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On topological degree for potential operators of class $(S)_+$

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Abstract

We extend some results of Amann about the topological degree for compact potential vector fields to potential operators of class $(S)_+$. Using these results, we obtain the existence of a third buckled state of a thin elastic shell in R^3 even in the case when the gradient of the potential energy of the shell might not be compact.

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1. Introduction

In [9] Rabinowitz established some results about the topological degree for compact potential vector fields and applied them to prove the existence of the third buckled state possessed by a thin elastic shell. However, the above results were obtained under the assumption that the functional whose gradient is a compact vector field was of class C^2 . Later, Amann [1] showed that the degree results of Rabinowitz hold for C^1 -mappings. In the present paper, we extend some results of [1] to a class of continuous potential operators of class $(S)_+$ and use them to relax conditions on the compactness of operators and the shallowness of shells in [9]. We also give an explicit example illustrating the relaxation of the compactness of operators (see (iii) of Lemma 3.3).

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The remainder of our paper consists of two sections. In Section 2, we improve results in [1]. Then we apply them to study the number of buckled states possessed by a thin elastic shell in the last section.

2. Topological degree for potential operators of class $(S)_+$

In this section, we extend the results about the topological degree in [1] to potential operators of class $(S)_+$. Throughout our paper, H is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and its dual space H^* is always identified with H . We denote by $B(x_0, r)$ the set $\{x \in H \mid \|x - x_0\| < r\}$ for any x_0 in H . Now, we recall the class $(S)_+$ introduced by Browder (see [2,3]).

Definition 2.1. Let A be a subset of H and h be a mapping of A into H . We say:

- (i) h is demicontinuous if the sequence $\{h(x_n)\}$ converges weakly to $h(x)$ in H for any sequence $\{x_n\}$ converging strongly to x in H .
- (ii) h is of class $(S)_+$ if h is demicontinuous and has the following property:
Let $\{x_n\}$ be a sequence in A such that $\{x_n\}$ converges weakly to x in H . Then $\{x_n\}$ converges strongly to x in H if $\limsup_{n \rightarrow \infty} \langle h(x_n), x_n - x \rangle \leq 0$.

Let D be a bounded open subset in H with boundary ∂D and closure \bar{D} . Let h be a mapping of class $(S)_+$ on \bar{D} and p be in $H \setminus h(\partial D)$. By Theorems 4 and 5 in [2], the topological degree of h on D at p is defined as a family of integers and is denoted by $deg(h, D, p)$. In [10] Skrypnik showed that this topological degree is single valued (see also [3]). Combining the Galerkin approximation technique in the proof of Theorem 4 in [3] and results in [1] for continuous mappings on finite-dimensional spaces, we have the following result.

Theorem 2.1. Let U be an open subset of H and G be a real C^1 -function on U such that its gradient ∇G is of class $(S)_+$ on U . Denote ∇G by g and assume that there are x_0 in H and real numbers α, β and r such that $\alpha < \beta, r > 0$ and

- (i) $V \equiv G^{-1}((-\infty, \beta))$ is bounded and $\bar{V} \subset U$,
- (ii) $G^{-1}((-\infty, \alpha]) \subset \bar{B}(x_0, r) \subset V$ and
- (iii) $g(x) \neq 0$ for every x in $G^{-1}([\alpha, \beta])$.

Then $G^{-1}((-\infty, \beta])$ is bounded and $deg(g, V, 0) = 1$.

Proof. First, since $V \equiv G^{-1}((-\infty, \beta))$ is bounded, V is contained in a ball $B(0, R)$ for some positive number R . If there is z in $G^{-1}(\{\beta\}) \setminus B(0, 2R)$ then G has a local minimum at z since $G(x) \geq \beta$ for all $x \in U \setminus B(0, R)$. Thus $g(z) = 0$, which contradicts (iii). Hence

$$G^{-1}(\{\beta\}) \subset B(0, 2R).$$

This proves the first statement of the theorem.

By the continuity of G , V is an open subset and

$$\partial V \subset G^{-1}((-\infty, \beta]) \setminus G^{-1}((-\infty, \beta)) = G^{-1}(\{\beta\}).$$

Therefore, $0 \notin g(\partial V)$ and the topological degree $deg(g, V, 0)$ is well defined in the sense of topological degree for $(S)_+$ mappings. Denote by $\{X_\lambda\}_{\lambda \in A}$ the family of all finite-dimensional subspaces of H such that $V_\lambda \equiv V \cap X_\lambda$ is nonempty. We define

$$\lambda \succ \mu \Leftrightarrow X_\mu \subset X_\lambda \quad \forall \lambda, \mu \in A.$$

For each λ in A , let π_λ be the orthogonal projection of H onto X_λ . For $x \in V_\lambda$, put

$$G_\lambda(x) = G(x),$$

$$g_\lambda(x) = \pi_\lambda(g(x)).$$

Then, it is easy to see that, for any $y \in V_\lambda$ and $x \in X_\lambda$,

$$\langle \nabla G_\lambda(y), x \rangle = \langle \nabla G(y), x \rangle = \langle \pi_\lambda(\nabla G(y)), x \rangle = \langle \pi_\lambda(g(y)), x \rangle = \langle g_\lambda(y), x \rangle,$$

i.e.,

$$\nabla G_\lambda(y) = g_\lambda(y) \quad \forall y \in V_\lambda.$$

We shall show that there exists λ_0 in A such that

$$g_\lambda(x) \neq 0 \quad \forall x \in G_\lambda^{-1}([\alpha, \beta]), \quad \lambda \succ \lambda_0. \tag{2.1}$$

Suppose by contradiction that, for each λ , there exist a $\mu \succ \lambda$ and an x_μ in $G_\mu^{-1}([\alpha, \beta])$ such that $g_\mu(x_\mu) = 0$. Thus,

$$\langle g(x_\mu), x_\mu \rangle = \langle g_\mu(x_\mu), x_\mu \rangle = 0, \quad \langle g(x_\mu), v \rangle = \langle g_\mu(x_\mu), v \rangle = 0 \quad \forall v \in X_\mu. \tag{2.2}$$

We define

$$W_\lambda = \{u \in G^{-1}([\alpha, \beta]) \mid \langle g(u), u \rangle = 0 \text{ and } \langle g(u), v \rangle = 0 \quad \forall v \in X_\lambda\} \quad \forall \lambda \in A.$$

