Method of Moving Planes in Integral Forms

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Abstract

We will introduce the method of moving planes in integral forms and will explain how to use it to prove the symmetry and non-existence of solutions to integral equations and systems.

Let n be a positive integer and let $0 < \alpha < n$. We consider the integral equation

$$u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} K(y) u(y)^p dy. \tag{0.1}$$

We show that the solutions are symmetric under some appropriate integral conditions, which implies the non-existence of solutions in sub-critical case, the symmetry of solutions in critical case (This solved an open problem posed by Lieb), and the symmetry of solutions in certain supercritical case.

We also consider the integral system

$$\left\{ \begin{array}{l} u(x) = \int_{R^n} |x-y|^{\alpha-n} v(y)^q dy \\ v(x) = \int_{R^n} |x-y|^{\alpha-n} u(y)^p dy \end{array} \right. \tag{0.2}$$

with $\frac{1}{q+1} + \frac{1}{p+1} = \frac{n-\alpha}{n}$. Under the natural integrability conditions $u \in L^{p+1}(R^n)$ and $v \in L^{q+1}(R^n)$, we prove that all the solutions are radially symmetric and monotone decreasing about some point.

The technique we used is the method of moving planes in an integral form, which is new and quite different from those for differential equations. The latter relies on the local property of the PDEs, while the former bases on the global properties of the solutions. Instead of using maximum principles for PDEs, we estimate certain integral norms to compare the values of the solutions on both sides of the plane.