U if and only if curl $X = 0$.

* Which problems provided a worthwhile learning experience? How many hours did you spend on it?