Note that $W_\alpha \subset W_\beta$ if $\alpha \succ \beta$. Therefore, by (2.2), $\{W_\lambda\}_{\lambda \in A}$ is a family of nonempty sets and has the finite intersection property. Denote by K_λ the closure of W_λ in the weak topology of H for any λ in A . It is clear that $\{K_\lambda\}_{\lambda \in A}$ is a family having the finite intersection property. Since W_λ is contained in the bounded set $G^{-1}([\alpha, \beta])$, K_λ is a weakly compact subset of H for any λ in A . Thus

$$K \equiv \bigcap_{\lambda \in A} K_\lambda \neq \emptyset.$$

Let x be in K and v be in H . Choose λ in A such that x and v are in X_λ . Since x lies in K_λ and H is a Hilbert space, there exists a sequence $\{x_j\}$ in W_λ which converges weakly to x in H . By the definition of W_λ , x_j belongs to $G^{-1}([\alpha, \beta])$ and

$$\langle g(x_j), x_j \rangle = 0, \quad \langle g(x_j), x \rangle = 0, \quad \langle g(x_j), w \rangle = 0 \quad \forall j \in \mathbb{N}, \quad w \in X_\lambda.$$

It implies that

$$\limsup_{j \rightarrow \infty} \langle g(x_j), x_j - x \rangle = 0.$$

Since g is of class $(S)_+$ on $G^{-1}([\alpha, \beta])$, $\{x_j\}$ converges strongly to x and hence x lies in $G^{-1}([\alpha, \beta])$. Using the continuity of g , we obtain

$$\langle g(x), v \rangle = \lim_{j \rightarrow \infty} \langle g(x_j), v \rangle = 0.$$

Since v is an arbitrary element of H , it follows that $g(x)=0$, which is a contradiction to the assumption on g . Therefore, we obtain (2.1).

Now, let λ be in A such that $\lambda \succ \lambda_0$ and $x_0 \in X_\lambda$. Then

$$\begin{aligned} G_\lambda^{-1}((-\infty, \alpha]) &= G^{-1}((-\infty, \alpha]) \cap X_\lambda \subset \bar{B}(x_0, r) \cap X_\lambda \\ &\equiv \bar{B}_\lambda(x_0, r) \subset V_\lambda = G_\lambda^{-1}((-\infty, \beta)) \end{aligned}$$

and

$$\bar{V}_\lambda \subset \bar{V} \cap X_\lambda \subset U \cap X_\lambda \equiv U_\lambda.$$

Therefore, the mapping G_λ satisfies all conditions of the Theorem in [1] for the finite-dimensional Hilbert space X_λ . Thus, we obtain

$$\text{deg}(g_\lambda, V_\lambda, 0) = \text{deg}(\nabla G_\lambda, V_\lambda, 0) = 1 \quad \forall \lambda \succ \lambda_0.$$

Since g is of class $(S)_+$, by definition (see [3], p. 75), the theorem follows. \square

Theorem 2.1 was proved in [1] for compact potential operators. A result similar to Theorem 2.1 was proved by Klimov [6] (see also [11]).

We have the following corollaries of Theorem 2.1.

Corollary 2.1. *Let G be a real C^1 -function on H such that its gradient $g \equiv \nabla G$ is of class $(S)_+$ on H . Suppose that $G(x) \rightarrow \infty$ when $\|x\| \rightarrow \infty$ and G maps bounded sets into bounded sets. Moreover, suppose that $g(x) \neq 0$ for all $\|x\| \geq r_0$ and some $r_0 > 0$. Then there is a number $r_1 \geq r_0$ such that*

$$\text{deg}(g, B(0, r), 0) = 1 \quad \text{for all } r \geq r_1.$$

The proof of this corollary is similar to that of Corollary 1 in [1] and is omitted.

Corollary 2.2. *Let W be an open convex subset of H and G be a real C^1 -function on W such that its gradient $g \equiv \nabla G$ is of class $(S)_+$ on W . Let w_0 be an isolated critical point of G in W . Assume that G has a local minimum at w_0 . Then*

$$i(g, w_0) \equiv \lim_{s \rightarrow 0} \text{deg}(g, B(w_0, s), 0) = 1.$$

Proof. First, we prove that G is weakly sequentially lower semicontinuous on W . Suppose by contradiction that there exists a sequence $\{w_k\}$ in W converging weakly to w in W such that $G(w) > \liminf_{k \rightarrow \infty} G(w_k)$. Choose a subsequence $\{w_{k_m}\}$ of $\{w_k\}$ such that

$$\lim_{m \rightarrow \infty} G(w_{k_m}) = \liminf_{k \rightarrow \infty} G(w_k).$$

Then

$$\lim_{m \rightarrow \infty} (G(w_{k_m}) - G(w)) < 0.$$

On the other hand, for each m , there exists a $t_m \in (0, 1)$ such that

$$G(w_{k_m}) - G(w) = \langle g(w + t_m(w_{k_m} - w)), w_{k_m} - w \rangle.$$

Hence

$$\begin{aligned} \limsup_{m \rightarrow \infty} t_m [G(w_{k_m}) - G(w)] &= \limsup_{m \rightarrow \infty} \langle g(w + t_m(w_{k_m} - w)), w + t_m(w_{k_m} - w) - w \rangle \\ &\leq 0. \end{aligned}$$

Since g is of class $(S)_+$ on W and the sequence $\{w + t_m(w_{k_m} - w)\}$ converges weakly to w , the above inequality shows that $\{w + t_m(w_{k_m} - w)\}$ converges strongly to w . Since g is continuous, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} [G(w_{k_m}) - G(w)] &= \lim_{m \rightarrow \infty} \langle g(w + t_m(w_{k_m} - w)), w_{k_m} - w \rangle \\ &= \lim_{m \rightarrow \infty} \langle g(w), w_{k_m} - w \rangle = 0, \end{aligned}$$

which contradicts the inequality $\lim_{m \rightarrow \infty} (G(w_{k_m}) - G(w)) < 0$. Hence, G is weakly sequentially lower semicontinuous on W .

Without loss of generality, we can assume that $w_0 = 0$, $G(0) = 0$ and there is a positive real number r_0 such that $W = B(0, r_0)$ and 0 is the unique critical point of G in W . Let r_1 and r_2 be two positive real numbers such that $r_1 < r_2 < r_0$. We claim that

$$\beta \equiv \inf G(\bar{B}(0, r_2) \setminus B(0, r_1)) > 0. \tag{2.3}$$

Suppose by contradiction that there is a sequence $\{w_k\}$ in $\bar{B}(0, r_2) \setminus B(0, r_1)$ such that $\{G(w_k)\}$ converges to 0. We can (and shall) assume that $\{w_k\}$ converges weakly to w in $\bar{B}(0, r_2)$. Since G is weakly sequentially lower semicontinuous on W , we have

$$0 \leq G(w) \leq \liminf_{k \rightarrow \infty} G(w_k) = 0, \tag{2.4}$$

which implies $w = 0$. On the other hand, for any integer n , there is an s_n in $(0, 1)$ such that

$$G(w_n) - G\left(\frac{w_n}{2}\right) = \left\langle g\left(\frac{w_n}{2} + s_n \frac{w_n}{2}\right), \frac{w_n}{2}\right\rangle.$$

Since $G(w_n/2) \geq 0$ and $\lim_{n \rightarrow \infty} G(w_n) = 0$,

$$\limsup_{n \rightarrow \infty} \left\langle g\left(\frac{w_n}{2} + s_n \frac{w_n}{2}\right), (1 + s_n) \frac{w_n}{2}\right\rangle \leq 0.$$

As $\{(1 + s_n)w_n/2\}$ converges weakly to 0 and g is of class $(S)_+$ on W , it follows that $\lim_{n \rightarrow \infty} w_n = 0$, which contradicts the fact that $\|w_n\| \geq r_1 > 0$ for all n . Therefore, (2.3) holds.

