Math 451 HW # 10, due Friday 11/18/22 NAME:

1. Let M be a smooth manifold and $\alpha \in \Omega^1(M)$.
   (a) Show that a smooth distribution $\xi = \ker \alpha$ is maximally (e.g. nowhere) nonintegrable whenever $d\alpha|_\xi$ is nondegenerate. Remark: A maximally nonintegrable hyperplane distribution ξ is called a contact structure and its defining 1-form α is called a contact form. *(Hint: Use HW #9 (3))*
   (b) Show that if M admits a maximally nonintegrable hyperplane distribution then M must be odd dimensional.
   (c) Let $\dim M = 2n+1$. Show that nondegeneracy of $d\alpha|_\xi$ is equivalent to the condition that $\alpha \wedge (d\alpha)^n$ is a volume form.
   (d) A contact form α uniquely determines a Reeb vector field R_α by the equations
   $$\iota(R_\alpha)d\alpha = 0, \quad \alpha(R_\alpha) = 1.$$
   What is the geometric interpretation of each of these equations? Compute $\mathcal{L}_{R_\alpha}\alpha$ and deduce that the flow of R_α preserves the form α and hence the contact structure ξ.

2. Let X, Y be vector fields on a smooth manifold M that are pointwise linearly independent and $[X, Y] = 3X - 2Y$. For all $p \in M$, does there exist a submanifold through p with tangent space spanned by $\{X, Y\}$?

3. Lee 16-18 a, b, c [SECOND]
   Let (M, g) be an oriented Riemannian n-manifold. This problem outlines an important generalization of the operator
   $$* : C^\infty(M) \to \Omega^n(M) :$$
   (a) For each $k = 1, ..., n$, show that g determines a unique inner product on on $\Lambda^k(T_p^*M)$ (denoted by $\langle \cdot, \cdot, \rangle_g$ just like the inner product on T_pM) satisfying
   $$\langle \omega^1 \wedge ... \wedge \omega^k, \eta^1 \wedge ... \wedge \eta^k \rangle_g = \det\Big(\langle (\omega^i)^\#, (\eta^j)^\# \rangle_g \Big)$$
   whenever $\omega^1, ..., \omega^k, \eta^1, ..., \eta^k$ are covectors at p. *Hint given in Lee 16-18 (a).*
   (b) Show that the Riemannian volume form dV_g is the unique positively oriented n-form that has unit norm with respect to this inner product.
   (c) For each $k = 0, ..., n$ show that there is a unique smooth bundle homomorphism
   $$* : \Lambda^k T^*M \to \Lambda^{n-k}T^*M$$
   satisfying
   $$\omega \wedge *\eta = \langle \omega, \eta \rangle_g dV_g$$
   for all smooth k-forms ω, η. (For $k = 0$, interpret the inner product as ordinary multiplication.) This map is called the **Hodge star operator.** *Hint given in Lee 16-18 (c).*

* Optional: Let (M, g) be an oriented Riemannian manifold and X a smooth vector field on M. Show that
$$\begin{aligned} \iota_X dV_g &= *X^\flat \\ \text{div } X &= *d * X^\flat \end{aligned}$$
and, when $\dim M = 3$,
$$\text{curl } X = (*dX^\flat)^\sharp$$
Note that the explanation of the curl operator appears on pages 426-427 of Lee SECOND.

* Which problems provided a worthwhile learning experience? How many hours did you spend on it?