Images of Distributions

Jo Nelson

Columbia University

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2-plane fields

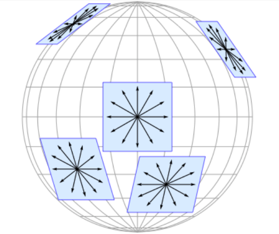
A **2-plane field** ξ on M^3 is the kernel of a 1-form α.

$$\xi = \ker \alpha := \{v \in T_pM \mid \alpha_p(v) = 0\}$$

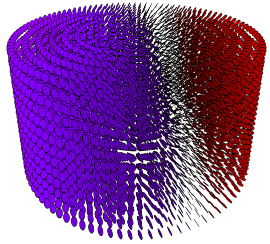
It is a smooth choice of an $\mathbb{R}^2$ subspace in T_pM at each point p.
Analogous to defining $S^2 \subset \mathbb{R}^3$ by $f^{-1}(0)$, $f = x^2 + y^2 + z^2 - 1$.

Definition

ξ is **integrable** if at each point p there is a small open chunk of a surface S in M containing p for which $T_pS = \xi_p$.



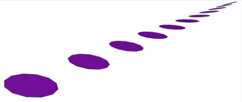
Nice and integrable



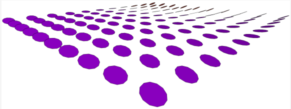
$\ker dz + r^2 d\theta$

First Contact with Contact Structures

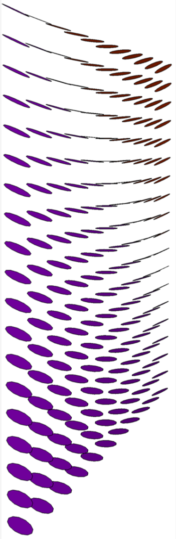
A 2-plane field ξ is a **contact structure** if it is nowhere integrable.

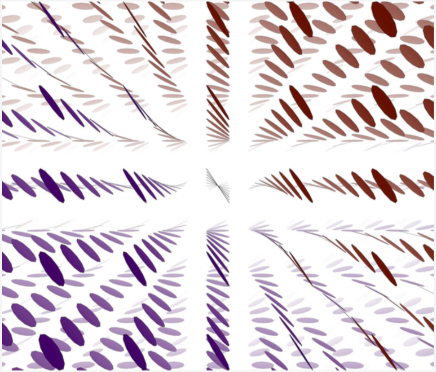


Take a line of planes rotating from $+\infty$ to $-\infty$.



Sweep these lines left-right and up-down.



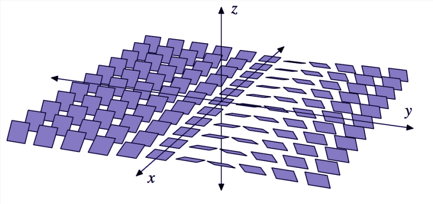


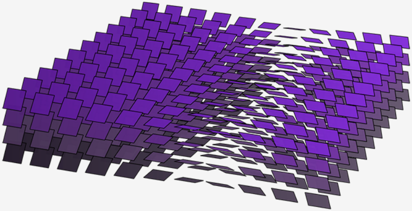
More Contact Structures

A **contact structure** is a maximally nonintegrable hyperplane field.

The kernel of a 1-form α on M^{2n-1} is a contact structure whenever

- $\alpha \wedge (d\alpha)^{n-1}$ is a volume form $\Leftrightarrow$ $d\alpha|_\xi$ is nondegenerate.





Above is ker $dz - ydx$.

Overtwisted

Take cylindrical coordinates (r, θ, z) on $\mathbb{R}^3$; define

$$\alpha_{OT} = \cos r dz + r \sin r d\theta.$$

α_{OT} is smooth since $r^2 d\theta$ and the following function are smooth

$$r \mapsto \begin{cases} \frac{\sin r}{r} & \text{for } r \neq 0, \\ 1 & \text{for } r = 0, \end{cases}$$

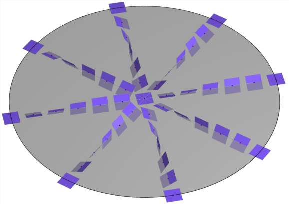
We see that α_{OT} is a contact form by computing

$$\alpha_{OT} \wedge d\alpha_{OT} = \left(1 + \frac{\sin r}{r} \cos r\right) rdr \wedge d\theta \wedge dz$$

$\xi_{OT} = \ker \alpha_{OT}$ is called the **overtwisted contact structure** on $\mathbb{R}^3$.

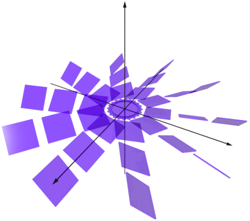
Overtwisted versus Tight

$\alpha_{OT} = \cos r dz + r \sin r d\theta$



Overtwisted

$\alpha_{std} = dz + r^2 d\theta$



Standard

ξ_{std} and ξ_{OT} are horizontal along the z-axis and all rays $\perp$ to the z-axis (with z and θ constant) are tangent to both ξ_{std} and ξ_{OT}.

Both turn $\circlearrowleft$ as one moves outwards from the z-axis along any ray. The rotation angle of ξ_{std} approaches (but never reaches) $\pi/2$, but the contact planes of ξ_{OT} make infinitely many complete turns.

Bennequin (82) proved $(\mathbb{R}^3, \xi)$ and $(\mathbb{R}^3, \xi_{OT})$ are NOT contactomorphic. This result can be regarded as the birth of contact topology!