Choose $r > 0$ such that $\bar{B}(0, r) \subset G^{-1}((-\infty, \beta))$ and put

$$\alpha = (1/2) \inf G(\bar{B}(0, r_2) \setminus B(0, r)).$$

By definitions and (2.3), we see that $r < r_1$, $0 < \alpha < \beta$, $V \equiv G^{-1}((-\infty, \beta)) \subset B(0, r_1)$, $\bar{V} \subset B(0, r_2)$, and $G^{-1}((-\infty, \alpha]) \subset \bar{B}(0, r)$.

Now, applying Theorem 2.1 for $U = B(0, r_2)$, we obtain

$$i(g, 0) = \text{deg}(g, V, 0) = 1. \quad \square$$

We have the following theorem, which will be useful for the application in Section 3.

Theorem 2.2. *Let g be a mapping of class $(S)_+$ on H and x_0 be an isolated zero of g in H . Suppose that g is Fréchet differentiable at x_0 , $\nabla g(x_0)$ is of class $(S)_+$ and $\nabla g(x_0)x \neq 0$ for all nonzero vector x in H . Then*

$$i(g, x_0) \equiv \lim_{s \rightarrow 0} \text{deg}(g, B(x_0, s), 0) \in \{-1, 1\}.$$

Proof. First, we prove that there exists a positive real number a such that

$$\|\nabla g(x_0)x\| \geq a\|x\| \quad \forall x \in H. \tag{2.5}$$

Suppose by contradiction that there exist a sequence $\{a_n\}$ in \mathbb{R} and a sequence $\{x_n\}$ in $H \setminus \{0\}$ such that $\lim_{n \rightarrow \infty} a_n = 0$ and

$$\|\nabla g(x_0)x_n\| \leq a_n\|x_n\|. \tag{2.6}$$

For any positive integer n , put $x_n^* = x_n/\|x_n\|$.

Then $\|x_n^*\| = 1$ and we can assume that the sequence $\{x_n^*\}$ converges weakly in H to x^* with $\|x^*\| \leq 1$. On the other hand, from (2.6) we have

$$|\langle \nabla g(x_0)x_n^*, x_n^* - x^* \rangle| \leq \left\| \nabla g(x_0) \frac{x_n}{\|x_n\|} \right\| \|x_n^* - x^*\| \leq 2a_n \rightarrow 0.$$

Since $\nabla g(x_0)$ is of class $(S)_+$, it follows that $\{x_n^*\}$ converges strongly to x^* . Hence $\|x^*\| = 1$, and $\nabla g(x_0)x^* = 0$, which is a contradiction to our assumption on $\nabla g(x_0)$. Thus we have (2.5).

Since g is Fréchet differentiable at x_0 , there is a mapping φ from a ball $B(0, r)$ into H such that $\lim_{h \rightarrow 0} \varphi(h) = 0$ and

$$\begin{aligned} g(x) &= g(x_0) + \nabla g(x_0)(x - x_0) + \|x - x_0\|\varphi(x - x_0) \\ &= \nabla g(x_0)(x - x_0) + \|x - x_0\|\varphi(x - x_0). \end{aligned}$$

For any (t, x) in $[0, 1] \times B(x_0, r)$, put

$$\begin{aligned} \tilde{g}(x) &= \nabla g(x_0)(x - x_0), \\ h(t, x) &= tg(x) + (1 - t)\tilde{g}(x) = \nabla g(x_0)(x - x_0) + t\|x - x_0\|\varphi(x - x_0). \end{aligned}$$

Then \tilde{g} is of class $(S)_+$ as $\nabla g(x_0)$ is linear and of class $(S)_+$. By (2.5), $h(t, x) \neq 0$ for all (t, x) in $[0, 1] \times \partial B(x_0, \rho)$ when ρ is a sufficiently small positive number. Since x_0

is an isolated zero of the continuous mappings g and \tilde{g} , by the homotopy invariance of the degree, we obtain

$$i(g, x_0) = \text{deg}(g, B(x_0, \rho), 0) = \text{deg}(\tilde{g}, B(x_0, \rho), 0). \tag{2.7}$$

Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be the family of all finite-dimensional subspaces of H as in the proof of Theorem 2.1 such that $x_0 \in X_\lambda$. Denote by $B_\lambda(x_0, \rho)$ the set $B(x_0, \rho) \cap X_\lambda$ and \tilde{g}_λ the restriction of $\pi_\lambda \circ \tilde{g}$ to $B_\lambda(x_0, \rho)$. Using the fact that \tilde{g} is of class $(S)_+$ and $0 \notin \tilde{g}(\partial B(x_0, \rho))$, and arguing as in the proof of Theorem 2.1, we can find a λ_0 in Λ such that $0 \notin \tilde{g}_\lambda(\partial B_\lambda(x_0, \rho))$ and

$$\text{deg}(\tilde{g}, B(x_0, \rho), 0) = \text{deg}(\tilde{g}_\lambda, B_\lambda(x_0, \rho), 0) \quad \forall \lambda \succ \lambda_0. \tag{2.8}$$

Since $\tilde{g}_\lambda(x) \neq 0$ for all $x \in \partial B_\lambda(x_0, \rho)$, it is obvious that $\tilde{g}_\lambda(x) \neq 0$ for all $x \neq x_0$. Furthermore, we have

$$\tilde{g}_\lambda(x_0 + h) - \tilde{g}_\lambda(x_0) = \pi_\lambda(\tilde{g}(x_0 + h) - \tilde{g}(x_0)) = \pi_\lambda(\nabla g(x_0)(h)) \quad \forall h \in X_\lambda.$$

It implies that

$$\nabla \tilde{g}_\lambda(x_0)(h) = \pi_\lambda(\nabla g(x_0)(h)) \quad \forall h \in X_\lambda.$$

