

ON THE CONVERGENCE OF THE OHTA-KAWASAKI EQUATION TO MOTION BY NONLOCAL MULLINS-SEKERKA LAW

NAM Q. LE

ABSTRACT. In this paper, we establish the convergence of the Ohta-Kawasaki equation to motion by nonlocal Mullins-Sekerka law on any smooth domain in space dimensions $N \leq 3$. These equations arise in modeling microphase separation in diblock copolymers. The only assumptions that guarantee our convergence result are (i) well-preparedness of the initial data and (ii) smoothness of the limiting interface. Our method makes use of the ‘‘Gamma-convergence’’ of gradient flows scheme initiated by Sandier and Serfaty and the constancy of multiplicity of the limiting interface due to its smoothness. For the case of radially symmetric initial data without well-preparedness, we give a new and short proof of the result of M. Henry for all space dimensions. Finally, we establish transport estimates for solutions of the Ohta-Kawasaki equation characterizing their transport mechanism.

1. INTRODUCTION

1.1. **The Ohta-Kawasaki equation.** This paper is concerned with the asymptotic limit, as $\varepsilon \searrow 0$, of the solutions to the Ohta-Kawasaki equation [30] with initial data u_0^ε

$$(1.1) \quad \left\{ \begin{array}{ll} \partial_t u^\varepsilon = -\Delta w^\varepsilon & (x, t) \in \Omega \times (0, \infty) \\ w^\varepsilon = \varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon) - \lambda v^\varepsilon & (x, t) \in \Omega \times [0, \infty) \\ -\Delta v^\varepsilon = u^\varepsilon - \bar{u}^\varepsilon_\Omega & (x, t) \in \Omega \times [0, \infty) \\ \bar{v}^\varepsilon_\Omega = 0 & (x, t) \in \Omega \times [0, \infty) \\ \frac{\partial u^\varepsilon}{\partial n}(x, t) = \frac{\partial v^\varepsilon}{\partial n}(x, t) = \frac{\partial w^\varepsilon}{\partial n}(x, t) = 0 & (x, t) \in \partial\Omega \times [0, \infty) \\ u^\varepsilon(x, 0) = u_0^\varepsilon(x) & x \in \Omega. \end{array} \right.$$

Here Ω is a bounded smooth domain in \mathbb{R}^N ($N \geq 2$), $f(u) = 2u(u^2 - 1)$ is the derivative of the double-well potential $W(u) = \frac{1}{2}(u^2 - 1)^2$ and $\lambda \geq 0$ is a fixed constant. Throughout, we denote \bar{u}_Ω the average of a function u over Ω : $\bar{u}_\Omega = \frac{1}{|\Omega|} \int_\Omega u dx$. Moreover, for any function u with average zero, we denote by $\|u\|_{H^{-1}(\Omega)} = \|\nabla \Delta^{-1} u\|_{L^2}$, where $\Delta^{-1} u$ is the

Date: April 24, 2010.

2000 Mathematics Subject Classification. 49J45, 35Q99, 35B25, 35K30, 35B40, 49S05.

Key words and phrases. Ohta-Kawasaki equation, nonlocal Mullins-Sekerka law, Cahn-Hilliard equation, diblock copolymers, Gamma convergence of gradient flows, De Giorgi conjecture, transport estimate.

unique solution of the elliptic problem

$$\begin{cases} -\Delta v = u & \text{in } \Omega, \\ \bar{v}_\Omega = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Associated with equation (1.1) is the Ohta-Kawasaki energy functional E_ε first introduced in [30] to model microphase separation in diblock copolymers' melts (*cf.* [3]):

$$(1.2) \quad E_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx + \frac{\lambda}{2} \|u - \bar{u}_\Omega\|_{H^{-1}(\Omega)}^2.$$

See also [12] for a derivation of E_ε from the statistical physics of interacting block copolymers. A diblock copolymer molecule is a linear chain consisting of two subchains made of two different monomers, say A and B . The function u^ε in (1.1) is related to the density parameter describing the diblock copolymers' melts: it is essentially the difference between the averaged densities of monomers A and B . The parameter ε is proportional to the thickness of the transition regions between two monomers and λ is a parameter related to the polymerization index. Outside the transition regions, $u^\varepsilon \approx \pm 1$.

There has been a vast literature on the analysis of (1.2). We refer the reader to [1, 9, 10, 32] for the study of minimizers of (1.2) and [29, 31, 33] for the existence and stability of stationary solutions of (1.2).

1.2. The nonlocal Mullins-Sekerka law. It is expected [28] that the Ohta-Kawasaki equation converges to motion by nonlocal Mullins-Sekerka law. This means that, as $\varepsilon \searrow 0$, $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ tends to a limit (u^0, v, w) , which, together with a free boundary $\cup_{0 \leq t \leq T} (\Gamma(t) \times \{t\})$, solves the following free-boundary problem in a time interval $[0, T]$ for some $T > 0$:

$$(1.3) \quad \begin{cases} u^0 = \pm 1 & \text{in } \Omega_t^\pm, \ t \in [0, T], \\ v = \Delta^{-1}(u^0 - \bar{u}_\Omega^0) & \text{in } \Omega \times [0, T], \\ \Delta w = 0 & \text{in } \Omega \setminus \Gamma(t), \ t \in [0, T], \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega \times [0, T], \\ w = \sigma\kappa - \lambda v & \text{on } \Gamma(t), \ t \in [0, T], \\ \partial_t \Gamma = \frac{1}{2} \left[\frac{\partial w}{\partial n} \right]_{\Gamma(t)} & \text{on } \Gamma(t), \ t \in [0, T], \\ \Gamma(0) = \Gamma_0. \end{cases}$$

Here $\kappa(t)$ is the mean curvature of the closed, connected hypersurface $\Gamma(t) \subset \Omega$ with the sign convention that the boundary of a convex domain has positive mean curvature (More generally, we will consider in this paper the case $\Gamma(t)$ is the union of a finite number of closed, connected hypersurfaces.); $\sigma = \int_{-1}^1 \sqrt{W(s)/2} ds = \frac{2}{3}$; $\partial_t \Gamma$ is the normal velocity

of the hypersurface $\Gamma(t)$ with the sign convention that the normal velocity on the boundary of an expanding domain is positive; \vec{n} is the unit outernormal either to Ω or $\Gamma(t)$; $\left[\frac{\partial w}{\partial n}\right]_{\Gamma(t)}$ denotes the jump in the normal derivative of w through the hypersurface $\Gamma(t)$, i.e., $\left[\frac{\partial w}{\partial n}\right]_{\Gamma(t)} = \frac{\partial w^+}{\partial n} - \frac{\partial w^-}{\partial n}$, where w^+ and w^- are respectively the restriction of w on Ω_t^+ and Ω_t^- , the exterior and interior of $\Gamma(t)$ in Ω ; and finally, $\Gamma_0 \subset\subset \Omega$ is the initial hypersurface separating the phases of the function $u_0 \in BV(\Omega, \{-1, 1\})$ which is the $L^2(\Omega)$ limit of the sequence $\{u_0^\varepsilon\}_{0 < \varepsilon < 1}$ (after extraction).

Associated with (1.3) is the nonlocal area functional E defined by

$$(1.4) \quad E(u) = \sigma \int_{\Omega} |\nabla u| + \frac{\lambda}{2} \|u - \bar{u}_{\Omega}\|_{H^{-1}(\Omega)}^2 \equiv E(\Gamma)$$

where Γ is the interface separating the phases of the function $u \in BV(\Omega, \{-1, 1\})$. This functional consists of competing short-range ($\sigma \int_{\Omega} |\nabla u|$) and long-range ($\frac{\lambda}{2} \|u - \bar{u}_{\Omega}\|_{H^{-1}(\Omega)}^2$) contributions. The former term is attractive, preferring large domains where $u = \pm 1$ with boundaries of minimal surface area. The latter term is repulsive, favoring small domains where $u = \pm 1$ which lead to cancellations.

Let us comment briefly on the well-posedness of (1.1) and (1.3). For each $\varepsilon > 0$, one can adapt the method in [15] to prove the existence and uniqueness of smooth solution to (1.1) for smooth initial data u_0^ε . The existence and uniqueness of classical solution for the free-boundary problem (1.3) with smooth initial data have been established in [16].

1.3. Related and previous results. When $\lambda = 0$, (1.1) and (1.3) are the Cahn-Hilliard equation [6, 15, 22] and Mullins-Sekerka law [26], respectively. The convergence of the Cahn-Hilliard equation to motion by Mullins-Sekerka law has been established in certain cases: for a class of very well-prepared initial data in [2, 7], in the presence of spherical symmetry in [41], for general initial data but for a weak varifold formulation of the Mullins-Sekerka law in [8], and under the validity of an H^1 -version of De Giorgi's conjecture in [23]. For the sake of completeness, we state here the key ingredient of our H^1 -version of De Giorgi's conjecture in [23]:

Conjecture (CH). *Let $\{u^\varepsilon\}_{0 < \varepsilon \leq 1}$ be a sequence of C^3 functions satisfying*

$$\int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) \right) dx \leq M < \infty, \quad \bar{u}_{\Omega}^\varepsilon = m_\varepsilon \in (-m, m) \quad (0 < m < 1).$$

and let $u \in BV(\Omega, \{-1, 1\})$ be its $L^2(\Omega)$ -limit (after extraction). Assume that $\Gamma = \partial^ \{u = 1\} \cap \Omega$ is C^2 and connected. Then*

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla(\varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon))|^2 dx \geq \sigma^2 \|\kappa\|_{H_n^{1/2}(\Gamma)}^2.$$

In the above conjecture, $\partial^* E$ denotes the reduced boundary of a set E of finite perimeter and for any function g defined on Γ , we denote by $\|g\|_{H_n^{1/2}(\Gamma)}^2$ the square of the homogeneous

Sobolev norm of g (see also Section 2.2)

$$\|g\|_{H_n^{1/2}(\Gamma)}^2 = \inf_{w \in H^1(\Omega), w=g \text{ on } \Gamma} \int_{\Omega} |\nabla w|^2 dx.$$

When $\lambda > 0$, there have been very few results justifying the convergence of (1.1) to (1.3) except in some special cases: in one space dimension by Fife and Hilhorst [17] and in higher dimensions with spherical symmetry by Henry [20]. See related results in [21]. On the other hand, there have been recent interesting works [11, 27] on the next order asymptotic limit of small volume fraction of (1.3) and (1.4). Concerning dynamics, assuming the initial component of small volume fraction, say $\{u^0(0) = 1\}$, consists of an ensemble of small spheres, the work [27] rigorously derives mean-field models for the evolution of such spheres under the nonlocal Mullins-Sekerka law (1.3).

Note that the proof of convergence of (1.1) to (1.3) with spherical symmetry in [20] was a nontrivial extension of the proof in [41] for the Cahn-Hilliard equation. In fact, (1.3) and Mullins-Sekerka dynamics are quite different. As observed in [16], in contrast to the Mullins-Sekerka law, (1.3) does not necessarily decrease the area of $\Gamma(t)$ and most importantly, spheres are not in general equilibria to (1.3) except for very special domains Ω like spherical ones. It has been an interesting and challenging problem to rigorously establish the convergence of (1.1) to (1.3) for general domains in higher space dimensions.

We are motivated by the question: is there any way to establish the convergence of (1.1) to (1.3), similar to the convergence of Cahn-Hilliard to motion by Mullins-Sekerka law, where the smooth nonlocal perturbations v^ε and v present no essential difficulty? We are also motivated by an open question in Glasner and Choksi [18] about the justification of the dynamic equations (1.3) (which have the gradient flow structure) from (1.1) via the recently established connection between Gamma-convergence and gradient flows [37].

It turns out that one can, at least formally, follow the ‘‘Gamma-convergence’’ of gradient flows scheme initiated by Sandier and Serfaty [37] to prove the convergence of (1.1) to (1.3) because of the following observations:

1. Equation (1.1) is the H^{-1} gradient flow of the Ohta-Kawasaki functional (see Sect. 2.1) E_ε .
2. The functional E_ε Gamma-converges to the nonlocal area functional E .
3. Equation (1.3) is the H^{-1} -gradient flow of E (see Sect. 2.2).

Concerning Gamma-convergence, what we will actually need is only the following liminf inequality in the definition of Gamma-convergence (denoted by Γ -convergence in what follows) [4]:

For any sequence u^ε such that $\limsup_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon) < \infty$, we can extract a subsequence, still labeled u^ε , such that u^ε converges in $L^2(\Omega)$ to a function $u^0 \in BV(\Omega, \{-1, 1\})$ and

$$\liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon) \geq E(u^0).$$

This inequality is well-known. It is a simple consequence of the Γ -convergence of the Allen-Cahn functional $\int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx$ to the area functional $\sigma \int_{\Omega} |\nabla u|$, due

to Modica-Mortola [25] (see also [40]), combined with the fact that the nonlocal term $\frac{\lambda}{2} \|u - \bar{u}_\Omega\|_{H^{-1}(\Omega)}^2$ is its continuous perturbation.

1.4. Main results. In this paper, following the ‘‘Gamma-convergence’’ of gradient flows scheme in [37], we prove the convergence of (1.1) to (1.3) on any smooth domain in space dimensions $N \leq 3$ under the following assumptions:

- (i) the initial data is well-prepared
and
- (ii) the limiting interface is smooth.

Note that the scheme in [37] when applied to Ginzburg-Landau equation with a finite number of vortices requires no smoothness of the limiting structure. This is due to its finite dimensionality character. Our setting is infinite dimensional and thus extra regularity is required to make sense of the gradient flow. It would be interesting to establish the smoothness of the limiting interface, maybe under some additional assumptions on the general initial data.

Throughout the paper, we always assume that the initial data u_0^ε satisfies the mass constraint

$$(1.5) \quad \overline{u_{0\Omega}^\varepsilon} = m_\varepsilon \in (-m, m) \quad (0 < m < 1).$$

Our first main theorem reads

Theorem 1.1. *Assume that the space dimensions $N \leq 3$. Let $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ be the smooth solution of (1.1) on $\Omega \times [0, \infty)$ with initial data u_0^ε . Assume that, after extraction, u_0^ε converges strongly in $L^2(\Omega)$ to $u^0(\cdot, 0) \in BV(\Omega, \{-1, 1\})$ with interface $\Gamma(0) = \partial\{x \in \Omega : u^0(x, 0) = 1\} \cap \Omega$ consisting of a finite number of closed, connected C^3 hypersurfaces. Let $T_* > 0$ be the minimum of the collision time and of the exit time from Ω of the hypersurfaces under the nonlocal Mullins-Sekerka law (1.3) with the initial interface $\Gamma(0)$.*

Then, after extraction, we have that for all $t \in [0, T_)$, $u^\varepsilon(\cdot, t)$ converges strongly in $L^2(\Omega)$ to $u^0(\cdot, t) \in BV(\Omega, \{-1, 1\})$ with interface $\Gamma(t) = \partial\{x \in \Omega : u^0(x, t) = 1\} \cap \Omega$. Moreover, under the following assumptions*

- (A1) *The initial data u_0^ε is well-prepared, i.e., $\lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_0^\varepsilon) = E(u^0)$,*
- (A2) *$\cup_{t \in [0, T_*)} (\Gamma(t) \times t)$ is a $C^{3,\alpha}$ ($\alpha > 0$) space-time hypersurface, that is, this hypersurface is C^α in time and for each $t \in [0, T_*)$, $\Gamma(t)$ is C^3 ,*

the Ohta-Kawasaki equation converges to motion by nonlocal Mullins-Sekerka law. That is, w^ε converges strongly in $L^2((0, T_), H^1(\Omega))$ to w solving (1.3) with the initial interface $\Gamma(0)$.*

Remark 1.1. *The restriction $N \leq 3$ on the space dimension enables us to apply Tonegawa’s convergence theorem [42] for diffused interface whose chemical potential belongs to $W^{1,p}(\Omega)$ with $p > \frac{N}{2}$. See the proof of Proposition 4.1. In our case, $p = 2$.*

Remark 1.2. *There is a large class of initial data u_0^ε for which the solutions to (1.1) satisfy (A1) and (A2). This class includes very well-prepared initial data for general domains Ω constructed similarly as in [2, 7] in the context of Cahn-Hilliard equation and radially*

symmetric initial data for spherical domains Ω . In the later case, the Hölder continuity in time of u^ε (as in (4.7)) implies the Hölder continuity in time of $\Gamma(t)$.

Remark 1.3. The interface $\Gamma(t)$ is contained in the limit measure $\mu(t)$ of $(|\nabla u^\varepsilon(t)|^2 + \frac{1}{\varepsilon}W(u^\varepsilon(t))) dx$. Throughout, we use the notation $u^\varepsilon(t) = u^\varepsilon(\cdot, t)$ etc. In general, $\text{supp}\mu(t) \setminus \Gamma(t)$ is not empty. The presence of hidden boundary outside the interface is responsible for this. However, under (A1) – (A2), hidden boundaries will be prevented during the evolution of (1.1).

In the process of proving Theorem 1.1, we also prove Conjecture (CH) for space dimensions $N \leq 3$. We state here as

Theorem 1.2. Let $\{u^\varepsilon\}_{0 < \varepsilon \leq 1}$ be a sequence of C^3 functions satisfying

$$(1.6) \quad \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) \right) dx \leq M < \infty, \quad \overline{u_\Omega^\varepsilon} = m_\varepsilon \in (-m, m) \quad (0 < m < 1).$$

and let $u \in BV(\Omega, \{-1, 1\})$ be its $L^2(\Omega)$ -limit (after extraction). Assume that $\Gamma = \partial^* \{u = 1\} \cap \Omega$ is C^2 and connected. Furthermore, assume that the space dimension $N = 2$ or 3 . Then the following inequality holds

$$(1.7) \quad \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla(\varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon))|^2 dx \geq \sigma^2 \|\kappa\|_{H_n^{1/2}(\Gamma)}^2.$$

For the case of radially symmetric initial data without well-preparedness, we give a new and short proof of the result of Henry [20] for all space dimensions in our next main theorem

Theorem 1.3. Assume that the space dimensions $N \geq 2$ and $\Omega = B_1 \subset \mathbb{R}^N$. Let $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ be the smooth solution of (1.1) on $\Omega \times [0, \infty)$ with radially symmetric initial data u_0^ε . Assume that, after extraction, u_0^ε converges strongly in $L^2(\Omega)$ to $u^0(\cdot, 0) \in BV(\Omega, \{-1, 1\})$ with interface $\Gamma(0) = \partial\{x \in \Omega : u^0(x, 0) = 1\} \cap \Omega$ consisting of a finite number of spheres. We assume that

- (B) the initial data u_0^ε has uniformly bounded energy $E_\varepsilon(u_0^\varepsilon) \leq M < \infty$,
- (BC) there exist $\alpha, \delta, \varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$, $|u_0^\varepsilon(x)| \geq \alpha$ for $x \in S_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \delta\}$.

