Math 451 HW # 6, due Friday 10/7/22 **NAME:**

1. Lee 8-16 [Second Edition]
For each of the following pairs of vector fields $X,\ Y$ defined on $\mathbb{R}^3$, compute the Lie bracket $[X, Y]$.

$$\begin{array}{lrcl lrcl} \text{(a)} & X_1 & = & y\dfrac{\partial}{\partial z} - 2xy^2\dfrac{\partial}{\partial y} & & Y_1 & = & \dfrac{\partial}{\partial y} \\[2ex] \text{(b)} & X_2 & = & x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x} & & Y_2 & = & y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y} \end{array}$$

Brute force computation is perhaps less enjoyable than remembering what the Lie Bracket of two vector fields encodes.

2. Lee 7-2 [Second Edition]
Let G be a Lie group.

(a) Let $m : G \times G \to G$ denote the multiplication map. *If needed you can assume that m is a smooth submersion.* Using Prop 3.14 to identify $T_{(e,e)}(G \times G)$ with $T_eG \oplus T_eG$, show that the differential $dm_{(e,e)} : T_eG \oplus T_eG \to T_eG$ is given by

$$dm_{(e,e)}(X, Y) = X + Y$$

Hint: compute $dm_{(e,e)}(X, 0)$ and $dm_{(e,e)}(0, Y)$ separately.

(b) Let $i : G \to G$ denote the inversion map. Show that $di_e : T_eG \to T_eG$ is given by $di_e(X) = -X$.

3. Lee 7-11 [Second Edition]
Considering $\mathbb{S}^{2n+1}$ as the unit sphere in $\mathbb{C}^{n+1}$, define an action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1}$, called the **Hopf action**, by

$$z \cdot \left(w^1, ..., w^{n+1}\right) = \left(zw^1, ..., zw^{n+1}\right).$$

Show that this action is smooth and its orbits are disjoint unit circles in $\mathbb{C}^{n+1}$ whose union is $\mathbb{S}^{2n+1}$.

4. Lee 7-16 [Second Edition]
Prove that $\mathrm{SU}(2)$ is diffeomorphic to $\mathbb{S}^3$.