Since $\tilde{g}_\lambda(x) = \pi_\lambda(\tilde{g}(x)) = \pi_\lambda[\nabla g(x_0)(x - x_0)] \neq 0$ for all $x \in X_\lambda \setminus \{x_0\}$, it follows that

$$\nabla \tilde{g}_\lambda(x_0)(h) \neq 0 \quad \forall h \in X_\lambda \setminus \{0\}.$$

Thus, by the Leray–Schauder index formula [7, Theorem 4.7, pp. 136–137]

$$\text{deg}(\tilde{g}_\lambda, B_\lambda(x_0, \rho), 0) \in \{-1, 1\}. \tag{2.9}$$

By (2.7)–(2.9), we obtain the theorem. \square

3. An application to shell buckling

Let S be a thin elastic shell in R^3 which has a single-valued projection Ω on the xy plane. S is assumed to be in equilibrium under the effects of edge stresses and of an applied force. In [9] Rabinowitz used the results of the topological degree for compact vector fields to prove the existence of three buckled states of S . In this section, we apply our results in Section 2 to obtain a similar result and show that we can relax some conditions on the compactness of mappings and the shallowness of the shells.

First, we need some definitions. Let H be the usual Sobolev space $W_0^{2,2}(\Omega)$ with the scalar product and norm

$$\langle u, v \rangle_H = \int_\Omega \Delta u \Delta v \, dx \, dy,$$

$$\|u\|_H = \left\{ \int_\Omega |\Delta u|^2 \, dx \, dy \right\}^{1/2} \quad \forall u, v \in H,$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

In this section, let $(k_{ij})_{1 \leq i, j \leq 2}$ be a symmetric matrix whose elements are given measurable functions on Ω . We put

$$[u, v] = \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y},$$

$$N(u) = \frac{\partial}{\partial x} \left(k_{11} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial x} \left(k_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left(k_{21} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_{22} \frac{\partial u}{\partial y} \right) \quad \forall u, v \in H.$$

Using the Cauchy–Schwarz inequality and Corollary 9.10 in [5], we obtain

$$\int_{\Omega} |[u, v]| \, dx \, dy \leq \|u\|_H \|v\|_H \quad \forall u, v \in H. \tag{3.1}$$

Integrating by parts, we see that

$$\int_{\Omega} [w, \psi] \varphi \, dx \, dy = \int_{\Omega} [w, \varphi] \psi \, dx \, dy, \tag{3.2}$$

$$\int_{\Omega} N(w) \varphi \, dx \, dy = \int_{\Omega} N(\varphi) w \, dx \, dy \quad \forall w, \psi, \varphi \in H. \tag{3.3}$$

We assume that there is a positive real number C such that

$$\left| \int_{\Omega} N(w) \varphi \, dx \, dy \right| \leq C \|w\|_H \|\varphi\|_H \quad \forall w, \varphi \in H. \tag{3.4}$$

Hereafter, by C we mean a general constant.

Let $w, f, Z(x, y), \Psi_1, \Psi_2$ and λ be, respectively, the deflection of the shell from its initial state, the Airy stress function, the applied force, the edge stresses, and a measure of the magnitude of Ψ_1 and Ψ_2 . After appropriate dimensional scalings have been made, w and f satisfy the von Kármán equations:

$$\Delta^2 f = -\frac{1}{2} [w, w] - N(w), \tag{3.5}$$

$$\Delta^2 w = [f, w] + N(f) + Z, \tag{3.6}$$

$$w = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0; \quad \frac{\partial^2 f}{\partial n \partial \tau} = \lambda \Psi_1, \quad \frac{\partial^2 f}{\partial \tau \partial \tau} = \lambda \Psi_2 \quad \text{on } \partial\Omega, \tag{3.7}$$

where $n(x, y)$ and $\tau(x, y)$ are the normal and tangent to $\partial\Omega$, respectively.

Assume that Ω, Ψ_1 and Ψ_2 are sufficiently smooth so that there is a solution F_0 in $C^4(\bar{\Omega})$ to the following boundary problem:

$$\begin{cases} \Delta^2 F_0 = 0 & \text{in } \Omega, \\ \frac{\partial^2 F_0}{\partial n \partial \tau} = \Psi_1, \quad \frac{\partial^2 F_0}{\partial \tau \partial \tau} = \Psi_2 & \text{on } \partial\Omega. \end{cases}$$

We further require that $Z = -\lambda N(F_0)$, i.e., the external force just balances the edge stresses so that $w = 0, f = \lambda F_0$ is an equilibrium state of the shell and satisfies

(3.5)–(3.7). Putting $F = f - \lambda F_0$, we can reduce the systems (3.5)–(3.7) to the following system:

$$\Delta^2 F = -\frac{1}{2} [w, w] - N(w), \tag{3.8}$$

$$\Delta^2 w = [F + \lambda F_0, w] + N(F), \tag{3.9}$$

$$w = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \quad \text{on } \partial\Omega. \tag{3.10}$$

Put

$$I_w(\varphi) = - \int_{\Omega} \left\{ \frac{1}{2} [w, w] + N(w) \right\} \varphi \, dx \, dy \quad \forall w, \varphi \in H. \tag{3.11}$$

By (3.1), (3.4) and the Sobolev embedding theorem, we have

$$\begin{aligned} |I_w(\varphi)| &\leq \frac{1}{2} \|w\|_H^2 \|\varphi\|_{C(\bar{\Omega})} + C \|w\|_H \|\varphi\|_H \\ &\leq C (\|w\|_H^2 + \|w\|_H) \|\varphi\|_H \quad \forall w, \varphi \in H. \end{aligned} \tag{3.12}$$

Thus, by the Riesz representation theorem, for each w in H , there exists a unique $F(w)$ in H such that

$$I_w(\varphi) = \langle F(w), \varphi \rangle_H \quad \forall \varphi \in H. \tag{3.13}$$

Choosing F as $F(w)$, by (3.13), we see that system (3.9) becomes

$$\Delta^2 w - [F(w) + \lambda F_0, w] - N(F(w)) = 0,$$

or, in the weak form

$$\int_{\Omega} \{ \Delta w \Delta \varphi - [F(w) + \lambda F_0, w] \varphi - N(F(w)) \varphi \} \, dx \, dy = 0 \quad \forall \varphi \in H. \tag{3.14}$$

In order to solve (3.14), we consider the following potential energy of possible shell states:

$$G(\lambda, w) = \frac{1}{2} \int_{\Omega} (|\Delta w|^2 + |\Delta F(w)|^2 - \lambda [F_0, w] w) \, dx \, dy.$$

The following lemmas contain properties of $G(\lambda, \cdot)$ and its derivatives. First, let us prove

Lemma 3.1. *Let A, B be two weakly continuous operators of H into H , that is, for any sequence $\{w_n\}$ converging weakly to w in H , the sequences $\{A(w_n)\}$ and $\{B(w_n)\}$ converge weakly to $A(w)$ and $B(w)$, respectively. Define an operator T on H as follows:*

$$\langle T(w), \varphi \rangle_H = \int_{\Omega} [A(w), B(w)] \varphi \, dx \, dy \quad \forall \varphi \in H.$$

Then T is a compact operator on H .