Then there exists $T_* > 0$ such that, after extraction, we have that for all $t \in [0, T_*)$, $u^\varepsilon(\cdot, t)$ converges strongly in $L^2(\Omega)$ to $u^0(\cdot, t) \in BV(\Omega, \{-1, 1\})$ with interface $\Gamma(t) = \partial\{x \in \Omega : u^0(x, t) = 1\} \cap \Omega$ and (1.1) converges to (1.3) on the time interval $[0, T_*)$. In fact, T_* can be chosen to be the minimum of the collision time and of the exit time from Ω of the spheres under the nonlocal Mullins-Sekerka law (1.3) with initial interface $\Gamma(0)$.

Remark 1.4. We are not seeking optimal conditions on the initial data u_0^ε to make the proof more transparent. In fact, (BC) can be replaced by the following condition

- (BC') The limit measure $\mu(0)$ of $(\frac{\varepsilon}{2} |\nabla u_0^\varepsilon|^2 + \frac{1}{\varepsilon} W(u_0^\varepsilon)) dx$ (in the sense of Radon measures) does not concentrate on the boundary $\partial\Omega$: $\mu(0)(\partial\Omega) = 0$.

As a by-product of our proofs and inspired by a deformation argument in [37], we are able to provide a transport estimate for the Ohta-Kawasaki equation by establishing a convergence of the velocity in its natural energy space. For this purpose, we need a new function space $H_n^{-1}(\Omega)$. It is a modification of the usual $H^{-1}(\Omega)$ and defined as follows.

Let $\langle \cdot, \cdot \rangle$ denote the pairing between $(H^1(\Omega))^*$ and $H^1(\Omega)$. Then, define

$$H_n^{-1}(\Omega) = \{f \in (H^1(\Omega))^* \mid \exists g \in H^1(\Omega) \text{ such that } \langle f, \varphi \rangle = \int_{\Omega} \nabla g \cdot \nabla \varphi dx \forall \varphi \in H^1(\Omega)\}.$$

The function g in the above definition is unique up to a constant. We denote by $-\Delta_n^{-1}f$ the one with mean 0 over Ω . Then, $H_n^{-1}(\Omega)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{H_n^{-1}(\Omega)} = \int_{\Omega} \nabla(\Delta_n^{-1}u) \cdot \nabla(\Delta_n^{-1}v) dx \quad \forall u, v \in H_n^{-1}(\Omega).$$

Our final main result states

Theorem 1.4. *Let $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ be the smooth solution of (1.1) on $\Omega \times [0, \infty)$ as in Theorem 1.1 or Theorem 1.3. Let $u^0(\cdot, t)$ be the limit in $L^2(\Omega)$ of $u^\varepsilon(\cdot, t)$ with smooth interface $\Gamma(t)$ satisfying (1.3). Let $\partial_t \Gamma \in (C_c^1(\Omega))^N$ be any smooth extension of $(\partial_t \Gamma) \vec{n}$ where \vec{n} is the unit outernormal to $\Gamma(t)$. Then we can find a small perturbation $\partial_t \Gamma^\varepsilon$ of $\partial_t \Gamma$ such that*

$$(1.8) \quad \lim_{\varepsilon \rightarrow 0} \|\partial_t \Gamma^\varepsilon - \partial_t \Gamma\|_{C_0^1(\Omega)} = 0 \text{ for each time slice } t \geq 0$$

and

$$(1.9) \quad \lim_{\varepsilon \rightarrow 0} \int_{t_1}^{T^*} \|\partial_t u^\varepsilon + \partial_t \Gamma^\varepsilon \cdot \nabla u^\varepsilon\|_{H_n^{-1}(\Omega)}^2 dt = 0 \text{ for all } t_1 > 0.$$

In the case of well-prepared initial data, (1.9) also holds for $t_1 = 0$.

Remark 1.5. *To our knowledge, in the context of the Cahn-Hilliard and Ohta-Kawasaki equations, the transport estimate (1.9) is new. It expresses that u^ε is very close to being simply transported at the velocity $\partial_t \Gamma$ around Γ . The space $L^2((0, T^*), H_n^{-1}(\Omega))$ is the natural energy space for the velocity $\partial_t u^\varepsilon$. From the definition of $H_n^{-1}(\Omega)$ (see also section 2.1), we have*

$$\int_0^{T^*} \|\partial_t u^\varepsilon\|_{H_n^{-1}(\Omega)}^2 dt = \int_0^{T^*} \|\nabla w^\varepsilon(t)\|_{L^2(\Omega)}^2 dt = E_\varepsilon(u^\varepsilon(0)) - E_\varepsilon(u^\varepsilon(T^*)) \leq M.$$

Remark 1.6. *In general, $\partial_t \Gamma \cdot \nabla u^\varepsilon$ does not belong to $H_n^{-1}(\Omega)$. Thus, we need a small perturbation $\partial_t \Gamma^\varepsilon$ of $\partial_t \Gamma$ as in (1.8) such that $\partial_t \Gamma^\varepsilon \cdot \nabla u^\varepsilon \in H_n^{-1}(\Omega)$.*

Remark 1.7. *Setting $\lambda = 0$ in Theorems 1.1, 1.3&1.4, we recover convergence results for the Cahn-Hilliard equation to motion by Mullins-Sekerka law. Note that, due to the validity of Conjecture (CH) established in Theorem 1.2 for space dimensions $N \leq 3$, we are able to remove condition (A3) of Theorem 1.3 in our previous paper [23].*

1.5. Ideas of the proofs. We conclude this introduction with some remarks on the proofs of the main theorems.

1.

(i) The structure of the proof of Theorem 1.1 is essentially the same as that of the convergence of Cahn-Hilliard equation to motion by Mullins-Sekerka law in [23] with the nonlocal term added. However, the main ingredient and difficulty, Lemma 4.1, is not assumed as it was in Theorem 1.3 of [23]. To prove this lemma, we make use of Tonegawa's convergence theorem, Theorem 4.1; Röger's locality theorem, Theorem 4.2; and finally, Schätzle's constancy theorem, Theorem 4.3, on the multiplicity of the smooth limiting interface. Our proof reveals that the fundamental difference between (1.1) and the Cahn-Hilliard equation lies in the potential higher multiplicity of the short-range contribution in E_ε . Precisely speaking, in the limit as $\varepsilon \rightarrow 0$ (after extraction), $u^\varepsilon(t) \rightarrow u^0(t) \in BV(\Omega, \{-1, 1\})$, the long-range contribution always has multiplicity one, i.e.,

$$\lim_{\varepsilon \rightarrow 0} \frac{\lambda}{2} \left\| u^\varepsilon(t) - \overline{u^\varepsilon(t)}_\Omega \right\|_{H^{-1}(\Omega)}^2 = \frac{\lambda}{2} \left\| u^0(t) - \overline{u^0(t)}_\Omega \right\|_{H^{-1}(\Omega)}^2.$$

Meanwhile, the short-range contribution may have higher multiplicity, that is,

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon(t)|^2 + \frac{1}{\varepsilon} W(u^\varepsilon(t)) \right) dx = m(t) \sigma \int_\Omega |\nabla u^0(t)|.$$

Here the multiplicity $m(t)$ is an odd integer, possibly larger than 1. The statement of Lemma 4.1 is only true for $m(t) = 1$. See also Remark 4.4. If $m(t) > 1$, which corresponds to the case $u^\varepsilon(t)$ folds $m(t)$ times around the interface $\Gamma(t)$, then our approach using the scheme in [37] completely breaks down.

(ii) As mentioned above, the proof of Lemma 4.1 only works for single multiplicity ($m(t) = 1$) of the limiting interface and for short time. Similar result in the Cahn-Hilliard case (see Theorem 1.2 in [23] or Theorem 1.2 in this paper) works for any constant multiplicity and long time. Nevertheless, we are able to get around this higher multiplicity issue. Our idea is to use the time continuity of the limiting interface to prove single multiplicity of the short-range contribution for short time, thus establishing Lemma 4.1. Then, to prove Theorem 1.1, we will first use the Γ -convergence scheme to prove well-preparedness of solution to (1.1) for short time. The process will be iterated until the hypersurfaces in the interface $\Gamma(t)$ collide or exit to the boundary.

(iii) Our proof of inequality (4.2) in Lemma 4.1 relies heavily on the well-preparedness of the initial data. In the original gradient flows scheme [37] and for the local evolution laws like Allen-Cahn and Cahn-Hilliard, we do not have to resort to dynamics (see (C2) in Section 3.1 and Theorem 1.2). With the presence of the nonlocal terms, a purely static statement similar to (4.2) may be false except when the multiplicity one theorem of Röger-Tonegawa [36] can be improved to the case of $W^{1,p}$ ($N/2 < p \leq N$) chemical potentials. As far as we know, this issue has not been resolved yet.

2. In Theorem 1.3, the crucial observation that allows us to apply the Γ -convergence of gradient flows scheme is that, in the presence of spherical symmetry, the evolution equation (1.1) creates well-preparedness of the evolving interface almost instantaneously.

See (7.2) and Theorem 7.1.

3. The proof of Theorem 1.4 is based on the well-preparedness in time of the evolving interface and a deformation argument presented in Proposition 8.1. Its basic idea is to “lift” a curve in the limiting space to a curve in the original space in such a way that the slope of the lifted curve is that of the original one, and that the energy decreases by that of the limiting energy; see (8.5) and (8.6). This deformation argument was first proposed in the abstract setting in [37]. The idea and proof of transport estimate based on this deformation argument are easy to state and prove. The difficulty is displaced into carrying on a concrete construction for each specific problem.

The rest of the paper is organized as follows. In Section 2, we interpret the Ohta-Kawasaki and nonlocal Mullins-Sekerka equations as gradient flows and introduce necessary notations and function spaces. In Section 3, we briefly recall the Γ -convergence of gradient flows scheme in [37] and its particularization to our problem. Then we prove a main inequality à la De Giorgi in Section 4 that will be crucial in the proof of Theorem 1.1. We will present the proof of Theorem 1.2 in Section 5. Section 6 is devoted to the proof of Theorem 1.1. The proof of Theorem 1.3 will be carried out in Section 7. In the final section, Section 8, we will prove Theorem 1.4.

Note on constants and notations. In this paper, we denote by M a universal upper bound for the energy of the initial data $E_\varepsilon(u_0^\varepsilon) \leq M$ and C a generic constant that may change from line to line but does not depend on ε . For any function f of space time variables (x, t) , we will write $f(t)$ for $f(\cdot, t)$.

Acknowledgements. The author would like to thank Professor Sylvia Serfaty for her useful comments and suggestions and for communicating the proof of Theorem 4.3 during the preparation of this paper. I am grateful to Professor Mark A. Peletier for his constructive comments and interesting discussion on an earlier version of the article. The author is very grateful to the referees for their careful reading, useful comments and sharp criticisms which resulted in a hopefully improved version of the original manuscript.

2. OHTA-KAWASAKI AND NONLOCAL MULLINS-SEKERKA AS GRADIENT FLOWS

In this section, we introduce some notations used throughout the paper. In Section 2.1, we derive the gradient flow of the Ohta-Kawasaki functional defined in (1.2) with respect to an appropriately defined H^{-1} structure. In Section 2.2, we present derivations of the gradient flows of the nonlocal area functional $E(u)$ defined in (1.4) with respect to different structures. These derivations allow us to interpret (1.1) and (1.3) as gradient flows. See [19] for a different approach in interpreting (1.3) as a gradient flow.

The notion of gradient flow alluded to in this paper should be understood as follows. Let F be a C^1 functional defined over \mathcal{M} , an open subset of an affine space associated to a Hilbert space X with inner product $\langle \cdot, \cdot \rangle_X$. By the C^1 character of F , we can define the differential $dF(u)$ of F at $u \in \mathcal{M}$ and denote by $\nabla_X F(u)$ the vector of X that represents it. That is, for all $\varphi \in \mathcal{M}$, we have

$$\frac{d}{dt} \Big|_{t=0} F(u + t\varphi) = dF(u)\varphi = \langle \nabla_X F(u), \varphi \rangle_X .$$

The gradient flow of F with respect to the structure X is the evolution equation

$$\partial_t u = -\nabla_X F(u).$$

2.1. The gradient flows of the Ohta-Kawasaki functional. Recall from the Introduction that $H_n^{-1}(\Omega)$ is a Hilbert space with inner product

$$(2.1) \quad \langle u, v \rangle_{H_n^{-1}(\Omega)} = \int_{\Omega} \nabla(\Delta_n^{-1}u) \cdot \nabla(\Delta_n^{-1}v) dx \quad \forall u, v \in H_n^{-1}(\Omega).$$

The gradient of the functional E_ε defined by (1.2) with respect to the structure $H_n^{-1}(\Omega)$ is

$$(2.2) \quad \nabla_{H_n^{-1}(\Omega)} E_\varepsilon(u) = -\Delta(-\varepsilon\Delta u + \varepsilon^{-1}f(u) + \lambda\Delta^{-1}(u - \bar{u}_\Omega)).$$

Therefore, equation (1.1) is the gradient flow of E_ε with respect to the $H_n^{-1}(\Omega)$ structure.

2.2. The gradient flows of the nonlocal area functional. Consider a subdomain Ω^- of Ω with smooth boundary Γ . Assume further that Γ is the union of a finite number of disjoint closed surfaces. This is the case of the interface $\Gamma(t)$ in our Theorems. Denote by Ω^+ the set $\Omega \setminus \overline{\Omega^-}$. Let $H^{1/2}(\Gamma)$ be the space of traces on Γ of $H^1(\Omega^-)$ functions. For $f \in H^{1/2}(\Gamma)$, let $X(f)$ be the set of extensions of f into $H^1(\Omega)$ functions over Ω . Then there exists a unique function $\tilde{f} \in X(f)$ minimizing the Dirichlet functional $\int_{\Omega} |\nabla u|^2 dx$ over $X(f)$. The function \tilde{f} satisfies

$$(2.3) \quad \Delta \tilde{f} = 0 \text{ in } \Omega \setminus \Gamma, \quad \tilde{f} = f \text{ on } \Gamma, \quad \text{and} \quad \frac{\partial \tilde{f}}{\partial n} = 0 \text{ on } \partial\Omega.$$

With this \tilde{f} , we let $\Delta_\Gamma(f) = -\left[\frac{\partial \tilde{f}}{\partial n}\right]_\Gamma$ (The reader will have not failed to note that, with abuse of notation, Δ_Γ in our definition is not the Laplace-Beltrami operator of Γ). Then, in the sense of distributions

$$(2.4) \quad \Delta \tilde{f} = \Delta_\Gamma(f) \delta_\Gamma.$$

Now, for $f, u, v \in H^{1/2}(\Gamma)$, define

$$(2.5) \quad \|f\|_{H_n^{1/2}(\Gamma)} = \left\| \nabla \tilde{f} \right\|_{L^2(\Omega)}, \quad \langle u, v \rangle_{H_n^{1/2}(\Gamma)} = \langle \nabla \tilde{u}, \nabla \tilde{v} \rangle_{L^2(\Omega)} \equiv - \int_{\Gamma} (\Delta_\Gamma u) v \, d\mathcal{H}^{N-1}.$$

Observe that $\|f\|_{H_n^{1/2}(\Gamma)} = 0$ iff f is a constant on Γ . So we can define the equivalence relation \sim in $H^{1/2}(\Gamma)$: $f_1 \sim f_2$ iff $\|f_1 - f_2\|_{H_n^{1/2}(\Gamma)} = 0$.

Notation. Let $H_n^{1/2}(\Gamma)$ be the quotient space $H^{1/2}(\Gamma)/\sim$.

Then, $H_n^{1/2}(\Gamma)$ with inner product $\langle \cdot, \cdot \rangle_{H_n^{1/2}(\Gamma)}$ is a Hilbert space. Let $H_n^{-1/2}(\Gamma)$ be the dual of $H_n^{1/2}(\Gamma)$ with the usual dual norm $\|\cdot\|_{H_n^{-1/2}(\Gamma)}$. Then, we have

Lemma 2.1. ([23]) (i) For each $u \in H_n^{-1/2}(\Gamma)$, there exists a unique $u^* \in H_n^{1/2}(\Gamma)$, denoted $\Delta_\Gamma^{-1}u$, such that $u = \Delta_\Gamma u^*$ and $\|u\|_{H_n^{-1/2}(\Gamma)} = \|u^*\|_{H_n^{1/2}(\Gamma)}$. Moreover, for all $v \in H_n^{1/2}(\Gamma)$,

$$\langle u, v \rangle_{H_n^{-1/2}(\Gamma) \times H_n^{1/2}(\Gamma)} = - \langle u^*, v \rangle_{H_n^{1/2}(\Gamma)}.$$

(ii) $H_n^{-1/2}(\Gamma)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{H_n^{-1/2}(\Gamma)} = \langle \Delta_\Gamma^{-1}u, \Delta_\Gamma^{-1}v \rangle_{H_n^{1/2}(\Gamma)} \quad \forall u, v \in H_n^{-1/2}(\Gamma).$$

Now, for any $u \in BV(\Omega, \{-1, 1\})$ with the interface $\Gamma = \partial\{x \in \Omega : u(x) = 1\} \cap \Omega$, let $E(\Gamma)$ be the nonlocal area functional defined in (1.4), which arises as the Γ -limit of the Ohta-Kawasaki functional E_ε . Denote by $v = \Delta^{-1}(u - \overline{u_\Omega})$.

