Math 222 HW#7, due Friday 4/2/21 NAME:

Do 1-5 or 3-6. Prof Jo is happy to give hints on #6; please ask via a discussion in Canvas.

1. Compute $\int_C (y^2 dx - 2x dy)$ where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$, oriented counterclockwise (Folland: $-4/3$). Do so by (a) Directly parametrizing C; (b) Green's theorem.

2. Compute $\int_C F \cdot d\mathbf{x}$, where $F = (x^2 y, x^3 y^2)$ and C is the closed curve formed by portions of the line $y = 4$ and the parabola $y = x^2$, oriented counterclockwise. Do not use Green's theorem. (Folland: $2(\frac{32}{5} + \frac{1024}{9} - \frac{32}{3})$)

3. Jones Problem 11-1
   Suppose $C \subset \mathbb{R}^2$ is a curve described in polar coordinates by an equation $r = g(\theta)$, where $a \leq \theta \leq b$. Show that the length of C is
   $$\int_a^b \sqrt{g'(\theta)^2 + g(\theta)^2}\, d\theta$$
   Suggestion: Using polar coordinates, we obtain a parametrization of C: $x(\theta) = g(\theta)\cos\theta,\ \ y(\theta) = g(\theta)\sin\theta$.
   Remark: This formula is usually written as
   $$\int_a^b \sqrt{\left(\frac{dr}{d\theta}^2 + r^2\right)}\, d\theta$$

4. Curves and Surfaces computation (made easier by Jones 11-1 if you convert to polar and use formula)
   Consider the logarithmic spiral $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ given by
   $$\mathbf{r}(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$$
   with $a > 0,\ b < 0$. Compute the arc length function $S : \mathbb{R} \to \mathbb{R}$ where
   $$S(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)| d\tau$$
   for t_0 corresponding to an arbitrary choice of $t_0 \in \mathbb{R}$. Describe/sketch the curve.

5. Jones 12.E, 12-7
   Define a 1-form α on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ by $\alpha = \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy$.
   (a) Calculate $\int_C \alpha$ for any circle C of radius R around the origin.
   (b) Prove that in the half plane $\{x > 0\}$, α is the differential of a function.

6. Curves and Surfaces short proof
   Let $\mathbf{r} : I \to \mathbb{R}^3$ be a C^1 curve and $[a, b] \subset I$. Prove that
   $$|\mathbf{r}(a) - \mathbf{r}(b)| \leq \int_a^b |\mathbf{r}'(t)| dt,$$
   e.g. straight lines are the shortest curves joining two given points. Hooray!
   Suggestion: First, prove that for any constant vector $u \in \mathbb{R}^3$, that
   $$\left\langle \int_a^b \mathbf{r}'(t) dt, u \right\rangle \leq |u| \int_a^b |\mathbf{r}'(t)| dt.$$
   Then use the Fundamental Theorem of Calculus after plugging in $u = \int_a^b \mathbf{r}'(t) dt$ to the preceding inequality that you proved. Be careful with your inequalities and absolute values!

* Assignment Reflections
   How difficult was this assignment? How many hours did you spend on it? Which problems did you find to provide a worthwhile learning experience? Should I be assigning a similar number of problems, fewer problems, or more problems in the future? Is there a good mix of theory and computations?