Proof. Fix a w in H . By (3.1) and the Sobolev embedding theorem, we have

$$\begin{aligned} \left| \int_{\Omega} [A(w), B(w)]\varphi \, dx \, dy \right| &\leq \|A(w)\|_H \|B(w)\|_H \|\varphi\|_{C(\bar{\Omega})} \\ &\leq C \|A(w)\|_H \|B(w)\|_H \|\varphi\|_H \quad \forall \varphi \in H. \end{aligned}$$

Therefore, T is well defined by the Riesz representation theorem. Let $\{w_n\}$ be a sequence in H converging weakly to w in H . Then the sequences $\{A(w_n)\}$ and $\{B(w_n)\}$ converge weakly to $A(w)$ and $B(w)$ in H , respectively. By the Sobolev embedding theorem, the sequences $\{A(w_n)\}$ and $\{B(w_n)\}$ converge strongly to $A(w)$ and $B(w)$ in $C(\bar{\Omega})$, respectively. Moreover, by (3.1), we have

$$\begin{aligned} \|T(w_n) - T(w)\|_H^2 &= \langle T(w_n) - T(w), T(w_n) - T(w) \rangle_H \\ &= \int_{\Omega} ([A(w_n), B(w_n)] - [A(w), B(w)])(T(w_n) - T(w)) \, dx \, dy \\ &\leq \left| \int_{\Omega} [A(w_n) - A(w), B(w_n)](T(w_n) - T(w)) \, dx \, dy \right| \\ &\quad + \left| \int_{\Omega} [A(w), B(w_n) - B(w)](T(w_n) - T(w)) \, dx \, dy \right| \\ &= \left| \int_{\Omega} [T(w_n) - T(w), B(w_n)](A(w_n) - A(w)) \, dx \, dy \right| \\ &\quad + \left| \int_{\Omega} [A(w), T(w_n) - T(w)](B(w_n) - B(w)) \, dx \, dy \right| \\ &\leq \|T(w_n) - T(w)\|_H \|B(w_n)\|_H \|A(w_n) - A(w)\|_{C(\bar{\Omega})} \\ &\quad + \|T(w_n) - T(w)\|_H \|A(w)\|_H \|B(w_n) - B(w)\|_{C(\bar{\Omega})}. \end{aligned}$$

Hence

$$\begin{aligned} \|T(w_n) - T(w)\|_H &\leq (\|A(w_n)\|_H + \|B(w_n)\|_H)(\|A(w_n) - A(w)\|_{C(\bar{\Omega})} \\ &\quad + \|B(w_n) - B(w)\|_{C(\bar{\Omega})}). \end{aligned}$$

Therefore, the sequence $\{T(w_n)\}$ converges strongly to $T(w)$ in H and we obtain the lemma. \square

Lemma 3.2. (i) For any fixed λ , $G(\lambda, \cdot)$ is Fréchet differentiable on H and its gradient $g(\lambda, \cdot) \equiv \nabla G(\lambda, \cdot)$ is given by the following formula:

$$g(\lambda, w) = w - \lambda L(w) - L_1(M_1(w)) - M(w) + L_1^2(w) \quad \forall w \in H,$$

where L, L_1, M_1 and M are operators on H defined as follows:

$$\langle L(w), \varphi \rangle_H = \int_{\Omega} [F_0, w] \varphi \, dx \, dy,$$

$$\langle L_1(w), \varphi \rangle_H = \int_{\Omega} N(w) \varphi \, dx \, dy,$$

$$\langle M_1(w), \varphi \rangle_H = -\frac{1}{2} \int_{\Omega} [w, w] \varphi \, dx \, dy,$$

$$\langle M(w), \varphi \rangle_H = \int_{\Omega} [F(w), w] \varphi \, dx \, dy \quad \forall w, \varphi \in H.$$

Furthermore, the operator L_1 is continuous; the operators L, M_1 and M are compact.

(ii) $G(\lambda, \cdot)$ is a C^1 -function on H .

(iii) $g(\lambda, \cdot)$ is of class $(S)_+$.

(iv) $\nabla g(\lambda, 0)$ is given by the formula

$$\nabla g(\lambda, 0)(w) = w - \lambda L(w) + L_1^2(w) \quad \forall w \in H$$

and is of class $(S)_+$.

Proof. (i) Consider functionals $G_a(\lambda, \cdot)$ and $G_b(\cdot)$ defined as follows:

$$G_a(\lambda, w) = \frac{1}{2} \int_{\Omega} \{ |\Delta w|^2 - \lambda [F_0, w] w \} \, dx \, dy,$$

$$G_b(w) = \frac{1}{2} \int_{\Omega} |\Delta F(w)|^2 \, dx \, dy = \frac{1}{2} \|F(w)\|_H^2 \quad \forall w, \varphi \in H.$$

Then

$$G(\lambda, w) = G_a(\lambda, w) + G_b(w) \quad \forall w \in H.$$

It is clear that $G_a(\lambda, \cdot)$ is Fréchet differentiable on H and its gradient $g_a(\lambda, \cdot) \equiv \nabla G_a(\lambda, \cdot)$ is defined as follows:

$$\langle g_a(\lambda, w), \psi \rangle_H = \langle w, \psi \rangle_H - \lambda \int_{\Omega} [F_0, w] \psi \, dx \, dy \quad \forall w, \psi \in H. \tag{3.15}$$

Fix $w, \psi \in H$ and put

$$S(w, \psi) = F(w + \psi) - F(w).$$

By (3.11) and (3.13), we have

$$\begin{aligned} \langle S(w, \psi), \varphi \rangle_H &= \langle F(w + \psi), \varphi \rangle_H - \langle F(w), \varphi \rangle_H = l_{w+\psi}(\varphi) - l_w(\varphi) \\ &= - \int_{\Omega} \left\{ \frac{1}{2} [w + \psi, w + \psi] \varphi + N(w + \psi) \varphi \right\} \, dx \, dy \end{aligned}$$

$$\begin{aligned}
 & + \int_{\Omega} \left\{ \frac{1}{2} [w, w] \varphi + N(w) \varphi \right\} dx dy \\
 & = \int_{\Omega} \left\{ -[w, \psi] \varphi - \frac{1}{2} [\psi, \psi] \varphi - N(\psi) \varphi \right\} dx dy \quad \forall \varphi \in H. \quad (3.16)
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \|S(w, \psi)\|_H^2 & \leq \left| \int_{\Omega} \left\{ [w, \psi] S(w, \psi) + \frac{1}{2} [\psi, \psi] S(w, \psi) + N(\psi) S(w, \psi) \right\} dx dy \right| \\
 & \leq C[\|w\|_H \|\psi\|_H + \|\psi\|_H^2 + \|\psi\|_H] \|S(w, \psi)\|_H.
 \end{aligned}$$