Then, with the choice of $\|\cdot\|_Y^2 = 4\|\cdot\|_{H_n^{-1/2}(\Gamma)}^2$, we have

Proposition 2.1. *Assume that Γ is C^3 . Then the gradient of E with respect to the structure Y at Γ is $\nabla_Y E(\Gamma) = \frac{1}{2}\Delta_\Gamma(\sigma\kappa - \lambda v) \vec{n}$, where κ is the mean curvature and \vec{n} the unit outernormal vector to Γ . So if $\Gamma(t)$ is C^3 in space-time then the gradient flow of E with respect to the structure $Y(t)$ at $\Gamma(t)$ is the nonlocal Mullins-Sekerka law (1.3).*

Proof. Because Γ is C^3 , κ is C^1 on Γ and thus $\kappa \in H^{1/2}(\Gamma)$. Consider a smooth volume preserving deformation $\Gamma(t)$ of Γ and let $V = (\partial_t \Gamma) \vec{n}$ be its normal velocity vector at $t = 0$. The volume preserving condition implies that

$$(2.6) \quad \int_\Gamma \partial_t \Gamma d\mathcal{H}^{N-1} = 0$$

and the first variation formula gives

$$(2.7) \quad \left. \frac{d}{dt} \right|_{t=0} E(\Gamma(t)) = -2 \langle \mathbf{K}, V \rangle_{L^2(\Gamma)}$$

where $\mathbf{K} = (\sigma\kappa - \lambda v) \vec{n}$. This formula can be found in [13]; see formula (2.47) in the proof of Theorem 2.3 and Remark 2.8. For completeness, we indicate a simple derivation using only (2.6). *This derivation will be used later in the proof of the construction of the deformation in Proposition 8.1.* Let $\Omega^-(t)$ be the region enclosed by $\Gamma(t)$ and $\Omega^+(t) = \Omega \setminus \overline{\Omega^-(t)}$. Set $u(x, t) = 2\chi_{\Omega^+(t)}(x) - 1$. Then $\left. \frac{d}{dt} \right|_{t=0} u(x, t) = 2\delta_\Gamma(x) \partial_t \Gamma(x)$. Recall that

$$(2.8) \quad E(\Gamma(t)) = \sigma \int_\Omega |\nabla u(t)| + \frac{\lambda}{2} \int_\Omega |\nabla v(t)|^2 dx$$

where $v(t) = \Delta^{-1}(u(t) - \overline{u(t)_\Omega})$. It is well-known that

$$(2.9) \quad \left. \frac{d}{dt} \right|_{t=0} \sigma \int_\Omega |\nabla u(t)| = \left. \frac{d}{dt} \right|_{t=0} 2\sigma \mathcal{H}^{N-1}(\Gamma(t)) = -2\sigma \langle \kappa, \partial_t \Gamma \rangle_{L^2(\Gamma)}.$$

For the variation of the second term on the left hand side of (2.8), we note that

$$v(x, t) = \int_\Omega G(x, y)(u(y, t) - \overline{u_\Omega(t)}) dy + C(t)$$

for some constant $C(t)$, where G is the Green's function of the operator $-\Delta$ on Ω with Neumann boundary condition. Integrating by parts gives

$$(2.10) \quad \begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla v(t)|^2 dx &= \frac{1}{2} \int_{\Omega} -\Delta v(t)v(t) dx \\ &= \frac{1}{2} \int_{\Omega} \int_{\Omega} G(x, y)(u(x, t) - \overline{u_{\Omega}(t)})(u(y, t) - \overline{u_{\Omega}(t)}) dy dx. \end{aligned}$$

By (2.6),

$$\left. \frac{d}{dt} \right|_{t=0} \overline{u(t)_{\Omega}} = \frac{1}{|\Omega|} \int_{\Omega} 2\delta_{\Gamma} \partial_t \Gamma dx = \frac{2}{|\Omega|} \int_{\Gamma} \partial_t \Gamma d\mathcal{H}^{N-1} = 0.$$

Hence, differentiating (2.10), we obtain

$$(2.11) \quad \begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \frac{1}{2} \int_{\Omega} |\nabla v(t)|^2 dx &= \int_{\Omega} \int_{\Omega} G(x, y)(u(y, 0) - \overline{u_{\Omega}(0)}) \left(\left. \frac{d}{dt} \right|_{t=0} u(x, t) \right) dy dx \\ &= \int_{\Omega} (v(x, 0) - C(0)) 2\delta_{\Gamma} \partial_t \Gamma dx = 2 \langle v, \partial_t \Gamma \rangle_{L^2(\Gamma)}. \end{aligned}$$

Combining (2.9) and (2.11), we get (2.7).

Therefore, the gradient of E with respect to the structure $L^2(\Gamma)$ at Γ is

$$(2.12) \quad \nabla_{L^2(\Gamma)} E(\Gamma) = -2\mathbf{K} = -2(\sigma\kappa - \lambda v) \vec{n}.$$

Now, we calculate the $H_n^{-1/2}$ -gradient $\nabla_{H_n^{-1/2}(\Gamma)} E(\Gamma) = D \vec{n}$ of $E(\Gamma)$ with respect to $H_n^{-1/2}(\Gamma)$. To do this, it suffices to express the quantity $\left. \frac{d}{dt} \right|_{t=0} E(\Gamma(t))$ as an inner product in $H_n^{-1/2}(\Gamma)$: $\left. \frac{d}{dt} \right|_{t=0} E(\Gamma(t)) = \langle D, \partial_t \Gamma \rangle_{H_n^{-1/2}(\Gamma)}$. By Lemma 2.1, and (2.5), we have

$$\begin{aligned} \langle D, \partial_t \Gamma \rangle_{H_n^{-1/2}(\Gamma)} &= \langle \Delta_{\Gamma}^{-1} D, \Delta_{\Gamma}^{-1} \partial_t \Gamma \rangle_{H_n^{1/2}(\Gamma)} = - \int_{\Gamma} (\Delta_{\Gamma}^{-1} D) \cdot \Delta_{\Gamma} (\Delta_{\Gamma}^{-1} \partial_t \Gamma) d\mathcal{H}^{N-1} \\ &= - \int_{\Gamma} (\Delta_{\Gamma}^{-1} D) \cdot \partial_t \Gamma d\mathcal{H}^{N-1}. \end{aligned}$$

It follows from (2.12) that $\Delta_{\Gamma}^{-1} D = 2(\sigma\kappa - \lambda v)$. In other words, the $H_n^{-1/2}$ -gradient $\nabla_{H_n^{-1/2}(\Gamma)} E(\Gamma)$ of E at Γ is given by $\nabla_{H_n^{-1/2}(\Gamma)} E(\Gamma) = D \vec{n} = \Delta_{\Gamma} (2(\sigma\kappa - \lambda v)) \vec{n}$. Recalling $\|\cdot\|_Y^2 = 4 \|\cdot\|_{H_n^{-1/2}(\Gamma)}^2$, we find that

$$(2.13) \quad \nabla_Y E(\Gamma) = \frac{1}{4} \nabla_{H_n^{-1/2}(\Gamma)} E(\Gamma) = \frac{1}{2} \Delta_{\Gamma} (\sigma\kappa - \lambda v) \vec{n}$$

and thus the gradient flow of $E(\Gamma)$ with respect to the structure Y at Γ is $V = -\nabla_Y E(\Gamma) = -\frac{1}{2} \Delta_{\Gamma} (\sigma\kappa - \lambda v) \vec{n}$. Recall the definition of Δ_{Γ} to find that $\partial_t \Gamma = \frac{1}{2} \left[\frac{\partial(\widetilde{\sigma\kappa - \lambda v})}{\partial n} \right]_{\Gamma}$ and this is equivalent to the nonlocal Mullins-Sekerka law (1.3). \square

3. GAMMA-CONVERGENCE OF GRADIENT FLOWS AND KEY INEQUALITIES

In this section we briefly recall the Γ -convergence of gradient flows scheme in [37] and discuss how to apply this scheme to prove the convergence of (1.1) to (1.3).

3.1. General framework. First, we recall from [37] the following general strategy.

If E_ε Γ -converges to E , then the key conditions for which the gradient flow of E_ε with respect to the structure X_ε Γ -converges to the gradient flow of E with respect to the structure Y are the following inequalities for general functions u^ε , not necessarily solving $\partial_t u^\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u^\varepsilon)$.

(C1) (*Lower bound on the velocity*) For a subsequence such that $u^\varepsilon(t) \xrightarrow{S} u(t)$, we have $u \in H^1((0, T), Y)$ and for every $s \in [0, T]$, $\liminf_{\varepsilon \rightarrow 0} \int_0^s \|\partial_t u^\varepsilon(t)\|_{X_\varepsilon}^2 dt \geq \int_0^s \|\partial_t u(t)\|_Y^2 dt$.

(C2) (*Lower bound on the slope*) If $u^\varepsilon \xrightarrow{S} u$ then $\liminf_{\varepsilon \rightarrow 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u^\varepsilon)\|_{X_\varepsilon}^2 \geq \|\nabla_Y E(u)\|_Y^2$. In the above conditions, (S) is a sense of convergence to be specified in each problem.

3.2. The case of the Ohta-Kawasaki functional. Let us now particularize the above framework to (1.1) and (1.3). In our case, the sense (S) is understood as $L^2(\Omega)$ convergence and the functionals E_ε and E are defined by (1.2) and (1.4), respectively. The space X_ε and Y are respectively $X_\varepsilon = H_n^{-1}(\Omega)$ and

$$(3.1) \quad \|\cdot\|_Y^2 = 4 \|\cdot\|_{H_n^{-1/2}(\Gamma)}^2.$$

By the results of Section 2, we are in the framework of the general scheme in [37].

The first criterion (C1) in the scheme now becomes

Proposition 3.1. *Let u^ε be defined over $\Omega \times [0, T]$ such that $\int_\Omega |u^\varepsilon(t)|^2 dx \leq M < \infty$ for all $t \in [0, T]$ and all $\varepsilon > 0$. Assume that, after extraction, $u^\varepsilon(t) \rightarrow u(t)$ in $L^2(\Omega)$ for all $t \in [0, T]$ where $u(t) \in BV(\Omega, \{-1, 1\})$ with interface $\Gamma(t) = \partial\{x \in \Omega : u(x, t) = 1\} \cap \Omega$. Then, for all $t \in (0, T)$, we have*

$$(3.2) \quad \liminf_{\varepsilon \rightarrow 0} \int_0^t \|\partial_t u^\varepsilon(s)\|_{H_n^{-1}(\Omega)}^2 ds \geq \int_0^t \|\partial_t u(s)\|_{H_n^{-1}(\Omega)}^2 ds = 4 \int_0^t \|\delta_{\Gamma(s)} \partial_t \Gamma(s)\|_{H_n^{-1}(\Omega)}^2 ds.$$

The proof of this Proposition is identical to that of Proposition 1.1 in [23].

The second criterion (C2) is equivalent to the following inequality à la De Giorgi: if u^ε converges strongly in $L^2(\Omega)$ to $u \in BV(\Omega, \{-1, 1\})$ with interface $\Gamma = \partial\{x \in \Omega : u(x) = 1\} \cap \Omega$ then

$$(3.3) \quad \liminf_{\varepsilon \rightarrow 0} \int_\Omega |\nabla w^\varepsilon|^2 dx \geq \|\sigma\kappa - \lambda v\|_{H_n^{1/2}(\Gamma)}^2.$$

Here $w^\varepsilon = \varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon) - \lambda v^\varepsilon$ and $v^\varepsilon = \Delta^{-1}(u^\varepsilon - \overline{u_\Omega^\varepsilon})$; κ is the mean curvature of Γ and $v = \Delta^{-1}(u - \overline{u_\Omega})$. Indeed, from (2.1) and (2.2), one can calculate

$$\left\| \nabla_{H_n^{-1}(\Omega)} E_\varepsilon(u^\varepsilon) \right\|_{H_n^{-1}(\Omega)}^2 = \|\Delta w^\varepsilon\|_{H_n^{-1}(\Omega)}^2 = \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2.$$

On the other hand, from (2.13) and Lemma 2.1 (ii), one deduces that

$$\|\nabla_Y E(\Gamma)\|_Y^2 = \left\| \frac{1}{2} \Delta_\Gamma(\sigma\kappa - \lambda v) \right\|_Y^2 = \|\Delta_\Gamma(\sigma\kappa - \lambda v)\|_{H_n^{-1/2}(\Gamma)}^2 = \|\sigma\kappa - \lambda v\|_{H_n^{1/2}(\Gamma)}^2.$$

We will prove (3.3) in Lemma 4.1 in Section 4.

3.3. Time-dependent limiting space. Let us emphasize that in [37], the limiting space Y is fixed. Assuming the validity of (C1) and (C2), the proof of the convergence of the gradient flow of E_ε with respect to the structure X_ε to the gradient flow of E with respect to the structure Y is quite short. In our case, we will apply (C2) (and (3.3)) to $u^\varepsilon(t)$ where u^ε is the solution of (1.1). Thus, Y is time-dependent and it is not entirely clear how to carry out the scheme in [37]. Let us say right away that we just formally follow [37] and the time-dependent nature of Y in our case is very special. The most crucial point is that the term $\|\sigma\kappa - \lambda v\|_{H_n^{1/2}(\Gamma)}^2$ on the left hand side of (3.3) can be expressed by a quantity defined globally on the whole domain Ω . Precisely, we have

$$(3.4) \quad \|\sigma\kappa - \lambda v\|_{H_n^{1/2}(\Gamma)}^2 = \inf_{\omega \in H^1(\Omega), \omega = \sigma\kappa - \lambda v \text{ on } \Gamma} \int_\Omega |\nabla \omega|^2 dx.$$

For each time slice t , (3.3) is a static statement. When considering the dynamics of (1.1), we use the function $\omega(x, t)$ such that for each time slice t , $\omega(\cdot, t) \in H^1(\Omega)$ and realizes the infimum in (3.4) for the quantity $\|\sigma\kappa - \lambda v\|_{H_n^{1/2}(\Gamma(t))}^2$. The smoothness assumption (A2) on the time-track interface $\cup_{0 \leq t \leq T} (\Gamma(t) \times \{t\})$ allows us to connect the values of ω on different time slices. See, e.g, (6.5) and (6.6) in the proof of Theorem 1.1. Thus the suspicion of the applicability of the scheme in [37] to our problem with the time-dependence nature of Y can be more or less lifted in our proofs.

4. AN INEQUALITY À LA DE GIORGI

In this section, we prove a main technical result, Lemma 4.1, that turns out to be crucial in the proof of Theorem 1.1.

Notes on notations. In this section, we consider the smooth solution $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ of (1.1) on $\Omega \times [0, \infty)$ with well-prepared initial data u_0^ε . By Proposition 6.1, we can actually choose a subsequence of ε such that $u^\varepsilon(\cdot, t)$ converges to $u^0(\cdot, t) \in BV(\Omega, \{-1, 1\})$ in $L^2(\Omega)$ for all time slice t . For ease of notation, we drop the superscript 0 in u^0 . Denote $\Gamma(t) = \partial\{x \in \Omega : u^0(x, t) = 1\} \cap \Omega$ and $\kappa(t)$ its mean curvature. Note that, due to the mass-preserving nature of (1.1), we have for all $t \in [0, \infty)$

$$(4.1) \quad \overline{u_\Omega^\varepsilon}(t) = \overline{u_{\Omega}^0} = m_\varepsilon \in (-m, m) \quad (0 < m < 1).$$

As always, we denote $\Delta^{-1}(u^0(s) - \overline{u_\Omega^0}(s))$ by $v(s)$. It is easy to see that $v^\varepsilon(t) \rightarrow v(t)$ in $H^1(\Omega)$ for each t .

Our main technical lemma reads

Lemma 4.1. (*Main Lemma*) *Assume the time-track interface $\cup_{0 \leq t \leq T} (\Gamma(t) \times \{t\})$ is $C^{3,\alpha}$. Then, there exists a positive constant $\delta(0) > 0$ depending only on the initial data $u(0)$ such that for L^1 a.e time slice $t \in [0, \delta(0)]$ we have*

$$(4.2) \quad \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla w^\varepsilon(t)|^2 dx \geq \|\sigma\kappa(t) - \lambda v(t)\|_{H_n^{1/2}(\Gamma(t))}^2.$$

Remark 4.1. *This is a nonlocal variant of an H^1 -version of De Giorgi's conjecture [14]. For more information on De Giorgi's conjectures and inequalities, we refer the reader to [23]. As explained by the end of the introduction and in Remark 4.4, a static statement similar to (4.2) may be false. However, when $\lambda = 0$, we have a purely static result as in Conjecture (CH) and Theorem 1.2.*

The rest of this section is devoted to proving Lemma 4.1. The proof of this lemma relies on

- (i) Tonegawa's convergence theorem for diffused interfaces whose chemical potentials are uniformly bounded in a Sobolev space;
- (ii) Röger's locality theorem for the weak mean curvature vector of an integral varifold; and
- (iii) Schätzle's constancy theorem for the density of an integral varifold with weak mean curvature in L^1 .

First, we recall

Theorem 4.1. (*Tonegawa's Convergence Theorem, Theorem 1 in [42]*) *Suppose $p > \frac{N}{2}$ and let $\{u^\varepsilon\}_{0 < \varepsilon \leq 1}$ be a sequence of $W^{3,p}(\Omega)$ functions satisfying*

(a) *The energy bound*

$$\int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) \right) dx \leq M < \infty,$$

(b) *The following uniform bound on the chemical potentials $\varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon)$*

$$\|\varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon)\|_{W^{1,p}(\Omega)} \leq M.$$

Then, after extraction,

(i)

$$u^\varepsilon \rightarrow u \text{ in } L^2(\Omega), \quad u \in BV(\Omega, \{-1, 1\}),$$

(ii)

$$\varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon) \rightharpoonup F, \text{ weakly in } W^{1,p}(\Omega),$$

(iii) *there exists a Radon measure μ on Ω such that, in the sense of Radon measures,*

$$\left(\frac{\varepsilon |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx \rightharpoonup \mu.$$

(iv) Moreover, $(2\sigma)^{-1}\mu$ is $(N-1)$ -integer-rectifiable varifold with $(N-1)$ -dimensional density

$$\theta^{(n-1)}(\mu, \cdot) = \theta(\cdot)2\sigma$$

where $\theta(\cdot)$ is integer-valued.

(v) Furthermore, μ has weak mean curvature $\overrightarrow{H}_\mu \in L_{loc}^{\frac{2(N-1)}{N-2}}(\mu)$ and

$$\overrightarrow{H}_\mu = \frac{F}{\theta\sigma}\nu \in L_{loc}^{\frac{2(N-1)}{N-2}}(\mu).$$

which holds μ -almost everywhere, where $\nu = \frac{\nabla u}{|\nabla u|}$ on $\partial^*\{u=1\}$ and $\nu=0$ elsewhere.

(vi) For \mathcal{H}^{N-1} a.e. $x \in \partial^*\{u=1\}$, $\theta(x)$ is an odd integer.

Theorem 4.2. (Röger's locality theorem, Proposition 3.1 in [35]) Let $E \subset \Omega$ be a set of finite perimeter, i.e. $\chi_E \in BV(\Omega)$. Assume that there are two $(N-1)$ -integer-rectifiable varifolds μ_1, μ_2 on Ω such that for $i=1, 2$, the following hold:

(a)

$$\partial^*E \subset \text{supp } \mu_i$$

(b) μ_i has locally bounded first variation with weak mean curvature vector $\overrightarrow{H}_{\mu_i}$,

(c)

$$\overrightarrow{H}_{\mu_i} \in L_{loc}^s(\mu_i), \quad s > \max\{N-1, 2\}.$$

Then

$$\overrightarrow{H}_{\mu_1}|_{\partial^*E} = \overrightarrow{H}_{\mu_2}|_{\partial^*E}$$

The above theorem justifies the definition of the weak curvature of ∂^*E if there is an $(N-1)$ -integer-rectifiable varifold μ satisfying (a)-(c).

Finally, we state the following result due to Reiner Schätzle [38] whose proof was communicated to us by Sylvia Serfaty.

Theorem 4.3. (Schätzle's Constancy Theorem) Let $\mu = \theta\mathcal{H}^n \llcorner M$ be an integral n -varifold in the open set $\Omega \subset \mathbb{R}^{n+m}$, $M \subset \Omega$ a connected C^1 - n -manifold, $\theta : M \rightarrow N_0$ be \mathcal{H}^n -measurable with weak mean curvature $\overrightarrow{H}_\mu \in L^1(\mu)$, that is

$$(4.3) \quad \int \text{div}_\mu \eta d\mu = \int_M \text{div}_M \eta \theta d\mathcal{H}^n = - \int \langle \overrightarrow{H}_\mu, \eta \rangle d\mu \quad \forall \eta \in C_0^1(\Omega, \mathbb{R}^{n+m}).$$

Then θ is a constant: $\theta \equiv \theta_0 \in N_0$. Here N_0 is the set of all nonnegative integers and $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on \mathbb{R}^{n+m} .

Proof. We consider locally C^1 -vector fields ν^1, \dots, ν^m on M , which are an orthonormal basis of the orthogonal complement TM^\perp of the tangent bundle TM in $T\mathbb{R}^{n+m}$. For $x \in M$, we choose an orthonormal basis τ_1, \dots, τ_n of the tangent space $T_x M$ of M at x . We decompose $\eta \in C_0^1(\Omega, \mathbb{R}^{n+m})$ into $\eta = \eta^{tan} + \eta^\perp$, where

$$\eta^{tan}(x) = \pi_{T_x M}(\eta(x)) \in T_x M, \quad \eta^\perp(x) = \pi_{T_x M^\perp}(\eta(x)) = \sum_{j=1}^m \langle \nu^j, \eta(x) \rangle \nu^j \in T_x M^\perp.$$

Here, we have denoted π_V the orthogonal projection operator on the subspace V of \mathbb{R}^{n+m} . In particular, $\eta^{tan}, \eta^\perp \in C_0^1(\Omega)$. Then, we have $\operatorname{div}_M \eta = \operatorname{div}_M \eta^{tan} + \operatorname{div}_M \eta^\perp$. Let D be the standard differentiation operator on \mathbb{R}^{n+m} and A_M the second fundamental form of M . Denote by \vec{H}_M the weak mean curvature of M . Then

$$\vec{H}_M = \sum_{i=1}^n A_M(\tau_i, \tau_i).$$

We have

$$\begin{aligned} \operatorname{div}_M \eta^\perp &= \sum_{i=1}^n \langle \tau_i, \nabla_{\tau_i}^M \eta^\perp \rangle = \sum_{i=1}^n \sum_{j=1}^m \langle \tau_i, D_{\tau_i} (\langle \nu^j, \eta(x) \rangle \nu^j) \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \langle \nu^j, \eta \rangle \langle \tau_i, D_{\tau_i} \nu^j \rangle = - \langle \eta, \sum_{i=1}^n A_M(\tau_i, \tau_i) \rangle = - \langle \eta, \vec{H}_M \rangle. \end{aligned}$$

From (4.3), we can calculate

$$\begin{aligned} - \int \langle \vec{H}_\mu, \eta \rangle d\mu &= - \int_M \langle \vec{H}_\mu, \eta \rangle \theta d\mathcal{H}^n = \int_M \operatorname{div}_M \eta \theta d\mathcal{H}^n \\ &= \int_M \operatorname{div}_M \eta^{tan} \theta d\mathcal{H}^n + \int_M \operatorname{div}_M \eta^\perp \theta d\mathcal{H}^n \\ &= \int_M \operatorname{div}_M \eta^{tan} \theta d\mathcal{H}^n - \int_M \langle \vec{H}_M, \eta \rangle \theta d\mathcal{H}^n. \end{aligned}$$

Let us make some special choices of η . First, for $\eta = \eta^\perp \in TM^\perp$, we conclude that the projection \vec{H}_μ^\perp of \vec{H}_μ on TM^\perp satisfies $\vec{H}_\mu^\perp = \vec{H}_M$. Since μ is integral, we get $\vec{H}_\mu^\perp \perp T\mu = TM$ by Theorem 5.8 in Brakke [5] and conclude $\vec{H}_\mu = \vec{H}_M$. Finally, if we choose η such that $\eta = \eta^{tan} \in TM$ then

$$\int_M \operatorname{div}_M \eta^{tan} \theta d\mathcal{H}^n = 0.$$

Calculating in local coordinates, this yields $\nabla_M \theta = 0$ weakly. Hence $\theta \equiv \theta_0$ is constant, as M is connected. \square

From the liminf inequality of Γ -convergence, we know that, for all t

$$\liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(t)) \geq E(u(t)) \equiv \sigma \int_\Omega |\nabla u(t)| + \frac{\lambda}{2} \int_\Omega |\nabla v(t)|^2 dx.$$

Using Schätzle's constancy theorem and Tonegawa's convergence theorem, we will improve the above inequality in (4.17) as follows

$$(4.4) \quad \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(t)) \geq \theta_0(t) \sigma \int_\Omega |\nabla u(t)| + \frac{\lambda}{2} \int_\Omega |\nabla v(t)|^2 dx,$$

where $\theta_0(t)$ is an odd integer. In order to establish the convergence of (1.1) to (1.3) using the Γ -convergence of gradient flows scheme, we must rule out the higher multiplicity (i.e.,

the case where $\theta_0(t) > 1$) of the interface $\Gamma(t)$ for all t (see Remark 4.4). Therefore, it is natural to find an upper bound for the left hand side of (4.4) to ensure, with possibly extra conditions, that $\theta_0(t) = 1$.

As a first step to rule out the higher multiplicity issue of the limiting interfaces $\Gamma(t)$, we will use Theorem 4.3 to establish the following important result concerning (1.1).

Proposition 4.1. *Suppose that for each $t \in [0, T]$, $\Gamma(t)$ is C^2 and that the interface $\Gamma(t)$ is C^α in time (cf. (A2) of Theorem 1.1), i.e.,*

$$(4.5) \quad \left| \int_{\Omega} |\nabla u(t)| - |\nabla u(s)| \right| \leq C |t - s|^\alpha \text{ for some } \alpha > 0.$$

Then, there exists $\delta(0) > 0$ depending only on the initial data $u(0)$ such that the well-preparedness of initial data guarantees for L^1 a.e. $t \in (0, \delta(0)]$, the interface $\Gamma(t)$ has multiplicity one. Precisely, there exists a Radon measure $\mu(t)$ on Ω such that, up to extracting a subsequence, we have the following convergence in the sense of Radon measures,

$$\left(\frac{\varepsilon |\nabla u^\varepsilon(t)|^2}{2} + \frac{W(u^\varepsilon(t))}{\varepsilon} \right) dx \rightharpoonup \mu(t).$$

Moreover, $\Gamma(t) \subset \text{supp } \mu(t)$; $(2\sigma)^{-1}\mu(t)$ is $(N-1)$ -integer-rectifiable varifold with $(N-1)$ -dimensional density

$$\theta^{(N-1)}(\mu(t), \cdot) = \theta(t)(\cdot)2\sigma$$

and

$$(4.6) \quad \theta(t)(\cdot) \equiv 1 \text{ on } \Gamma(t).$$

Here we call that the $(N-1)$ -dimensional density $\theta^{(N-1)}(\mu(t), x)$ alluded to above is defined as follows

$$\theta^{(N-1)}(\mu(t), x) = \lim_{r \rightarrow 0} \frac{\mu(t)(B(x, r))}{\omega_{N-1} r^{N-1}}$$

where ω_{N-1} is the volume of the unit ball in \mathbb{R}^{N-1} .

The idea of the proof is very simple. Hölder continuous hypersurfaces can not change much length in a short time. If we have higher constant integer multiplicity at a later time then to some extent, we will have more energy in E_ε . But this is a contradiction because the energy is decreasing in time for (1.1). Key to our proof is the following inequality for t sufficiently small

$$\limsup_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(t)) < 2\sigma \int_{\Omega} |\nabla u(t)| + \frac{\lambda}{2} \int_{\Omega} |\nabla v(t)|^2 dx.$$

As a preparation for the proof, we prove the following time-continuity estimates for u^ε in $L^2(\Omega)$ and v^ε in $H^1(\Omega)$.

Lemma 4.2. (i) For all $s, t \in [0, T]$

$$(4.7) \quad \|u^\varepsilon(s) - u^\varepsilon(t)\|_{L^2(\Omega)} \leq C |t - s|^{1/8}.$$

(ii). For all $s, t \in [0, T]$

$$(4.8) \quad \left| \int_{\Omega} (|\nabla v^\varepsilon(s)|^2 - |\nabla v^\varepsilon(t)|^2) dx \right| \leq C |t - s|^{1/8}.$$

Proof. Item (i) can be proved similarly as in the proof of Lemma 3.2 in [8]. Now we prove (ii). We have

$$\left| \int_{\Omega} (|\nabla v^\varepsilon(s)|^2 - |\nabla v^\varepsilon(t)|^2) dx \right| \leq (\|\nabla v^\varepsilon(s)\|_{L^2(\Omega)} + \|\nabla v^\varepsilon(t)\|_{L^2(\Omega)}) \|\nabla(v^\varepsilon(s) - v^\varepsilon(t))\|_{L^2(\Omega)}.$$

The standard estimate

$$(4.9) \quad \|\nabla v^\varepsilon\|_{L^2(\Omega)} \leq C \|u^\varepsilon - \overline{u^\varepsilon}\|_{L^2(\Omega)}$$

combined with (4.1) implies that

$$\|\nabla(v^\varepsilon(s) - v^\varepsilon(t))\|_{L^2(\Omega)} \leq C \|u^\varepsilon(s) - u^\varepsilon(t)\|_{L^2(\Omega)}$$

Recalling (i), we obtain the desired inequality. \square

Now, we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. To simplify the proof of our Proposition, we can assume further that $\Gamma(t)$ consists of one closed, connected hypersurface. Our proof can be modified easily to cover the case $\Gamma(t)$ consists of finitely many closed, connected hypersurfaces as in Theorem 1.1. For each time slice $t \in [0, T]$, we have

$$E_\varepsilon(u^\varepsilon(t)) = E_\varepsilon(u^\varepsilon(0)) - \int_0^t \|\nabla w^\varepsilon(s)\|_{L^2(\Omega)}^2 ds \leq M.$$

In particular

$$(4.10) \quad \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon(t)|^2 + \frac{1}{\varepsilon} W(u^\varepsilon(t)) \right) dx \leq E_\varepsilon(u^\varepsilon(t)) \leq M$$

and by Fatou's lemma, for L^1 a.e $t \in [0, T]$,

$$(4.11) \quad \liminf_{\varepsilon \rightarrow 0} \|\nabla w^\varepsilon(t)\|_{L^2(\Omega)}^2 < \infty.$$

Recall that

$$\varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon) = w^\varepsilon + \lambda v^\varepsilon := k^\varepsilon(t).$$

From the energy bound and the mass constraint (4.1) and in view of Lemma 3.4 in [8], which gives an upper bound for $\|k^\varepsilon(t)\|_{H^1(\Omega)}$ in terms of the energy $E_\varepsilon(u^\varepsilon(t))$ and the

homogeneous H^1 -norm $\|\nabla k^\varepsilon(t)\|_{L^2(\Omega)}$, we have for all ε sufficiently small

$$\begin{aligned} \|w^\varepsilon(t) + \lambda v^\varepsilon(t)\|_{H^1(\Omega)} = \|k^\varepsilon(t)\|_{H^1(\Omega)} &\leq C(E_\varepsilon(u^\varepsilon(t)) + \|\nabla k^\varepsilon(t)\|_{L^2(\Omega)}) \\ &= C(E_\varepsilon(u^\varepsilon(t)) + \|\nabla w^\varepsilon(t) + \lambda \nabla v^\varepsilon(t)\|_{L^2(\Omega)}) \\ &\leq C(M + \|\nabla w^\varepsilon(t)\|_{L^2(\Omega)} + \|\nabla v^\varepsilon(t)\|_{L^2(\Omega)}). \end{aligned}$$

Moreover, (4.10) gives a uniform upper bound for $u^\varepsilon(t)$ in $L^4(\Omega)$ and hence

$$\|u^\varepsilon(t) - \overline{u^\varepsilon}(t)\|_{L^2(\Omega)} \leq CM.$$

Because $v^\varepsilon(t)$ has average $\overline{v^\varepsilon} = 0$ for each t , the Poincare inequality and (4.9) gives

$$\begin{aligned} \|w^\varepsilon(t)\|_{H^1(\Omega)} &\leq \|w^\varepsilon(t) + \lambda v^\varepsilon(t)\|_{H^1(\Omega)} + \|-\lambda v^\varepsilon(t)\|_{H^1(\Omega)} \\ &\leq C(M + \|\nabla w^\varepsilon(t)\|_{L^2(\Omega)} + \|u^\varepsilon(t) - \overline{u^\varepsilon}(t)\|_{L^2(\Omega)}) \\ (4.12) \quad &\leq C(M + \|\nabla w^\varepsilon(t)\|_{L^2(\Omega)}). \end{aligned}$$

and

$$(4.13) \quad \|k^\varepsilon(t)\|_{H^1(\Omega)} = \|w^\varepsilon(t) + \lambda v^\varepsilon(t)\|_{H^1(\Omega)} \leq C(M + \|\nabla w^\varepsilon(t)\|_{L^2(\Omega)}).$$

By (4.11), we have the uniform bound in $H^1(\Omega)$ of $k^\varepsilon(t)$ for a.e $t \in [0, T]$. This combined with (4.10) allows us to apply Tonegawa's convergence theorem (see Theorem 1 in [42]). *For ease of notation, we drop a.e for the moment.* Up to extracting a subsequence, $k^\varepsilon(t)$ converges weakly to $k(t)$ in $H^1(\Omega)$ and there exists a Radon measure $\mu(t)$ on Ω such that, in the sense of Radon measures,

$$\left(\frac{\varepsilon |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx \rightharpoonup \mu(t).$$

Moreover, $(2\sigma)^{-1}\mu(t)$ is $(N-1)$ -integer-rectifiable varifold with $(N-1)$ -dimensional density

$$(4.14) \quad \theta^{(n-1)}(\mu(t), \cdot) = \theta(t)(\cdot)2\sigma$$

where $\theta(t)(\cdot)$ is integer-valued. Furthermore, $\mu(t)$ has weak mean curvature $\overrightarrow{H}_\mu(t) \in L_{loc}^{\frac{2(N-1)}{N-2}}(\mu)$ and

$$(4.15) \quad \overrightarrow{H}_\mu(t) = \frac{k(t)}{\theta(t)\sigma} \nu \in L^2(\mu(t)).$$

which holds μ -almost everywhere, where $\nu = \frac{\nabla u}{|\nabla u|}$ on $\partial^*\{u = 1\} \cap \Omega = \Gamma(t)$ and $\nu = 0$ elsewhere.