Hence

$$\lim_{\psi \rightarrow 0} \|S(w, \psi)\|_H = 0. \quad (3.17)$$

Using the definition of $G_b(\cdot)$ and substituting $\varphi = F(w) + \frac{1}{2} S(w, \psi)$ into (3.16) yield

$$\begin{aligned}
 G_b(w + \psi) - G_b(w) & = \frac{1}{2} \|F(w) + S(w, \psi)\|_H^2 - \frac{1}{2} \|F(w)\|_H^2 \\
 & = \langle S(w, \psi), F(w) + \frac{1}{2} S(w, \psi) \rangle_H \\
 & = \int_{\Omega} \{ -[w, \psi] F(w) - N(\psi) F(w) \} dx dy \\
 & \quad + \frac{1}{2} \int_{\Omega} \left\{ -[w, \psi] S(w, \psi) - \frac{1}{2} [\psi, \psi] (2F(w) \right. \\
 & \quad \left. + S(w, \psi)) - N(\psi) S(w, \psi) \right\} dx dy. \quad (3.18)
 \end{aligned}$$

By (3.1), (3.4) and the Sobolev embedding theorem, we have

$$\begin{aligned}
 & \left| \int_{\Omega} \{ -[w, \psi] F(w) - N(\psi) F(w) \} dx dy \right| \\
 & \leq \|w\|_H \|\psi\|_H \|F(w)\|_{C(\bar{\Omega})} + C \|\psi\|_H \|F(w)\|_H \\
 & \leq C \{ \|w\|_H \|F(w)\|_H + \|F(w)\|_H \} \|\psi\|_H, \quad (3.19)
 \end{aligned}$$

$$\begin{aligned}
 & \left| \int_{\Omega} \left\{ -[w, \psi] S(w, \psi) - \frac{1}{2} [\psi, \psi] (2F(w) + S(w, \psi)) - N(\psi) S(w, \psi) \right\} dx dy \right| \\
 & \leq \|w\|_H \|\psi\|_H \|S(w, \psi)\|_{C(\bar{\Omega})} + \frac{1}{2} \|\psi\|_H^2 \|2F(w) \\
 & \quad + S(w, \psi)\|_{C(\bar{\Omega})} + C \|\psi\|_H \|S(w, \psi)\|_H \\
 & \leq C \|\psi\|_H \{ \|w\|_H \|S(w, \psi)\|_H + \|2F(w) \\
 & \quad + S(w, \psi)\|_H \|\psi\|_H + \|S(w, \psi)\|_H \}. \quad (3.20)
 \end{aligned}$$

Combining (3.2), (3.3) and (3.17)–(3.20) and the equation

$$F(w) = M_1(w) - L_1(w),$$

we see that G_b is Fréchet differentiable at w and its gradient $g_b \equiv \nabla G_b$ is defined as follows:

$$\begin{aligned} \langle g_b(w), \psi \rangle_H &= \int_{\Omega} \{-[w, \psi]F(w) - N(F(w))\psi\} \, dx \, dy \\ &= \int_{\Omega} \{-[F(w), w]\psi - N(F(w))\psi\} \, dx \, dy \\ &= \int_{\Omega} \{-[F(w), w]\psi - N(M_1(w))\psi \\ &\quad + N(L_1(w))\psi\} \, dx \, dy \quad \forall w, \psi \in H. \end{aligned} \tag{3.21}$$

Therefore, the formula of $g(\lambda, \cdot)$ follows from (3.15) and (3.21).

By (3.4), L_1 is continuous on H . Now, let $\{w_n\}$ be a sequence converging weakly to w in H . By Lemma 3.1, the sequence $\{M_1(w_n)\}$ converges strongly to $M_1(w)$ in H . Now, fix a φ in H . By (3.4) and the weak convergence of $\{w_n\}$ to w , we have

$$|\langle L_1(w_n) - L_1(w), \varphi \rangle_H| = \left| \int_{\Omega} N(\varphi)(w_n - w) \right| \rightarrow 0.$$

Hence, the sequence $\{L_1(w_n)\}$ converges weakly to $L_1(w)$. Thus, the operators L_1 and F are weakly continuous. Hence, by Lemma 3.1, the operators L, M_1, M are compact. The proof of (i) is complete.

(ii) It is easy to see that (ii) is derived from (i).

(iii) Let $\{w_n\}$ be a sequence converging weakly to w in H such that

$$\limsup_{n \rightarrow \infty} \langle g(\lambda, w_n), w_n - w \rangle_H \leq 0. \tag{3.22}$$

We shall prove that $\{w_n\}$ converges strongly to w . Note that

$$\begin{aligned} \langle g(\lambda, w_n), w_n - w \rangle_H &= \langle w_n, w_n - w \rangle_H - \langle \lambda L(w_n) + M(w_n), w_n - w \rangle_H \\ &\quad - \langle L_1(M_1(w_n)), w_n - w \rangle_H + \langle L_1^2(w_n), w_n - w \rangle_H. \end{aligned} \tag{3.23}$$

By (i), the sequences $\{\lambda L(w_n) + M(w_n)\}$ and $\{M_1(w_n)\}$ converge strongly in H and $\{L_1(w_n)\}$ converges weakly to $L_1(w)$ in H . Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} - \langle \lambda L(w_n) + M(w_n), w_n - w \rangle_H &= 0, \\ \lim_{n \rightarrow \infty} - \langle L_1(M_1(w_n)), w_n - w \rangle_H &= \lim_{n \rightarrow \infty} - \langle L_1(w_n) - L_1(w), M_1(w_n) \rangle_H = 0, \\ \limsup_{n \rightarrow \infty} \langle L_1^2(w_n), w_n - w \rangle_H &= \limsup_{n \rightarrow \infty} \|L_1(w_n) - L_1(w)\|_H^2 \geq 0. \end{aligned}$$

Therefore, by (3.22) and (3.23), we have

$$\limsup_{n \rightarrow \infty} \|w_n - w\|_H^2 = \limsup_{n \rightarrow \infty} \langle w_n, w_n - w \rangle_H \leq 0.$$

Thus, $\{w_n\}$ converges in H to w , and we obtain (iii).