It follows from our assumption $N \leq 3$ that $\frac{2(N-1)}{N-2} > \max\{N-1, 2\}$. Thus, the locality result of Röger in Theorem 4.2 applies. Because $\Gamma(t) \subset \text{supp}\mu(t)$, we see that $\theta(t) : \Gamma(t) \rightarrow N_0$ is \mathcal{H}^{N-1} -measurable and $2\sigma\theta(t)\mathcal{H}^{N-1} \llcorner \Gamma(t)$ has weak mean curvature

$$(4.16) \quad \overrightarrow{H}_\mu(t) = \frac{k(t)}{\theta(t)\sigma} \frac{\nabla u}{|\nabla u|} \in L^2(2\sigma\theta(t)\mathcal{H}^{N-1} \llcorner \Gamma(t)).$$

By Schätzle's Theorem, $\theta(t)(\cdot)$ is a constant $\theta_0(t)$ on $\Gamma(t)$. Moreover, [42] shows that $\theta_0(t)$ is an odd integer.

Now, the constancy of θ on $\Gamma(t)$ gives

$$(4.17) \quad \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(t)) \geq 2\theta_0(t)\sigma \mathcal{H}^{N-1}(\Gamma(t)) + \frac{\lambda}{2} \int_{\Omega} |\nabla v(t)|^2 dx \\ = \theta_0(t)\sigma \int_{\Omega} |\nabla u(t)| + \frac{\lambda}{2} \int_{\Omega} |\nabla v(t)|^2 dx.$$

Moreover, from the proof of Theorem 4.3, one has $\overrightarrow{H}_\mu(t) = \overrightarrow{H}_{\Gamma(t)}$. Because $\Gamma(t)$ is C^2 , by Corollary 4.3 in [39], the weak mean curvature vector coincides with the classical mean curvature vector. Hence, (4.16) gives

$$(4.18) \quad \kappa(t) = \frac{k(t)}{\theta_0(t)\sigma}.$$

From (4.5) and (4.8), we can estimate

$$2\sigma \int_{\Omega} |\nabla u(t)| + \frac{\lambda}{2} \int_{\Omega} |\nabla v(t)|^2 dx - \left(\sigma \int_{\Omega} |\nabla u(s)| + \frac{\lambda}{2} \int_{\Omega} |\nabla v(s)|^2 dx \right) \\ \geq -2C\sigma |t-s|^\alpha - C|t-s|^{1/8} + \sigma \int_{\Omega} |\nabla u(s)|.$$

Thus, we can find $\delta = \delta(u^0, s) > 0$ depending only on the initial data and s such that for all $t \in [s, s + \delta)$

$$(4.19) \quad 2\sigma \int_{\Omega} |\nabla u(t)| + \frac{\lambda}{2} \int_{\Omega} |\nabla v(t)|^2 dx > \sigma \int_{\Omega} |\nabla u(s)| + \frac{\lambda}{2} \int_{\Omega} |\nabla v(s)|^2 dx.$$

Assuming we have the well-preparedness at time $s \geq 0$. Then

$$(4.20) \quad \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(s)) = \sigma \int_{\Omega} |\nabla u(s)| + \frac{\lambda}{2} \int_{\Omega} |\nabla v(s)|^2 dx.$$

Because the Ohta-Kawasaki functional is decreasing along the flow, one has for $t > s$

$$(4.21) \quad \limsup_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(t)) \leq \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(s)).$$

Thus from (4.17), (4.20) and (4.21), one finds that, for L^1 a.e $t \in [s, T]$,

$$(4.22) \quad \theta_0(t)\sigma \int_{\Omega} |\nabla u(t)| + \frac{\lambda}{2} \int_{\Omega} |\nabla v(t)|^2 dx \leq \sigma \int_{\Omega} |\nabla u(s)| + \frac{\lambda}{2} \int_{\Omega} |\nabla v(s)|^2 dx.$$

Revoking (4.19) and (4.22), we conclude that the interface $\Gamma(t)$ has single multiplicity $\theta_0(t) = 1$ for L^1 a.e. $t \in [s, s + \delta)$, i.e., (4.6) is satisfied. Therefore, the proof of Proposition 4.1 is complete by setting $s = 0$. \square

Remark 4.2. *The inequality (4.17) can only be strict in the presence of hidden boundary, i.e, the set $\text{supp}\mu(t) \setminus \Gamma(t)$ is not empty and has positive $(N-1)$ -dimensional Hausdorff measure. The set $\text{supp}\mu(t) \setminus \Gamma(t)$ is one where $\nu = 0$ in (4.15).*

Remark 4.3. *Our proof shows that well-preparedness of the data at any time s will ensure (4.6) for all $t \in [s, s + \delta(s)]$ with single multiplicity for $\Gamma(t)$.*

Finally, we give the proof of Lemma 4.1.

Proof of Lemma 4.1. Consider $t \in [0, \delta(0)]$ where $\delta(0)$ is defined in the proof of Proposition 4.1. We can assume that $\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla w^\varepsilon(t)|^2 \leq C$, otherwise the inequality (4.2) is trivial. Let $k^\varepsilon(t) = w^\varepsilon(t) + \lambda v^\varepsilon(t)$. Recall from (4.12) and (4.13) that

$$(4.23) \quad \|w^\varepsilon(t)\|_{H^1(\Omega)} + \|k^\varepsilon(t)\|_{H^1(\Omega)} \leq C(M + \|\nabla w^\varepsilon(t)\|_{L^2(\Omega)}) \leq C.$$

Now, up to extraction, we have $w^\varepsilon(t)$ and $k^\varepsilon(t)$ weakly converge in $H^1(\Omega)$ to some $w(t)$ and $k(t)$, respectively. Inspecting the proof of Proposition 4.1, one observes that well-preparedness of the initial data together with (4.23) implies (4.6) at the time slice t , that is, the interface $\Gamma(t)$ has constant multiplicity $\theta_0(t) = 1$. Thus, from (4.18) with the constant $\theta \equiv 1$, one deduces $k(t) = \sigma\kappa(t)$ on $\Gamma(t)$. Letting $\varepsilon \rightarrow 0$ in $k^\varepsilon(t) = w^\varepsilon(t) + \lambda v^\varepsilon(t)$, one gets $k(t) = w(t) + \lambda v(t)$. Hence $w(t) = \sigma\kappa(t) - \lambda v(t)$ on $\Gamma(t)$.

By lower semicontinuity, one has

$$(4.24) \quad \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla w^\varepsilon(t)|^2 dx \geq \int_{\Omega} |\nabla w(t)|^2 dx \geq \inf_{\omega \in H^1(\Omega), \omega = \sigma\kappa - \lambda v \text{ on } \Gamma(t)} \int_{\Omega} |\nabla \omega|^2 dx.$$

The latter minimization problem has a unique solution $\omega = \sigma\kappa(t) - \lambda v(t)$ as defined in Section 2. Therefore, from (4.24) and (2.5), we obtain

$$(4.25) \quad \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla w^\varepsilon(t)|^2 dx \geq \|\sigma\kappa(t) - \lambda v(t)\|_{H_n^{1/2}(\Gamma(t))}^2.$$

□

Remark 4.4. *It is very important to obtain the single multiplicity of the interface $\Gamma(t)$ in the proof of Lemma 4.1. In general, if $\Gamma(t)$ has constant multiplicity m then $k(t) = m\sigma\kappa(t)$ on $\Gamma(t)$ and the best inequality one can get is the following*

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla w^\varepsilon(t)|^2 dx \geq \|m\sigma\kappa(t) - \lambda v(t)\|_{H_n^{1/2}(\Gamma(t))}^2$$

where the quantity on the right hand side can be much smaller than the expected quantity $\|\sigma\kappa(t) - \lambda v\|_{H_n^{1/2}(\Gamma(t))}^2$. This is in contrast to an H^1 -version of De Giorgi's conjecture (see Theorem 1.2 in [23] and Theorem 1.2 in this paper) where any constant multiplicity suffices the proof.

5. PROOF OF THEOREM 1.2

In this section, we present the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $k^\varepsilon = \varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon)$. We can assume that $\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla k^\varepsilon|^2 dx \leq C$, otherwise the inequality (1.7) is trivial. From the energy bound and the mass constraint

(1.6) and in view of Lemma 3.4 in [8], we have for all ε sufficiently small

$$\|k^\varepsilon\|_{H^1(\Omega)} \leq C \left(\int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) \right) dx + \|\nabla k^\varepsilon\|_{L^2(\Omega)} \right) \leq C < \infty.$$

Now, up to extraction, we have that k^ε weakly converges to some k in $H^1(\Omega)$. As in the proof of Proposition 4.1, especially following (4.13)-(4.18), we can find an odd integer θ_0 such that

$$k = \theta_0 \sigma \kappa \text{ on } \Gamma \text{ a.e } \mathcal{H}^{N-1}.$$

Now, by lower semicontinuity, one has

$$(5.1) \quad \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla k^\varepsilon|^2 dx \geq \int_{\Omega} |\nabla k|^2 dx \geq \inf_{w \in H^1(\Omega), w = \theta_0 \sigma \kappa \text{ on } \Gamma} \int_{\Omega} |\nabla w|^2 dx = \theta_0^2 \sigma^2 \|\kappa\|_{H_n^{1/2}(\Gamma)}^2.$$

Because θ_0 is an odd integer, $|\theta_0| \geq 1$. This combined with (5.1) gives (1.7) as desired. \square

Remark 5.1. *In view of a recent result by Röger and Tonegawa [36], we might expect θ_0 to be exactly 1.*

6. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1, formally following [37] (see also [23] for related results for the Cahn-Hilliard equation).

First, we briefly discuss the selection result alluded to in Section 4.

For the rest of the section, $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ denotes the solution of (1.1) on $\Omega \times [0, \infty)$. Let $T > 0$ be any finite number. We define the following norm on distributions u on Ω

$$(6.1) \quad \|u\|_1 = \sup_{\varphi \in C_0^\infty(\Omega), |\nabla \varphi| \leq 1} \left| \int_{\Omega} u \varphi \right|,$$

i.e., the norm in the dual of Lipschitz functions. Then, we have the following

Proposition 6.1. *There exists $u^0 \in L^4(\Omega \times [0, T])$ such that u^0 is $C^{0,1/2}$ in time for the $\|\cdot\|_1$ -norm, and that, after extraction,*

$$(6.2) \quad u^\varepsilon \rightharpoonup u^0 \text{ in } L^4(\Omega \times [0, T]).$$

Moreover, for all $t \in [0, T]$, we have $u^0(t) \in BV(\Omega, \{-1, 1\})$ and

$$(6.3) \quad u^\varepsilon(t) \rightharpoonup u^0(t) \text{ in } L^4(\Omega), \quad u^\varepsilon(t) \rightarrow u^0(t) \text{ in } L^2(\Omega).$$

The proof of this Proposition is similar to that of Proposition 4.1 in [23] and is thus omitted.

Remark 6.1. *For each t , from the energy bound $E_\varepsilon(u^\varepsilon(t)) \leq E_\varepsilon(u^\varepsilon(0)) \leq M$ and the compactness of BV functions in $L^1(\Omega)$, we can obtain (6.3) for a subsequence of ε 's. In general, this subsequence depends on t . The main point of Proposition 6.1 is that this subsequence can be chosen independent of t . This follows from the time-continuity of u^0 in the $\|\cdot\|_1$ -norm. See Proposition 4.1 in [23] for more details.*

Now, we are in a position to present the proof of Theorem 1.1.

Proof of Theorem 1.1. **1.** First, we note that the nonlocal Mullins-Sekerka law (1.3) with smooth initial interface $\Gamma(0)$ has unique smooth solution [16]. Thus, if T_* is the minimum of the collision time and of the exit time from Ω of the hypersurfaces under the motion law (1.3), then $T_* > 0$. By the selection result in Proposition 6.1, after extraction, we have that for all $t \in [0, T_*]$, $u^\varepsilon(\cdot, t)$ converges strongly in $L^2(\Omega)$ to $u^0(\cdot, t) \in \text{BV}(\Omega, \{-1, 1\})$ with interface $\Gamma(t) = \partial\{x \in \Omega : u^0(x, t) = 1\} \cap \Omega$. By our assumption (A2) on the regularity of the time-track interface $\cup_{0 \leq t \leq T_*} (\Gamma(t) \times \{t\})$, Lemma 4.1 can be applied. Choose $\delta(0) > 0$ as in Lemma 4.1. Without loss of generality, one can assume that $\delta(0) < T_*$. We proceed as follows. First, we confirm the evolution law on $[0, \delta(0)]$. Then we can easily iterate to continue the dynamics up to time T_* .

Let us prove that the interfaces $\Gamma(t)$ ($t \in [0, \delta(0)]$) evolve by the nonlocal Mullins-Sekerka law (1.3). Indeed, we have $\partial_t u^\varepsilon = -\nabla_{H_n^{-1}(\Omega)} E_\varepsilon(u^\varepsilon)$ and, for all $t \in (0, \delta(0))$,

$$\begin{aligned} E_\varepsilon(u^\varepsilon(0)) - E_\varepsilon(u^\varepsilon(t)) &= - \int_0^t \langle \nabla_{H_n^{-1}(\Omega)} E_\varepsilon(u^\varepsilon(s)), \partial_t u^\varepsilon(s) \rangle_{H_n^{-1}(\Omega)} ds \\ &= \frac{1}{2} \int_0^t \left(\left\| \nabla_{H_n^{-1}(\Omega)} E_\varepsilon(u^\varepsilon(s)) \right\|_{H_n^{-1}(\Omega)}^2 + \|\partial_t u^\varepsilon(s)\|_{H_n^{-1}(\Omega)}^2 \right) ds \\ &= \frac{1}{2} \int_0^t \left(\|\nabla w^\varepsilon(s)\|_{L^2(\Omega)}^2 + \|\partial_t u^\varepsilon(s)\|_{H_n^{-1}(\Omega)}^2 \right) ds. \end{aligned}$$

For each $s \in (0, t)$, recall that $\kappa(s)$ is the mean curvature of $\Gamma(s)$. Let $w(\cdot, s) \in H^1(\Omega)$ be the function $\sigma\kappa(s) - \lambda v(s)$, i.e., $w(\cdot, s)$ satisfies $\Delta w(\cdot, s) = 0$ in $\Omega \setminus \Gamma(s)$, $w(\cdot, s) = \sigma\kappa(s) - \lambda v(s)$ on $\Gamma(s)$ and finally $\frac{\partial w}{\partial n} = 0$ on $\partial\Omega$. By Proposition 6.1, all assumptions of Proposition 3.1 are satisfied for u^ε and u^0 . Thus, by Lemma 4.1, the lower bound on velocity (3.2) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} (6.4) \quad E_\varepsilon(u^\varepsilon(0)) - E_\varepsilon(u^\varepsilon(t)) &\geq \frac{1}{2} \int_0^t \left(\|\sigma\kappa - \lambda v\|_{H_n^{1/2}(\Gamma(s))}^2 + 4 \|\delta_{\Gamma(s)} \partial_t \Gamma(s)\|_{H_n^{-1}(\Omega)}^2 \right) ds - o(1) \\ &= \frac{1}{2} \int_0^t \int_\Omega \left(|\nabla w(x, s)|^2 + 4 |\nabla \Delta_n^{-1}(\delta_{\Gamma(s)} \partial_t \Gamma(x, s))|^2 \right) dx ds - o(1) \\ (6.5) \quad &\geq -2 \int_0^t \int_\Omega \nabla w(x, s) \cdot \nabla(\Delta_n^{-1}(\delta_{\Gamma(s)} \partial_t \Gamma(x, s))) dx ds - o(1). \end{aligned}$$

In view of the definition of Δ_n^{-1} in (2.1), the right hand side of (6.5) becomes

$$\begin{aligned} 2 \int_0^t \langle \partial_t \Gamma(s), w \rangle_{L^2(\Gamma(s))} ds - o(1) &= \int_0^t \int_{\Gamma(s)} 2(\sigma\kappa(s) - \lambda v) \partial_t \Gamma(s) d\mathcal{H}^{N-1} ds - o(1) \\ (6.6) \quad &= - \int_0^t \frac{d}{ds} E(\Gamma(s)) ds - o(1) = E(\Gamma(0)) - E(\Gamma(t)) - o(1). \end{aligned}$$

Equality (6.6) follows from the smoothness assumption (A2). From (6.4)-(6.6), one gets

$$E_\varepsilon(u^\varepsilon(t)) - E(\Gamma(t)) \leq E_\varepsilon(u^\varepsilon(0)) - E(\Gamma(0)) + o(1).$$

By (A1), we deduce that $\limsup_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(t)) \leq E(\Gamma(t))$. However, since E_ε Γ -converges to E , we have $\liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(t)) \geq E(\Gamma(t))$. Therefore, we must have

$$(6.7) \quad \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(t)) = E(\Gamma(t)).$$

This means that well-prepared initial data remains “well-prepared” in time for all $t \in [0, \delta(0)]$ and there are no hidden boundaries in the limit measure of $E_\varepsilon(u^\varepsilon(t))$ (see Remark 4.2). Furthermore, this also shows that the inequality (6.5) is actually an equality. This implies that for each $s \in (0, t)$ and for a.e. $x \in \Omega$, we have $\nabla w(x, s) = -2\nabla \Delta_n^{-1}(\delta_{\Gamma(s)} \partial_t \Gamma(x, s))$. So $w(x, s) = -2\Delta_n^{-1}(\delta_{\Gamma(s)} \partial_t \Gamma(x, s)) + c(s)$ for some function c depending only on time. Thus, in the sense of distributions $\delta_{\Gamma(s)} \partial_t \Gamma(x, s) = -\frac{1}{2} \Delta w(x, s)$. By (2.4) and the definition of the function w , this relation is exactly the limiting dynamical law we wish to establish. Our proof of this nonlocal Mullins-Sekerka law is valid as long as $\Gamma(t) \subset \Omega$ and hypersurfaces contained in $\Gamma(t)$ do not collide for all $t < T_*$.

Now, starting from the time $\delta(0)$ with well-preparedness, we can use Remark 4.3 and Lemma 4.1 to confirm the evolution law on $[\delta(0), \delta(1)]$ where $\delta(1) = \delta(\delta(0))$ defined in the proof of Proposition 4.1. Define $\delta(k) = \delta(\delta(k-1))$. Due to the strict positivity of the area $\int_\Omega |\nabla u(t)|$ for any t , and from the construction of $\delta(k)$, we can show that

$$\lim_{k \rightarrow \infty} \delta(k) = T_*$$

where T_* can be chosen to be the minimum of the collision time and of the exit time from Ω of the hypersurfaces under the nonlocal Mullins-Sekerka law.

2. Second, we show that w^ε converges weakly in $L^2((0, T_*), H^1(\Omega))$ to w . Indeed, for all $t \in (0, T_*)$ we have

$$(6.8) \quad \int_0^t \|\nabla w^\varepsilon(s)\|_{L^2(\Omega)}^2 ds = E_\varepsilon(u^\varepsilon(0)) - E_\varepsilon(u^\varepsilon(t)) \leq M.$$

Recall from (4.12) that

$$\|w^\varepsilon(s)\|_{H^1(\Omega)} \leq C(M + \|\nabla w^\varepsilon\|_{L^2(\Omega)} + 1) \leq C.$$

It follows that for ε sufficiently small, we have

$$\int_0^t \|w^\varepsilon(s)\|_{H^1(\Omega)}^2 ds \leq C(M^2 + \int_0^t \|\nabla w^\varepsilon(s)\|_{L^2(\Omega)}^2 ds + \|u^\varepsilon\|_{L^2(\Omega \times [0, T])}^2) \leq C < \infty.$$

Therefore, up to a further extraction, we have that w^ε weakly converges to some z in $L^2((0, T_*), H^1(\Omega))$. We are going to prove that for a.e. $t \in (0, T_*)$,

$$(6.9) \quad z(x, t) = \sigma \kappa(x, t) - \lambda v(x, t) = w(x, t) \text{ for } \mathcal{H}^{N-1} \text{ a.e. } x \in \Gamma(t).$$

Indeed, from (6.7) and $\lim_{\varepsilon \rightarrow 0} \left\| u^\varepsilon(t) - \overline{u^\varepsilon(t)}_\Omega \right\|_{H^{-1}(\Omega)}^2 = \left\| u(t) - \overline{u(t)}_\Omega \right\|_{H^{-1}(\Omega)}^2$, one deduces the single-multiplicity property of the limiting interface $\Gamma(t)$ on each time slice t . That is, in the sense of Radon measures

$$\left(\frac{\varepsilon |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx \rightharpoonup 2\sigma d\mathcal{H}^{N-1} \llcorner \Gamma(t).$$

Moreover, we have the uniform bound on the energy $E_\varepsilon(u^\varepsilon(t)) \leq M$ for all $t \in [0, T_*]$ and all $\varepsilon > 0$. Combining these facts with the dominated convergence theorem, we get

- The single-multiplicity in space-time, i.e, in the sense of Radon measures,

$$\left(\frac{\varepsilon |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dxdt \rightharpoonup 2\sigma d\mathcal{H}^{N-1} \llcorner \Gamma(t) dt.$$

- The limiting equipartition of energy in space-time, i.e, in the sense of Radon measures

$$\left| \frac{\varepsilon |\nabla u^\varepsilon|^2}{2} - \frac{W(u^\varepsilon)}{\varepsilon} \right| dxdt \rightharpoonup 0.$$

Arguing as in the proof of Lemma 3.1 in [23], we get (6.9). Now, we pass to the limit in the equation $\partial_t u^\varepsilon = -\Delta w^\varepsilon$. Recalling that w^ε weakly converges to z in $L^2((0, T_*), H^1(\Omega))$ and that w^ε satisfies the zero Neumann boundary condition, we find that $2\delta_{\Gamma(s)} \partial_t \Gamma(s) = -\Delta z(s)$ in $\Omega \times (0, T_*)$ and $\frac{\partial z}{\partial n} = 0$ on $\partial\Omega \times (0, T_*)$ in the sense of distributions. To see this, fix $t \in (0, T)$. From the assumptions of our Theorem and the dominated convergence theorem, we find that $u^\varepsilon \rightarrow u$ in $L^1(\Omega \times [0, T])$. It follows that $\partial_t u^\varepsilon(x, s) \rightarrow \partial_t u(x, s)$ in the sense of distributions. Denote by $\Omega^+(s)$ the set $\{x \in \Omega : u(x, s) = 1\}$ and recall that $\Gamma(s) = \partial\{u(s) = 1\} \cap \Omega$ is the interface separating the phases -1 and $+1$. Then, $\partial_t u(s) = \partial_t(u(s) + 1) = \partial_t(2\chi_{\Omega^+(s)}) = 2\delta_{\Gamma(s)} \partial_t \Gamma(s) = -\Delta z(s)$.

Recall from **1.** that $2\delta_{\Gamma(s)} \partial_t \Gamma(s) = -\Delta w(s)$. Therefore, in the sense of distributions, $\Delta(z - w) = 0$ in $\Omega \times (0, T_*)$ and $\frac{\partial(z-w)}{\partial n} = 0$ on $\partial\Omega \times (0, T_*)$. From (6.9), we conclude that $z = w$ a.e. in $\Omega \times (0, T_*)$ and this shows that w^ε converges weakly to w in $L^2((0, T_*), H^1(\Omega))$.

3. Finally, we now complete the proof of the theorem by showing that w^ε actually converges strongly in $L^2((0, T_*), H^1(\Omega))$ to w . In fact, because of the equality (6.7), the inequality (6.4) is actually an equality. Therefore

$$(6.10) \quad \lim_{\varepsilon \rightarrow 0} \int_0^{T_*} \|\nabla w^\varepsilon(s)\|_{L^2(\Omega)}^2 ds = \int_0^{T_*} \int_\Omega |\nabla w(x, s)|^2 dx ds.$$

Since ∇w^ε converges weakly to ∇w in $L^2((0, T_*), L^2(\Omega))$, we conclude that ∇w^ε converges strongly to ∇w in $L^2((0, T_*), L^2(\Omega))$. It follows that w^ε converges strongly to w in $L^2((0, T_*), H^1(\Omega))$ and this completes the proof of Theorem 1.1. \square

7. PROOF OF THEOREM 1.3

In this section, we prove Theorem 1.3.

Proof of Theorem 1.3. By (4.7), u^ε is Hölder continuous in time. From its radial symmetry and the fact that $\Gamma(t)$ consists of a finite number of spheres, we have the Hölder continuity in time for the limiting interface $\Gamma(t)$. This together with (BC) implies the existence of $T_* > 0$ such that for all $t \in [0, T_*]$, the spheres contained in $\Gamma(t)$ do not collide and

(BC') The limit measure $\mu(t)$ of $\left(\frac{\varepsilon}{2} |\nabla u^\varepsilon(t)|^2 + \frac{1}{\varepsilon} W(u^\varepsilon(t))\right) dx$ (in the sense of Radon measures) does not concentrate on the boundary $\partial\Omega$: $\mu(t)(\partial\Omega) = 0$.

As in (4.13), denoting $k^\varepsilon(t) = \varepsilon \Delta u^\varepsilon(t) - \varepsilon^{-1} f(u^\varepsilon(t))$, we have

$$\|k^\varepsilon(t)\|_{H^1(\Omega)} \leq C(M + \|\nabla w^\varepsilon(t)\|_{L^2(\Omega)}).$$

Integrating from 0 to T^* , and recalling (6.8), we obtain

$$(7.1) \quad \int_0^{T^*} \|k^\varepsilon(t)\|_{H^1(\Omega)}^2 dt \leq C.$$

By Fatou's lemma, for L^1 a.e $t \in [0, T^*)$, we have

$$(7.2) \quad \liminf_{\varepsilon \rightarrow 0} \|k^\varepsilon(t)\|_{H^1(\Omega)} < \infty.$$

Let $t_0 \geq 0$ be any sufficiently small number such that (7.2) is satisfied. It suffices to prove the following

Proposition 7.1. *The limit function (u^0, v, w) and the interfaces $\Gamma(t)$ satisfy (1.3) on $[t_0, T^*)$. Furthermore, we have well-preparedness of the interface $\Gamma(t)$ for all time slice $t \geq t_0$, i.e.,*

$$\lim_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(t)) = E(\Gamma(t)).$$

Then (u^0, v, w) and $\Gamma(t)$ satisfy (1.3) on $[0, T^*)$ with the initial data $\Gamma(0)$ understood as the initial trace: $\lim_{t \searrow 0} \Gamma(t) = \Gamma(0)$. Indeed, for radial solution with interface consisting of a finite number of spheres, the Hölder continuity in time of u^ε in (4.7) implies the Hölder continuity in time of $\Gamma(t)$. Thus the above limit of $\Gamma(t)$ as $t \rightarrow 0$ exists.

The proof of Proposition 7.1 relies on the following theorem, which could be of independent interest.

Theorem 7.1. *Let (u^ε) be a sequence of smooth radially symmetric functions on $\Omega = B_1$ such that*

$$(1) \quad \frac{\partial u^\varepsilon}{\partial n} = 0 \text{ on } \partial\Omega, \quad (2) \quad \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) \right) dx \leq C,$$

$$(3) \quad \liminf_{\varepsilon \rightarrow 0} \|\varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon)\|_{H^1(\Omega)} \leq C.$$

(4) *The limit measure μ of $\left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon)\right) dx$ (in the sense of Radon measures) does not concentrate on the boundary $\partial\Omega$: $\mu(\partial\Omega) = 0$.*

Then, up to extracting a subsequence, u^ε converges in $L^2(\Omega)$ to $u \in BV(\Omega, \{-1, 1\})$ with interface Γ separating the phases. Then

$$(7.3) \quad \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon) = E(\Gamma).$$

Remark 7.1. *Our theorem is an elliptic refinement of Chen's result [8] (Theorem 5.3) for the time-dependent Cahn-Hilliard equation.*

Proof. For simplicity, let us denote $k^\varepsilon = \varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon)$ and the discrepancy measure by $\xi^\varepsilon = \frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 - \frac{1}{\varepsilon} W(u^\varepsilon)$. By (1) and (2) and following the argument of the proof of

Theorem 5.1 in [8], one can bound the discrepancy measure in term of the Allen-Cahn energy as follows

$$\int_{\Omega} |\xi^\varepsilon| dx \leq C_1 (\delta + \eta + \varepsilon + C(\delta, \eta)\sqrt{\varepsilon}) \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) \right) dx,$$

where δ, η are arbitrary small numbers and C_1 is independent of $\varepsilon, \delta, \eta$. Sending first ε to 0 and then δ and η to 0, we obtain the limiting equipartition of energy

$$(7.4) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\xi^\varepsilon| dx = 0.$$

It is easy to see from (2) that, up to extracting a subsequence, u^ε converges in $L^p(\Omega)$ ($1 \leq p < 4$) to $u \in BV(\Omega, \{-1, 1\})$ with interface Γ separating the phases, see, e. g. [40]. Moreover, Γ consists of a finite number of spheres with radii $0 < r_1 < r_2 < \dots < r_k \leq 1$. In the sequel, we will take $p = 10/3$. The limit measure μ of $e^\varepsilon = \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) \right) dx$ contains $\Gamma = \cup_{i=1}^k \partial B_{r_i}$ in its support. Because there is no energy concentrating on the boundary $\partial\Omega$ due to (4), we must have $r_k < 1$. Now we prove that μ concentrates exactly on Γ . Indeed, writing

$$\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) = \left(\sqrt{\frac{\varepsilon}{2}} |\nabla u^\varepsilon| - \sqrt{\frac{W(u^\varepsilon)}{\varepsilon}} \right)^2 + |\nabla u^\varepsilon| \sqrt{2W(u^\varepsilon)}$$

and keeping in mind that $W(u) = \frac{1}{2}(1 - u^2)^2$, one has

$$\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) = \left(\sqrt{\frac{\varepsilon}{2}} |\nabla u^\varepsilon| - \sqrt{\frac{W(u^\varepsilon)}{\varepsilon}} \right)^2 + \left| \nabla(u^\varepsilon - \frac{(u^\varepsilon)^3}{3}) \right|.$$

On the other hand, it is easy to see that

$$\left(\sqrt{\frac{\varepsilon}{2}} |\nabla u^\varepsilon| - \sqrt{\frac{W(u^\varepsilon)}{\varepsilon}} \right)^2 \leq \left| \sqrt{\frac{\varepsilon}{2}} |\nabla u^\varepsilon| - \sqrt{\frac{W(u^\varepsilon)}{\varepsilon}} \right| \left(\sqrt{\frac{\varepsilon}{2}} |\nabla u^\varepsilon| + \sqrt{\frac{W(u^\varepsilon)}{\varepsilon}} \right) = |\xi^\varepsilon|.$$

Therefore, it follows from (7.4) that the limit measure μ of e^ε is that of $\left| \nabla(u^\varepsilon - \frac{(u^\varepsilon)^3}{3}) \right| dx$. Because u^ε converges to u in $L^{10/3}(\Omega)$, $u^\varepsilon - \frac{(u^\varepsilon)^3}{3}$ converges in $L^{10/9}(\Omega)$ to $u - \frac{u^3}{3} = \frac{2}{3}u$ where $u \in BV(\Omega, \{-1, 1\})$. This together with the fact that $u^\varepsilon - \frac{(u^\varepsilon)^3}{3}$ is radial shows that the limit measure μ of $\left| \nabla(u^\varepsilon - \frac{(u^\varepsilon)^3}{3}) \right| dx$ concentrates on the support of $|\nabla u|$. Hence μ concentrates on $\Gamma = \cup_{i=1}^k \partial B_{r_i}$. More precisely, there are numbers $m_1, \dots, m_k > 0$ such that, in the sense of Radon measures

$$(7.5) \quad e^\varepsilon \rightharpoonup \mu = \sum_{i=1}^k m_i 2\sigma \mathcal{H}^{N-1} \llcorner \partial B_{r_i}.$$

We claim that $m_j = 1$ for all j . Note that the case $m_j > 1$ for some j , if exists, corresponds to the piling up of the interface.

The key of the proof is the following identity for $\varphi = (\varphi^1, \dots, \varphi^N) \in (C_0^1(\Omega))^N$

$$(7.6) \quad \int_{\Omega} \left(\operatorname{div} \varphi - \sum_{j,k} \frac{\partial_j u^\varepsilon}{|\nabla u^\varepsilon|} \frac{\partial_k u^\varepsilon}{|\nabla u^\varepsilon|} \partial_k \varphi^j \right) \varepsilon |\nabla u^\varepsilon|^2 dx = \int_{\Omega} (\xi^\varepsilon \operatorname{div} \varphi - u^\varepsilon \operatorname{div}(k^\varepsilon \varphi)) dx.$$

This identity can be obtained by multiplying both sides of the equation $k^\varepsilon = \varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon)$ by $\nabla u^\varepsilon \cdot \varphi$ and then integrating by parts twice.

For any j , choose a thin annulus A_j around ∂B_{r_j} such that $(\cup_{i \neq j} \partial B_{r_i}) \cap A_j = \emptyset$. Now, fix j . Choose $\varphi \in C_0^1(A_j)$ to localize (7.6). Because the limit measure of e^ε has constant multiplicity m_j in A_j and by the limiting equipartition of energy (7.4), we observe as in [23] that

$$\varepsilon \nabla u^\varepsilon \otimes \nabla u^\varepsilon dx \llcorner A_j \rightarrow 2m_j \sigma \vec{n} \otimes \vec{n} \mathcal{H}^{N-1} \llcorner \partial B_{r_j}.$$

Consequently, letting $\varepsilon \rightarrow 0$ in (7.6), we obtain

$$(7.7) \quad 2m_j \sigma \int_{\partial B_{r_j}} (\operatorname{div} \varphi - \partial_k \varphi^j \vec{n}_j \otimes \vec{n}_k) d\mathcal{H}^{N-1} = - \int_{A_j} u \operatorname{div}(k\varphi) dx,$$

where k is the weak limit in $H^1(\Omega)$ of k^ε and $\vec{n} = (\vec{n}_1, \dots, \vec{n}_N)$ is an outward unit normal to ∂B_{r_j} . Applying the divergence theorem to the left hand side of (7.7), we get

$$(7.8) \quad 2m_j \sigma \int_{\partial B_{r_j}} \varphi \frac{N-1}{r_j} \vec{n} d\mathcal{H}^{N-1} = - \int_{A_j} u \operatorname{div}(k\varphi) dx.$$

Now, we are ready to prove the Claim. Fix j where $1 \leq j \leq k$. Then $\partial B_{r_j} \subset \Gamma$ and $u = 1$ on one side of A_j and $u = -1$ on the other side of A_j (with respect to ∂B_{r_j}). Using the divergence theorem for the right hand side of (7.8), one finds that

$$(7.9) \quad 2m_j \sigma \int_{\partial B_{r_j}} \varphi \frac{N-1}{r_j} \vec{n} d\mathcal{H}^{N-1} = 2 \int_{\partial B_{r_j}} v \varphi \cdot \vec{n} d\mathcal{H}^{N-1}.$$

Hence $k = m_j \frac{\sigma(N-1)}{r_j}$ on ∂B_{r_j} . Combining this with Item 3. in Lemma 5.4 of [8], which says that on ∂B_{r_j} , $k = \pm \frac{\sigma(N-1)}{r_j}$, gives $m_j = 1$ and thus completing the proof of the Claim.