(iv) Using the formula of $g(\lambda, w)$ and arguing as in the proof of (i) and (iii), we find that $\nabla g(\lambda, 0)$ has the desired properties. \square

Lemma 3.3. *Let L_1 be as in Lemma 3.2. Then*

- (i) *The operator $I + L_1^2$ is an isomorphism from H onto H .*
- (ii) *If L_1^2 is compact then L_1 is also compact.*
- (iii) *If $\Omega = B(0, 1)$ and*

$$k_{ij}(z) = \begin{cases} 0 & \text{when } i \neq j, \\ (1 - \|z\|)^{-2} & \text{when } i = j, \end{cases}$$

where $z = (x, y) \in \Omega$ and $\|z\| = (x^2 + y^2)^{1/2}$, then $g(\lambda, \cdot)$ is a continuous but not compact vector field on H for any λ in $(0, \infty)$.

Proof. (i) Note that

$$\langle (I + L_1^2)(w), w \rangle_H = \|w\|_H^2 + \|L_1(w)\|_H^2 \geq \|w\|_H^2.$$

Therefore, by (3.4) and the Lax–Milgram theorem, we obtain (i).

(ii) Let $\{w_n\}$ be a sequence in H converging weakly to w in H such that $\{L_1^2(w_n)\}$ converges strongly in H . Since

$$\|L_1(w_n - w)\|_H^2 = \langle L_1^2(w_n - w), w_n - w \rangle_H = \langle L_1^2(w_n) - L_1^2(w), w_n - w \rangle_H,$$

the sequence $\{L_1(w_n)\}$ converges strongly to $L_1(w)$ in H . So, (ii) is proved.

(iii) By the Cauchy–Schwarz inequality and the weighted Sobolev inequality [8, Theorem 8.4], we have

$$\begin{aligned} \left| \int_{\Omega} N(w)\phi \, dx \, dy \right| &= \left| \int_{\Omega} (1 - \|z\|)^{-2} \nabla w \cdot \nabla \phi \, dx \, dy \right| \\ &\leq \left\{ \int_{\Omega} (1 - \|z\|)^{-2} |\nabla w|^2 \, dx \, dy \right\}^{1/2} \\ &\quad \times \left\{ \int_{\Omega} (1 - \|z\|)^{-2} |\nabla \phi|^2 \, dx \, dy \right\}^{1/2} \\ &\leq C \|w\|_H \|\phi\|_H. \end{aligned}$$

Consequently, (3.4) holds and L_1 is a linear continuous operator on H . For each positive integer i , let r_i and a_i be, respectively, the number 2^{-i-2} and the point $(1 - 2^{-i}, 0)$ in Ω . Then $\{B(a_i, r_i)\}$ is a family of disjoint subsets of Ω . Choose ϕ in $C_c^\infty(B(0, 1))$ such that

$$\left\{ \int_{\Omega} |\nabla^m \phi|^2 \, dx \, dy \right\}^{1/2} = A_m > 0 \quad \forall m = 1, 2.$$

We define

$$\phi_i(z) = r_i \phi \left(\frac{z - a_i}{r_i} \right) \quad \forall i \in \mathbb{N}.$$

Then, it is easy to see that ϕ_i is in $C_c^\infty(B(a_i, r_i))$ and

$$\left\{ \int_{\Omega} |\nabla^m \phi_i|^2 \, dx \, dy \right\}^{1/2} = r_i^{2-m} A_m > 0 \quad \forall m = 1, 2, \quad i \in \mathbb{N}.$$

Since the supports of ϕ_i are disjoint and $\|\phi_i\|_H = A_2$, $\{\phi_i\}$ is an orthogonal system in H . Therefore, $\{\phi_i\}$ converges weakly to 0 in H . Hence, $\{L_1(\phi_i)\}$ converges weakly in H to $L_1(0) = 0$ by the linearity of L_1 . Moreover, we note that

$$(1 - \|z\|)^{-2} \geq ((1 - \|a_i\|) + \|a_i - z\|)^{-2} \geq 5^{-2} r_i^{-2} \quad \forall z \in B(a_i, r_i), \quad i \in \mathbb{N},$$

$$\begin{aligned} |\langle L_1(\phi_i), \phi_i \rangle_H| &= \left| \int_{\Omega} N(\phi_i) \phi_i \, dx \, dy \right| = \int_{\Omega} (1 - \|z\|)^{-2} |\nabla \phi_i|^2 \, dx \, dy \\ &= \int_{B(a_i, r_i)} (1 - \|z\|)^{-2} |\nabla \phi_i|^2 \, dx \, dy \\ &\geq \frac{1}{25} r_i^{-2} \int_{\Omega} |\nabla \phi_i|^2 \, dx \, dy = \frac{A_1^2}{25} \end{aligned}$$

and

$$|\langle L_1(\phi_i), \phi_i \rangle_H| \leq \|L_1(\phi_i)\|_H \|\phi_i\|_H = \|L_1(\phi_i)\|_H A_2.$$

Thus, $\{L_1(\phi_i)\}$ does not converge strongly to 0 in H since $\|L_1(\phi_i)\|_H \geq A_1^2/25A_2 > 0$ for all $i \in \mathbb{N}$, and the operator L_1 is not compact. By (i) of Lemma 3.2 and (ii) of Lemma 3.3, we obtain (iii). \square

Arguing as in the proof of Lemma 2.1 in [9], we obtain the following result.

Lemma 3.4. *Let λ and β be two real numbers. Then the set $\{w \in H \mid G(\lambda, w) \leq \beta\}$ is bounded.*

By Lemmas 3.2 and 3.3, the operator $(I + L_1^2)^{-1} \circ L$ is compact. We shall suppose that the linear characteristic value problem

$$\psi + L_1^2 \psi = \lambda L \psi \tag{3.24}$$

has infinitely many positive characteristic values $\lambda_1 \leq \dots \leq \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$.