It follows from the Claim that

$$(7.10) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) \right) dx = 2\sigma \mathcal{H}^{N-1}(\Gamma).$$

Furthermore, because u^ε converges to u in $L^2(\Omega)$, one has $\lim_{\varepsilon \rightarrow 0} \|u^\varepsilon - \overline{u^\varepsilon}\|_{H^{-1}(\Omega)}^2 = \|u - \overline{u}\|_{H^{-1}(\Omega)}^2$. Combining this with (7.10), one obtains (7.3) as desired. \square

Now, we give the proof of Proposition 7.1. For ease of notation and by translating time, we can assume that $t_0 = 0$. By (7.2), (BC') and Theorem 7.1, the equation (1.1) has well-prepared initial data. We claim that, for all $t \in [0, T^*)$,

$$(7.11) \quad \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla w^\varepsilon(t)|^2 dx \geq \|\sigma \kappa(t) - \lambda v\|_{H_n^{1/2}(\Gamma(t))}^2.$$

Indeed, we only need to prove inequality for the case the right hand side of (7.11) is finite. Then, as in (6.8) and (4.13), we have

$$(7.12) \quad \liminf_{\varepsilon \rightarrow 0} \left(\|w^\varepsilon(t)\|_{H^1(\Omega)} + \|k^\varepsilon(t)\|_{H^1(\Omega)} \right) \leq C.$$

Thus, by Theorem 7.1, we have

$$(7.13) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon(t)|^2 + \frac{1}{\varepsilon} W(u^\varepsilon(t)) \right) dx = 2\sigma \mathcal{H}^{N-1}(\Gamma(t)).$$

Recall that $w^\varepsilon(t) = k^\varepsilon(t) - \lambda v^\varepsilon(t)$. By extracting a subsequence, $w^\varepsilon(t)$ and $k^\varepsilon(t)$ converge weakly to $w(t)$ and $k(t)$ respectively in $H^1(\Omega)$. It is well-known [24] that the single multiplicity of the interface $\Gamma(t)$ in (7.13) gives the Gibbs-Thompson relation $k(t) = \sigma\kappa(t)$ on $\Gamma(t)$. Thus $w(t) = \sigma\kappa(t) - \lambda v(t)$ on $\Gamma(t)$. Now (7.11) follows as in the proof of the Lemma 4.1. We remark that well-preparedness of initial data and (7.11) are all we need to complete the proof of Proposition 7.1, following the same lines of argument as in the proof of Theorem 1.1. Thus the proof of Theorem 1.3 is also complete. \square

8. PROOF OF THEOREM 1.4

In this section, we give the proof of Theorem 1.4.

Proof of Theorem 1.4. We recall the following notation for all $s \geq 0$

$$\|\cdot\|_{Y(s)}^2 = 4 \|\cdot\|_{H_n^{-1/2}(\Gamma(s))}^2$$

It follows from the proofs of Theorems 1.1 and 1.3 that for all $t_0 > 0$, we have

1. Well-preparedness of the evolving interface, i.e,

$$(8.1) \quad \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon(t_0)) = E(u(t_0))$$

2. The convergence of the velocity in its natural energy space (cf. (6.10))

$$(8.2) \quad \lim_{\varepsilon \rightarrow 0} \int_{t_0}^{T^*} \|\nabla w^\varepsilon(s)\|_{L^2(\Omega)}^2 ds = \int_{t_0}^{T^*} \int_{\Omega} |\nabla w(x, s)|^2 dx ds = \int_{t_0}^{T^*} \|\nabla_{Y(s)} E(\Gamma(s))\|_{Y(s)}^2 ds.$$

For the case of well-prepared initial data, as it can be seen from the proof of Theorem 1.1 that (8.1) and (8.2) also hold for $t_0 = 0$. The first equality, (8.1), allows us to construct a deformation presented in Proposition 8.1. The second equality, (8.2), allows us to apply the deformation to prove the transport estimate stated in (1.9). The proof of Theorem 1.4 will then follow from Lemma 8.1 and the transport estimate in Section 8.2.

8.1. Construction of the deformation. Our main result in this section is the construction of a deformation in the following

Proposition 8.1. *Let (u^ε) be a sequence of smooth functions on Ω satisfying $\frac{\partial u^\varepsilon}{\partial n} = 0$ on $\partial\Omega$, $E_\varepsilon(u^\varepsilon) \leq M$ and $u^\varepsilon \rightarrow u \in BV(\Omega, \{-1, 1\})$ in $L^2(\Omega)$ where u has Γ as its smooth*

interface separating the phases 1 and -1 . Furthermore, assume that Γ consists of a finite number of closed, connected hypersurfaces inside Ω and that

$$(8.3) \quad \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon) = E(\Gamma).$$

Let V be a smooth function defined on Γ so that $V \in H_n^{-1/2}(\Gamma)$. Let $w(t)$ be any smooth deformation of Γ with normal velocity vector \mathbf{V} at $t = 0$, i.e., $w(t)$ consists of a finite number of closed, connected hypersurfaces inside Ω satisfying

$$(8.4) \quad w(0) = \Gamma, \quad \partial_t w(0) = \mathbf{V}$$

where $\mathbf{V} = V \vec{n}$. Then, we can find $w^\varepsilon(t) \in C^1(\Omega)$ such that $w^\varepsilon(0) = u^\varepsilon$, and the following equalities hold

$$(8.5) \quad \lim_{\varepsilon \rightarrow 0} \|\partial_t w^\varepsilon(0)\|_{H_n^{-1}(\Omega)}^2 = \|\partial_t w(0)\|_Y^2 = \|V\|_Y^2 = 4 \|V\|_{H_n^{-1/2}(\Gamma)}^2,$$

$$(8.6) \quad \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \Big|_{t=0} E_\varepsilon(w^\varepsilon(t)) = \frac{d}{dt} \Big|_{t=0} E(w(t)).$$

Proof. We observe that V being smooth on Γ and belonging to $H_n^{-1/2}(\Gamma)$ imply, by Lemma 2.1, $\int_\Gamma V d\mathcal{H}^{N-1} = 0$. In fact, in Lemma 2.1, let $u = V$ and $v = 1$. Then, by (2.5),

$$\int_\Gamma V d\mathcal{H}^{N-1} = \langle V, 1 \rangle_{H_n^{-1/2}(\Gamma) \times H_n^{1/2}(\Gamma)} = - \langle V^*, 1 \rangle_{H_n^{1/2}(\Gamma)} = - \langle \nabla \tilde{V}^*, \nabla \tilde{1} \rangle_{L^2(\Omega)}.$$

It follows from (2.3) that $\tilde{1} = 1$. Thus

$$\int_\Gamma V d\mathcal{H}^{N-1} = - \langle \nabla \tilde{V}^*, \nabla 1 \rangle_{L^2(\Omega)} = 0.$$

Let us extend the vector field \mathbf{V} outside Γ in such a way that $\mathbf{V} \in (C_c^1(\Omega))^N$. Let $\Omega^+ = \{x \in \Omega : u(x) = 1\}$. Then the divergence theorem gives

$$(8.7) \quad \int_\Omega \operatorname{div} \mathbf{V} dx = 0; \quad \int_\Omega 2\chi_{\Omega^+} \operatorname{div} \mathbf{V} dx = 0.$$

We need the following simple lemma, which also implies the existence of a small perturbation $\partial_t \Gamma^\varepsilon$ of $\partial_t \Gamma$ satisfying (1.8).

Lemma 8.1. *There exists a vector field $\mathbf{V}^\varepsilon \in (C_0^1(\Omega))^N$ satisfying the following conditions (i) $\lim_{\varepsilon \rightarrow 0} \|\mathbf{V}^\varepsilon - \mathbf{V}\|_{C_0^1(\Omega)} = 0$; (ii) $\int_\Omega \nabla u^\varepsilon \cdot \mathbf{V}^\varepsilon dx = 0$.*

Proof. Let us consider a smooth vector field $\varphi \in (C_0^1(\Omega))^N$ satisfying $\int_\Gamma \varphi \cdot \vec{n} d\mathcal{H}^{N-1} \neq 0$. Let $\mathbf{V}^\varepsilon = \mathbf{V} + h(\varepsilon)\varphi$ where $h(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ to be chosen later. Then $\mathbf{V}^\varepsilon \in (C_0^1(\Omega))^N$. With this choice of \mathbf{V}^ε , (i) is clearly satisfied.

Concerning (ii), we have, by the divergence theorem and the fact that \mathbf{V}^ε has compact support

$$- \int_\Omega \nabla u^\varepsilon \cdot \mathbf{V}^\varepsilon dx = - \int_\Omega \operatorname{div}(u^\varepsilon \mathbf{V}^\varepsilon) dx + \int_\Omega u^\varepsilon \operatorname{div} \mathbf{V}^\varepsilon dx = \int_\Omega u^\varepsilon \operatorname{div} \mathbf{V}^\varepsilon dx.$$

Because $\int_{\Omega} \operatorname{div} \mathbf{V}^{\varepsilon} dx = \int_{\partial\Omega} \mathbf{V}^{\varepsilon} \cdot \vec{n} \, d\mathcal{H}^{N-1} = 0$, we see that

$$(8.8) \quad - \int_{\Omega} \nabla u^{\varepsilon} \cdot \mathbf{V}^{\varepsilon} dx = \int_{\Omega} (u^{\varepsilon} + 1) \operatorname{div} \mathbf{V}^{\varepsilon} dx = \int_{\Omega} (u^{\varepsilon} + 1) \operatorname{div} \mathbf{V} dx + h(\varepsilon) \int_{\Omega} (u^{\varepsilon} + 1) \operatorname{div} \varphi dx.$$

Therefore, (ii) will be satisfied by choosing

$$h(\varepsilon) = \frac{- \int_{\Omega} (u^{\varepsilon} + 1) \operatorname{div} \mathbf{V} dx}{\int_{\Omega} (u^{\varepsilon} + 1) \operatorname{div} \varphi dx}.$$

It remains to verify that $h(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Indeed, because $u^{\varepsilon} + 1 \rightarrow 2\chi_{\Omega^+}$ in $L^1(\Omega)$, the denominator of $h(\varepsilon)$, $\int_{\Omega} (u^{\varepsilon} + 1) \operatorname{div} \varphi dx$ converges to $\int_{\Omega} 2\chi_{\Omega^+} \operatorname{div} \varphi dx = 2 \int_{\Gamma} \varphi \cdot \vec{n} \, d\mathcal{H}^{N-1} \neq 0$, as $\varepsilon \rightarrow 0$. On the other hand, using (8.7), we see that the numerator of $h(\varepsilon)$, $- \int_{\Omega} (u^{\varepsilon} + 1) \operatorname{div} \mathbf{V} dx = - \int_{\Omega} (u^{\varepsilon} + 1 - 2\chi_{\Omega^+}) \operatorname{div} \mathbf{V} dx \rightarrow 0$ as $\varepsilon \rightarrow 0$. As a result, $h(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. \square

Consider t sufficiently small such that the map $\chi_{\varepsilon,t}(x) = x + t\mathbf{V}^{\varepsilon}(x)$ is a diffeomorphism of Ω into itself. By the construction of \mathbf{V}^{ε} in Lemma 8.1, the smallness of t can be chosen independent of ε . We define $w^{\varepsilon}(x, t)$ as follows

$$(8.9) \quad w^{\varepsilon}(x, t) = u^{\varepsilon}(\chi_{\varepsilon,t}^{-1}(x)).$$

Let us check that w^{ε} satisfies the desired properties. First, we confirm (8.5) by showing that

$$\lim_{\varepsilon \rightarrow 0} \|\partial_t w^{\varepsilon}(0)\|_{H_n^{-1}(\Omega)}^2 = \|\partial_t w(0)\|_Y^2 = \|V\|_Y^2 = \int_{\Gamma} \int_{\Gamma} G(x, y) V(x) V(y) d\mathcal{H}^{N-1}(x) d\mathcal{H}^{N-1}(y)$$

where $G(x, y)$ is the Green's function for $-\Delta$ on Ω with Neumann boundary conditions. To do so, we start by evaluating $\|\partial_t w^{\varepsilon}(0)\|_{H_n^{-1}(\Omega)}^2$. Note that, for each x , we have

$$x = \chi_{\varepsilon,t}(\chi_{\varepsilon,t}^{-1}(x)) = \chi_{\varepsilon,t}^{-1}(x) + t\mathbf{V}^{\varepsilon}(\chi_{\varepsilon,t}^{-1}(x)).$$

Hence

$$0 = \frac{d}{dt}(\chi_{\varepsilon,t}^{-1}(x)) + \mathbf{V}^{\varepsilon}(\chi_{\varepsilon,t}^{-1}(x)) + t\nabla \mathbf{V}^{\varepsilon} \cdot \frac{d}{dt}(\chi_{\varepsilon,t}^{-1}(x)).$$

Evaluating the above equation at $t = 0$ and noting that $\chi_{\varepsilon,0}^{-1}(x) = x$, one obtains

$$\left. \frac{d}{dt} \right|_{t=0} (\chi_{\varepsilon,t}^{-1}(x)) = -\mathbf{V}^{\varepsilon}(x).$$

Thus

$$(8.10) \quad \partial_t w^{\varepsilon}(0) = \nabla u^{\varepsilon} \cdot \left. \frac{d}{dt} \right|_{t=0} (\chi_{\varepsilon,t}^{-1}(x)) = -\nabla u^{\varepsilon} \cdot \mathbf{V}^{\varepsilon}.$$

By Lemma 8.1 (ii), there exists $g_{\varepsilon} \in H^1(\Omega)$ such that $-\Delta g_{\varepsilon} = \nabla u^{\varepsilon} \cdot \mathbf{V}^{\varepsilon} = \nabla u_{*}^{\varepsilon} \cdot \mathbf{V}^{\varepsilon}$ and $\frac{\partial g_{\varepsilon}}{\partial n} = 0$ where we have denoted $u_{*}^{\varepsilon} = u^{\varepsilon} + 1$. Then, by the definition of the $H_n^{-1}(\Omega)$ norm in Section 2.1

$$\|\partial_t w^{\varepsilon}(0)\|_{H_n^{-1}(\Omega)}^2 = \|\nabla u_{*}^{\varepsilon} \cdot \mathbf{V}^{\varepsilon}\|_{H_n^{-1}(\Omega)}^2 = \int_{\Omega} |\nabla g_{\varepsilon}|^2 dx.$$

Now, let $G(x, y)$ be the Green's function for $-\Delta$ on Ω with Neumann boundary conditions. Then

$$(8.11) \quad \int_{\Omega} |\nabla g_{\varepsilon}|^2 dx = \int_{\Omega} \int_{\Omega} G(x, y) \nabla u_{*}^{\varepsilon}(x) \cdot \mathbf{V}^{\varepsilon}(x) \nabla u_{*}^{\varepsilon}(y) \cdot \mathbf{V}^{\varepsilon}(y) dx dy.$$

Using integration by parts

$$\begin{aligned} \int_{\Omega} G(x, y) \nabla u_{*}^{\varepsilon}(x) \cdot \mathbf{V}^{\varepsilon}(x) dx &= \int_{\Omega} G(x, y) [\operatorname{div}_x [u_{*}^{\varepsilon}(x) \mathbf{V}^{\varepsilon}(x)] - u_{*}^{\varepsilon}(x) \operatorname{div}_x \mathbf{V}^{\varepsilon}(x)] dx \\ &= - \int_{\Omega} (\nabla_x G(x, y) \cdot \mathbf{V}^{\varepsilon}(x) u_{*}^{\varepsilon}(x) + u_{*}^{\varepsilon}(x) G(x, y) \operatorname{div}_x \mathbf{V}^{\varepsilon}(x)) dx \\ &\equiv -H(y). \end{aligned}$$

Using integration by parts one more time

$$\int_{\Omega} H(y) \nabla u_{*}^{\varepsilon}(y) \cdot \mathbf{V}^{\varepsilon}(y) dy = - \int_{\Omega} (\nabla_y H(y) \cdot \mathbf{V}^{\varepsilon}(y) u_{*}^{\varepsilon}(y) + u_{*}^{\varepsilon}(y) H(y) \operatorname{div}_y \mathbf{V}^{\varepsilon}(y)) dy.$$

Thus, (8.11) gives

$$\begin{aligned} (8.12) \quad \int_{\Omega} |\nabla g_{\varepsilon}|^2 dx &= - \int_{\Omega} H(y) \nabla u_{*}^{\varepsilon}(y) \cdot \mathbf{V}^{\varepsilon}(y) dy \\ &= \int_{\Omega} (\nabla_y H(y) \cdot \mathbf{V}^{\varepsilon}(y) u_{*}^{\varepsilon}(y) + u_{*}^{\varepsilon}(y) H(y) \operatorname{div}_y \mathbf{V}^{\varepsilon}(y)) dy \\ &= \int_{\Omega} u_{*}^{\varepsilon}(y) \mathbf{V}^{\varepsilon}(y) dy \cdot \int_{\Omega} (\nabla_y [\nabla_x G(x, y) \cdot \mathbf{V}^{\varepsilon}(x) u_{*}^{\varepsilon}(x)] + u_{*}^{\varepsilon}(x) \nabla_y G(x, y) \operatorname{div}_x \mathbf{V}^{\varepsilon}(x)) dx \\ &\quad + \int_{\Omega} u_{*}^{\varepsilon}(y) \operatorname{div}_y \mathbf{V}^{\varepsilon}(y) dy \int_{\Omega} (\nabla_x G(x, y) \mathbf{V}^{\varepsilon}(x) u_{*}^{\varepsilon}(x) + u_{*}^{\varepsilon}(x) G(x, y) \operatorname{div}_x \mathbf{V}^{\varepsilon}(x)) dx. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ in (8.12), taking into account Lemma 8.1 (i) and the fact that $u_{*}^{\varepsilon} = u^{\varepsilon} + 1 \rightarrow 2\chi_{\Omega^+}$ in $L^1(\Omega)$ as $\varepsilon \rightarrow 0$, we find that