Lemma 3.5. *Let λ be in the interval (λ_1, ∞) and w be an eigenvector of (3.24) corresponding to λ_1 . Then $G(\lambda, w) < 0$ when $\|w\|_H$ is sufficiently small.*

Proof. Let w be an arbitrary eigenvector of (3.24) corresponding to λ_1 . By the definitions of operators L, L_1, F and (3.24), we have

$$\int_{\Omega} [F_0, w] w \, dx \, dy = \langle L(w), w \rangle_H = \frac{1}{\lambda_1} \|w\|_H^2 + \frac{1}{\lambda_1} \langle L_1^2(w), w \rangle_H, \tag{3.25}$$

$$\begin{aligned}
 \langle L_1^2(w), w \rangle_H &= \int_{\Omega} [N(L_1(w))]w \, dx \, dy = \int_{\Omega} N(w)L_1(w) \, dx \, dy \\
 &= -\frac{1}{2} \int_{\Omega} [w, w]L_1(w) \, dx \, dy - \langle F(w), L_1(w) \rangle_H \\
 &= -\frac{1}{2} \int_{\Omega} [w, w]L_1(w) \, dx \, dy - \int_{\Omega} N(w)F(w) \, dx \, dy \\
 &= -\frac{1}{2} \int_{\Omega} [w, w]L_1(w) \, dx \, dy + \frac{1}{2} \int_{\Omega} [w, w]F(w) \, dx \, dy + \|F(w)\|_H^2 \\
 &= -\frac{1}{2} \int_{\Omega} [w, w](L_1(w) - F(w)) \, dx \, dy + \|F(w)\|_H^2, \tag{3.26}
 \end{aligned}$$

$$\begin{aligned}
 \left| \int_{\Omega} [w, w](L_1(w) - F(w)) \, dx \, dy \right| &\leq \|L_1(w) - F(w)\|_{C(\bar{\Omega})} \int_{\Omega} |[w, w]| \, dx \, dy \\
 &\leq C\|w\|_H^2(\|L_1(w)\|_{C(\bar{\Omega})} + \|F(w)\|_{C(\bar{\Omega})}) \\
 &\leq C\|w\|_H^2(\|L_1(w)\|_H + \|F(w)\|_H) \\
 &\leq C\|w\|_H^2(\|w\|_H + \|w\|_H^2). \tag{3.27}
 \end{aligned}$$

Combining (3.25)–(3.27), we obtain

$$\begin{aligned}
 2G(\lambda, w) &= \int_{\Omega} (|\Delta w|^2 + |\Delta F(w)|^2 - \lambda[F_0, w]w) \, dx \, dy \\
 &= \|w\|_H^2 + \|F(w)\|_H^2 - \lambda \int_{\Omega} [F_0, w]w \, dx \, dy \\
 &= \|w\|_H^2 + \|F(w)\|_H^2 - \frac{\lambda}{\lambda_1} \|w\|_H^2 - \frac{\lambda}{\lambda_1} \langle L_1^2(w), w \rangle_H \\
 &= \left(1 - \frac{\lambda}{\lambda_1}\right) (\|w\|_H^2 + \|F(w)\|_H^2) \\
 &\quad + \frac{\lambda}{2\lambda_1} \int_{\Omega} [w, w](L_1(w) - F(w)) \, dx \, dy \\
 &\leq \left(1 - \frac{\lambda}{\lambda_1}\right) (\|w\|_H^2 + \|F(w)\|_H^2) + C\|w\|_H^3(\|w\|_H + 1).
 \end{aligned}$$

Since $\lambda > \lambda_1 > 0$, we have $1 - \lambda/\lambda_1 < 0$ and thus $G(\lambda, w) < 0$ if $\|w\|_H$ is sufficiently small. \square

Lemma 3.6. *Let λ be in (λ_1, ∞) . Then there exists w_1 in H such that $G(\lambda, w_1)$ is a negative minimum of $G(\lambda, \cdot)$ in H .*

Proof. By Lemma 3.4, $G(\lambda, \cdot)$ is bounded from below. By Lemma 3.2, $G(\lambda, \cdot)$ is a C^1 -function and its gradient $g(\lambda, \cdot)$ is of class $(S)_+$. Hence, arguing as in the proof

of Corollary 2.2, we see that $G(\lambda, \cdot)$ is weakly sequentially lower semicontinuous. Therefore, by Theorem 9.2 in [12], $G(\lambda, \cdot)$ has a minimum in H . By Lemma 3.5, this minimum is negative. \square

By (3.14), Lemma 3.2 and standard arguments, we see that $(w, F(w))$ is a solution of (3.8) and (3.9) if and only if $(w, F(w))$ satisfies

$$\begin{cases} l_w(\varphi) = \langle F(w), \varphi \rangle_H & \forall \varphi \in H, \\ g(\lambda, w) = 0. \end{cases}$$

Since $F(w)$ is uniquely determined in terms of w from Eq. (3.13), it remains to solve the latter equation of the above system. Clearly, this equation implies that w is a critical point of $G(\lambda, \cdot)$. Consequently, the problem of finding solutions to the system (3.8)–(3.10) now turns to the problem of seeking critical points of $G(\lambda, \cdot)$.

We have the main result of this section as follows.

Theorem 3.1. *Let λ be in $(\lambda_1, \infty) \setminus \{\lambda_n : n \in \mathbb{N}\}$. Then the system (3.8)–(3.10) possesses at least three distinct solutions.*

Proof. A first solution is $(w_0, F(w_0)) = (0, 0)$ and a second solution $(w_1, F(w_1)) \neq (0, 0)$ is obtained from Lemma 3.6. If $G(\lambda, \cdot)$ has infinitely many critical points then there is nothing more to prove. Therefore, we only consider the case in which all critical points of $G(\lambda, \cdot)$ are isolated and there exists $r_0 > 0$ such that $g(\lambda, w) \neq 0$ whenever $\|w\| \geq r_0$. Since λ is not a characteristic value of (3.24), $\nabla g(\lambda, 0)(w) \neq 0$ if $w \neq 0$. By Lemma 3.2, parts (iii) and (iv), the conditions of Theorem 2.2 are satisfied. Hence

$$i(g(\lambda, \cdot), w_0) \in \{-1, 1\}. \tag{3.28}$$

Lemma 3.4 implies that $\lim_{\|w\| \rightarrow \infty} G(\lambda, w) = \infty$. By Lemma 3.2 and Corollary 2.1, there exists a number $r_1 \geq r_0$ such that

$$\text{deg}(g(\lambda, \cdot), B(0, r), 0) = 1 \quad \text{for all } r \geq r_1.$$

By Lemma 3.2, part (ii), Lemma 3.6 and Corollary 2.2,

$$i(g(\lambda, \cdot), w_1) = 1. \tag{3.29}$$

Suppose by contradiction that $G(\lambda, \cdot)$ has only critical points w_0 and w_1 in $B(0, r)$. Then by (3.28) and (3.29) and the additivity of the degree, we have

$$1 = \text{deg}(g(\lambda, \cdot), B(0, r), 0) = i(g(\lambda, \cdot), w_0) + i(g(\lambda, \cdot), w_1) \in \{0, 2\},$$

which is a contradiction. This proves our theorem. \square

Remark 3.1. Theorem 3.1 was showed by Rabinowitz in [9] if the shell is sufficiently shallow and the vector field $g(\lambda, \cdot)$ is compact. Here we relax both conditions. Lemma 3.3 gives us an example of the relaxation of the compactness. We can also get other examples by using weighted inequalities in [4].

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