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \|\partial_t w^{\varepsilon}(0)\|_{H_n^{-1}(\Omega)}^2 \\ &= 4 \int_{\Omega} \chi_{\Omega^+}(y) \mathbf{V}(y) dy \cdot \int_{\Omega} (\nabla_y [\nabla_x G(x, y) \chi_{\Omega^+}(x) \mathbf{V}(x)] + \chi_{\Omega^+}(x) \nabla_y G(x, y) \operatorname{div}_x \mathbf{V}(x)) dx \\ &\quad + 4 \int_{\Omega} \chi_{\Omega^+}(y) \operatorname{div}_y \mathbf{V}(y) dy \int_{\Omega} (\nabla_x G(x, y) \cdot \mathbf{V}(x) \chi_{\Omega^+}(x) + \chi_{\Omega^+}(x) G(x, y) \operatorname{div}_x \mathbf{V}(x)) dx \\ &= 4 \int_{\Omega^+} (\mathbf{V}(y) \cdot \nabla_y M(y) + \operatorname{div}_y \mathbf{V}(y) M(y)) dy, \end{aligned}$$

where

$$\begin{aligned}
M(y) &= \int_{\Omega^+} (\nabla_x G(x, y) \cdot \mathbf{V}(x) + \operatorname{div}_x \mathbf{V}(x) G(x, y)) dx \\
&= \int_{\Omega^+} \operatorname{div}_x (G(x, y) \mathbf{V}(x)) dx = \int_{\Gamma} G(x, y) \mathbf{V}(x) \cdot \vec{n} d\mathcal{H}^{N-1}(x) \\
&= \int_{\Gamma} G(x, y) V(x) d\mathcal{H}^{N-1}(x).
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_{\Omega^+} (\mathbf{V}(y) \cdot \nabla_y M(y) + \operatorname{div}_y \mathbf{V}(y) M(y)) dy &= \int_{\Omega^+} \operatorname{div}_y (\mathbf{V}(y) M(y)) dy \\
&= \int_{\Gamma} M(y) \mathbf{V}(y) \cdot \vec{n} d\mathcal{H}^{N-1}(y) \\
&= \int_{\Gamma} V(y) M(y) d\mathcal{H}^{N-1}(y) \\
&= \int_{\Gamma} \int_{\Gamma} G(x, y) V(x) V(y) d\mathcal{H}^{N-1}(x) d\mathcal{H}^{N-1}(y).
\end{aligned}$$

Hence

$$(8.13) \quad \lim_{\varepsilon \rightarrow 0} \|\partial_t w^\varepsilon(0)\|_{H_n^{-1}(\Omega)}^2 = 4 \int_{\Gamma} \int_{\Gamma} G(x, y) V(x) V(y) d\mathcal{H}^{N-1}(x) d\mathcal{H}^{N-1}(y).$$

Now, we will express $\|V\|_{H_n^{-1/2}(\Gamma)}^2$ in terms of the Green function $G(x, y)$ and V . To do this, let us denote $V^* = \Delta_{\Gamma}^{-1} V$ as in Lemma 2.1. Then $\Delta_{\Gamma} V^* = V$ and

$$(8.14) \quad \|V\|_{H_n^{-1/2}(\Gamma)}^2 = \|V^*\|_{H_n^{1/2}(\Gamma)}^2.$$

Recall from (2.5) that

$$(8.15) \quad \|V^*\|_{H_n^{1/2}(\Gamma)}^2 = \|\nabla \tilde{V}^*\|_{L^2(\Omega)}^2$$

where $\tilde{V}^* \in H^1(\Omega)$ satisfying $\frac{\partial \tilde{V}^*}{\partial n} = 0$ on $\partial\Omega$ and by (2.4), $\Delta \tilde{V}^* = \Delta_{\Gamma}(V^*) \delta_{\Gamma} = V \delta_{\Gamma}$. Thus, there is a constant C such that

$$\tilde{V}^*(x) = - \int_{\Omega} G(x, y) V(y) \delta_{\Gamma}(y) dy + C$$

and therefore,

$$\begin{aligned}
\|\nabla \tilde{V}^*\|_{L^2(\Omega)}^2 &= \int_{\Omega} \int_{\Omega} G(x, y) V(x) \delta_{\Gamma}(x) V(y) \delta_{\Gamma}(y) dx dy \\
&= \int_{\Gamma} \int_{\Gamma} G(x, y) V(x) V(y) d\mathcal{H}^{N-1}(x) d\mathcal{H}^{N-1}(y).
\end{aligned}$$

Combining the above equality with (8.14) and (8.15), we get

$$(8.16) \quad \|V\|_{H_n^{-1/2}(\Gamma)}^2 = \int_{\Gamma} \int_{\Gamma} G(x, y) V(x) V(y) d\mathcal{H}^{N-1}(x) d\mathcal{H}^{N-1}(y).$$

From (8.13) and (8.16), we obtain (8.5).

Next, we prove (8.6) by establishing

$$(8.17) \quad \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \Big|_{t=0} \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla w^\varepsilon(t)|^2 + \frac{1}{2\varepsilon} (1 - |w^\varepsilon(t)|^2)^2 \right) dx = \frac{d}{dt} \Big|_{t=0} \sigma \int_{\Omega} |\nabla w(t)|$$

and

$$(8.18) \quad \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \left\| w^\varepsilon(t) - \overline{w(t)^\varepsilon} \right\|_{H^{-1}(\Omega)}^2 = \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \left\| w(t) - \overline{w(t)_\Omega} \right\|_{H^{-1}(\Omega)}^2.$$

We first prove (8.17). Let us denote

$$E_\varepsilon^{loc}(w) = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla w|^2 + \frac{1}{2\varepsilon} (1 - |w|^2)^2 \right) dx.$$

We start by evaluating $\frac{d}{dt} \Big|_{t=0} E_\varepsilon^{loc}(w^\varepsilon(t))$. In view of the definition of $w^\varepsilon(x, t) = u^\varepsilon(\chi_{\varepsilon, t}^{-1}(x))$, with the change of variables $y = \chi_t(x)$, we have

$$E_\varepsilon(w^\varepsilon(t)) = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon \cdot \nabla \chi_{\varepsilon, t}^{-1}(\chi_{\varepsilon, t}(x))|^2 + \frac{1}{2\varepsilon} (1 - |u^\varepsilon|^2)^2 \right) |\det \nabla \chi_{\varepsilon, t}(x)| dx.$$

Since

$$\begin{aligned} \nabla \chi_{\varepsilon, t}^{-1}(\chi_{\varepsilon, t}(x)) &= [I + t \nabla \mathbf{V}^\varepsilon(x)]^{-1} = I - t \nabla \mathbf{V}^\varepsilon(x) + o(t), \\ \det \nabla \chi_{\varepsilon, t}(x) &= \det(I + t \nabla \mathbf{V}^\varepsilon(x)) = 1 + t \operatorname{div} \mathbf{V}^\varepsilon + o(t) \end{aligned}$$

we obtain after a simple calculation

$$\begin{aligned} E_\varepsilon^{loc}(w^\varepsilon(t)) &= \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{2\varepsilon} (1 - |u^\varepsilon|^2)^2 \right) (1 + t \operatorname{div} \mathbf{V}^\varepsilon) dx \\ &\quad - \int_{\Omega} \varepsilon t \langle \nabla u^\varepsilon, \nabla u^\varepsilon \cdot \nabla \mathbf{V}^\varepsilon(x) \rangle dx + o(t). \end{aligned}$$

Therefore

$$\frac{d}{dt} \Big|_{t=0} E_\varepsilon^{loc}(w^\varepsilon(t)) = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{2\varepsilon} (1 - |u^\varepsilon|^2)^2 \right) \operatorname{div} \mathbf{V}^\varepsilon dx - \int_{\Omega} \varepsilon \langle \nabla u^\varepsilon, \nabla u^\varepsilon \cdot \nabla \mathbf{V}^\varepsilon(x) \rangle dx.$$

We note that the convergence (8.3) corresponds to the case of single multiplicity of the limiting interface Γ . Now, the work of Reshetnyak [34] (see also [24]) tells us that

$$\varepsilon \nabla u^\varepsilon \otimes \nabla u^\varepsilon dx \rightharpoonup 2\sigma \vec{n} \otimes \vec{n} \mathcal{H}^{N-1} \llcorner \Gamma.$$

Thus, denoting $\mathbf{H} = \kappa \vec{n}$ the mean curvature vector of Γ , we can now calculate, using Lemma 8.1 (i), that

$$(8.19) \quad \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \Big|_{t=0} E_\varepsilon^{loc}(w^\varepsilon(t)) = \int_{\Gamma} 2\sigma \left(\operatorname{div} \mathbf{V} - \langle \vec{n}, \vec{n} \cdot \nabla \mathbf{V} \rangle \right) d\mathcal{H}^{N-1} = -2\sigma \langle \mathbf{H}, \mathbf{V} \rangle_{L^2(\Gamma)}.$$

On the other hand, we have

$$(8.20) \quad \frac{d}{dt} \Big|_{t=0} \sigma \int_{\Omega} |\nabla w(t)| = -2\sigma \langle \mathbf{H}, \mathbf{V} \rangle_{L^2(\Gamma)}.$$

Therefore, (8.17) follows from (8.19) and (8.20).

Thus, to obtain (8.6), it remains to establish (8.18). Let $v(t) = \Delta^{-1}(w(t) - \overline{w(t)}_{\Omega})$. Then, we recall from (2.11) that

$$(8.21) \quad \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \left\| w(t) - \overline{w(t)}_{\Omega} \right\|_{H^{-1}(\Omega)}^2 = \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \|\nabla v(t)\|_{L^2(\Omega)}^2 = 2 \langle v, V \rangle_{L^2(\Gamma)}$$

where $v = \Delta^{-1}(u - \overline{u}_{\Omega})$. As in the proof of (2.11), we see that

$$\left\| w^\varepsilon(t) - \overline{w^\varepsilon(t)}_{\Omega} \right\|_{H^{-1}(\Omega)}^2 = \int_{\Omega} \int_{\Omega} G(x, y) (w^\varepsilon(x, t) - \overline{w^\varepsilon_{\Omega}(t)}) (w^\varepsilon(y, t) - \overline{w^\varepsilon_{\Omega}(t)}) dx dy.$$

Differentiating, we get

$$(8.22) \quad \begin{aligned} \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \left\| w^\varepsilon(t) - \overline{w^\varepsilon(t)}_{\Omega} \right\|_{H^{-1}(\Omega)}^2 \\ = \int_{\Omega} \int_{\Omega} G(x, y) (w^\varepsilon(x, 0) - \overline{w^\varepsilon_{\Omega}(0)}) \frac{d}{dt} \Big|_{t=0} (w^\varepsilon(y, t) - \overline{w^\varepsilon_{\Omega}(t)}) dx dy \end{aligned}$$

By (8.10) and Lemma 8.1(ii),

$$(8.23) \quad \frac{d}{dt} \Big|_{t=0} \overline{w^\varepsilon(t)}_{\Omega} = -\overline{(\nabla u^\varepsilon \cdot \mathbf{V}^\varepsilon)}_{\Omega} = 0.$$

Let us denote $v^\varepsilon = \Delta^{-1}(u^\varepsilon - \overline{u^\varepsilon}_{\Omega})$. Because $w^\varepsilon(x, 0) = u^\varepsilon(x)$, there is some constant c^ε such that

$$(8.24) \quad \int_{\Omega} G(x, y) (w^\varepsilon(x, 0) - \overline{w^\varepsilon_{\Omega}(0)}) dy = v^\varepsilon(x) + c^\varepsilon.$$

Now, one has, using Lemma 8.1 (ii) again,

$$\frac{d}{dt} \Big|_{t=0} \frac{1}{2} \left\| w^\varepsilon(t) - \overline{w^\varepsilon(t)}_{\Omega} \right\|_{H^{-1}(\Omega)}^2 = \int_{\Omega} (v^\varepsilon + c^\varepsilon) (-\nabla u^\varepsilon \cdot \mathbf{V}^\varepsilon) dx = \int_{\Omega} v^\varepsilon (-\nabla u^\varepsilon \cdot \mathbf{V}^\varepsilon) dx.$$

Integrating by parts gives

$$(8.25) \quad \int_{\Omega} v^\varepsilon (-\nabla u^\varepsilon \cdot \mathbf{V}^\varepsilon) dx = \int_{\Omega} v^\varepsilon (-\nabla(u^\varepsilon + 1) \cdot \mathbf{V}^\varepsilon) dx = \int_{\Omega} (u^\varepsilon + 1) \operatorname{div}(v^\varepsilon \mathbf{V}^\varepsilon) dx.$$

Letting $\varepsilon \rightarrow 0$, one obtains

$$(8.26) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \left\| w^\varepsilon(t) - \overline{w^\varepsilon(t)}_{\Omega} \right\|_{H^{-1}(\Omega)}^2 \\ = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (u^\varepsilon + 1) \operatorname{div}(v^\varepsilon \mathbf{V}^\varepsilon) dx = \int_{\Omega} 2\chi_{\Omega^+} \operatorname{div}(v \mathbf{V}) dx = 2 \int_{\Gamma} v \mathbf{V} \cdot \vec{n} d\mathcal{H}^{N-1} = 2 \langle v, V \rangle_{L^2(\Gamma)} \end{aligned}$$

and (8.18) follows. \square

8.2. Transport estimate. In this section, we prove the existence of a small perturbation $\partial_t \Gamma^\varepsilon$ of $\partial_t \Gamma$ satisfying (1.8)- (1.9) and thus completing the proof of Theorem 1.4. Fix $t_1 > 0$. Let $t_0 \in [t_1, T^*)$. Then, the smoothness of $\Gamma(t_0)$ implies that of

$$\nabla_{Y(t_0)} E(\Gamma(t_0)) = \frac{1}{2} \Delta_{\Gamma(t_0)} (\sigma \kappa(t_0) - \lambda v(t_0)) \vec{n}$$

where \vec{n} is the unit outernormal vector to $\Gamma(t_0)$. By (8.1) and Proposition 8.1, for any z defined in a neighborhood of t_0 satisfying $z(t_0) = \Gamma(t_0)$, $\partial_t z(t_0) = -\nabla_{Y(t_0)} E(\Gamma(t_0))$, there exists $z^\varepsilon(t) = z_{t_0}^\varepsilon(t)$ such that $z^\varepsilon(t_0) = u^\varepsilon(t_0)$,

$$(8.27) \quad \limsup_{\varepsilon \rightarrow 0} \|\partial_t z^\varepsilon(t_0)\|_{X_\varepsilon}^2 = \|\partial_t z(0)\|_{Y(t_0)}^2 = \|\nabla_{Y(t_0)} E(\Gamma(t_0))\|_{Y(t_0)}^2$$

and

$$(8.28) \quad \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \Big|_{t=t_0} E_\varepsilon(z^\varepsilon(t)) = \frac{d}{dt} \Big|_{t=t_0} E(z(t)).$$

Here we recall that $X_\varepsilon = H_n^{-1}(\Omega)$.

In the following, we will use the notation $\partial_t z^\varepsilon(t_0) \equiv \partial_t z_{t_0}^\varepsilon(t_0)$. Note that (8.28) implies

$$(8.29) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} \langle \nabla_{X_\varepsilon} E_\varepsilon(u^\varepsilon(t_0)), \partial_t z^\varepsilon(t_0) \rangle_{X_\varepsilon} &= \langle \nabla_{Y(t_0)} E(z(t_0)), \partial_t z(t_0) \rangle_{Y(t_0)} \\ &= -\|\nabla_{Y(t_0)} E(\Gamma(t_0))\|_{Y(t_0)}^2. \end{aligned}$$

Now, upon expanding

$$\begin{aligned} \int_{t_1}^{T^*} \|\nabla_{X_\varepsilon} E_\varepsilon(u^\varepsilon) + \partial_t z^\varepsilon(t)\|_{X_\varepsilon}^2 dt &= \int_{t_1}^{T^*} \|\nabla_{X_\varepsilon} E_\varepsilon(u^\varepsilon)\|_{X_\varepsilon}^2 dt + \int_{t_1}^{T^*} \|\partial_t z^\varepsilon(t)\|_{X_\varepsilon}^2 dt \\ &\quad + \int_{t_1}^{T^*} 2 \langle \nabla_{X_\varepsilon} E_\varepsilon(u^\varepsilon), \partial_t z^\varepsilon(t) \rangle_{X_\varepsilon} dt, \end{aligned}$$

and letting $\varepsilon \rightarrow 0$, and using (8.2), (8.27) and (8.29), we find that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{t_1}^{T^*} \|\nabla_{X_\varepsilon} E_\varepsilon(u^\varepsilon) + \partial_t z^\varepsilon(t)\|_{X_\varepsilon}^2 dt \\ = \int_{t_1}^{T^*} \left(\|\nabla_{Y(t)} E(\Gamma(t))\|_{Y(t)}^2 + \|\nabla_{Y(t)} E(\Gamma(t))\|_{Y(t)}^2 - 2 \|\nabla_{Y(t)} E(\Gamma(t))\|_{Y(t)}^2 \right) dt = 0. \end{aligned}$$

This combined with the equation $\partial_t u^\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u^\varepsilon)$ shows that

$$(8.30) \quad \lim_{\varepsilon \rightarrow 0} \int_{t_1}^{T^*} \|\partial_t u^\varepsilon - \partial_t z^\varepsilon(t)\|_{X_\varepsilon}^2 dt = 0.$$

Recall from the construction of $z^\varepsilon(x, t)$, as in Proposition 8.1, that $\partial_t z^\varepsilon(x, t) = -\nabla u^\varepsilon \cdot \mathbf{V}^\varepsilon$ (see (8.10)). Here \mathbf{V}^ε is a small perturbation of the vector field \mathbf{V} satisfying $\mathbf{V} = \partial_t w(t) = -\nabla_Y E(\Gamma(t)) = (\partial_t \Gamma) \vec{n}$ on $\Gamma(t)$ in the sense that $\lim_{\varepsilon \rightarrow 0} \|\mathbf{V}^\varepsilon - \mathbf{V}\|_{C_0^1(\Omega)} = 0$. Thus, in

terms of the notations of Theorem 1.4, $\partial_t z^\varepsilon(x, t) = -\nabla u^\varepsilon \cdot \partial_t \Gamma^\varepsilon$ and (1.8) is satisfied. Consequently, we get from the estimate (8.30) that

$$\lim_{\varepsilon \rightarrow 0} \int_{t_1}^{T^*} \|\partial_t u^\varepsilon + \nabla u^\varepsilon \cdot \partial_t \Gamma^\varepsilon\|_{X_\varepsilon}^2 dt = 0.$$

Therefore, we have proved (1.9) and Theorem 1.4. □

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DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NY 10027, USA
E-mail address: namle@math.columbia.edu