THE MEAN CURVATURE AT THE FIRST SINGULAR TIME OF THE MEAN CURVATURE FLOW

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Abstract. Consider a family of smooth immersions \( F(\cdot, t) : M^n \rightarrow \mathbb{R}^{n+1} \) of closed hypersurfaces in \( \mathbb{R}^{n+1} \) moving by the mean curvature flow \( \frac{\partial F(p, t)}{\partial t} = -H(p, t) \cdot \nu(p, t), \) for \( t \in [0, T) \). We prove that the mean curvature blows up at the first singular time \( T \) if all singularities are of type I. In the case \( n = 2 \), regardless of the type of a possibly forming singularity, we show that at the first singular time the mean curvature necessarily blows up provided that either the Multiplicity One Conjecture holds or the Gaussian density is less than two. We also establish and give several applications of a local regularity theorem which is a parabolic analogue of Choi-Schoen estimate for minimal submanifolds.

1. Introduction

Let \( M^n \) be a compact \( n \)-dimensional hypersurface without boundary, and let \( F_0 : M^n \rightarrow \mathbb{R}^{n+1} \) be a smooth immersion of \( M^n \) into \( \mathbb{R}^{n+1} \). Consider a smooth one-parameter family of immersions \( F(\cdot, t) : M^n \rightarrow \mathbb{R}^{n+1} \) satisfying \( F(\cdot, 0) = F_0(\cdot) \) and

\[
\frac{\partial F(p, t)}{\partial t} = -H(p, t) \nu(p, t), \quad \forall (p, t) \in M \times [0, T).
\]

Here \( H(p, t) \) and \( \nu(p, t) \) denote the mean curvature and a choice of unit normal for the hypersurface \( M_t = F(M^n, t) \) at \( F(p, t) \), respectively. We will sometimes also write \( x(p, t) = F(p, t) \) and refer to (1.1) as to the mean curvature flow equation. Furthermore, for any compact \( n \)-dimensional hypersurface \( M^n \) which is smoothly embedded in \( \mathbb{R}^{n+1} \) by \( F : M^n \rightarrow \mathbb{R}^{n+1} \), let us denote by \( g = (g_{ij}) \) the induced metric, \( A = (h_{ij}) \) the second fundamental form, \( d\mu = \sqrt{\det(g_{ij})} \, dx \) the volume form, \( \nabla \) the induced Levi-Civita connection. Then the mean curvature of \( M^n \) is given by \( H = g^{ij}h_{ij} \).

Without any special assumptions on \( M_0 \), the mean curvature flow (1.1) will in general develop singularities in finite time, characterized by a blow up of the second fundamental form \( A(\cdot, t) \).

Theorem 1.1 (Huisken [10]). Suppose \( T < \infty \) is the first singularity time for a compact mean curvature flow. Then \( \sup_{M_t} |A| (\cdot, t) \rightarrow \infty \) as \( t \rightarrow T \).

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By the work of Huisken and Sinestrari [13] the blow up of $H$ near a singularity is known for mean convex hypersurfaces. They show that when $H \geq 0$ one has a pinching curvature estimate stating that $|A|^2 \leq C_1 H^2 + C_2$, for uniform constants $C_1, C_2$. In [21] a similar pinching estimate has been proven for star shaped hypersurfaces. The present article establishes the blow up of the mean curvature in the case of type I singularities.

**Definition 1.1.** We say that the mean curvature flow (1.1) develops a singularity of type I at $T < \infty$ if the blow-up rate of the curvature satisfies an upper bound of the form

$$\max_{\mathcal{M}_t} |A|^2 (\cdot, t) \leq \frac{C_0}{T-t}, \quad 0 \leq t < T.$$ 

In this paper, we prove the following

**Theorem 1.2.** Assume (1.2) for the mean curvature flow (1.1). If

$$\max_{\mathcal{M}_t} |H|^2 (\cdot, t) \leq C_0$$

then the flow can be extended past time $T$.

In fact, the above theorem is a consequence of the following result.

**Theorem 1.3.** Assume (1.2) for the mean curvature flow (1.1). If for some $\alpha \geq n+2$

$$\|H\|_{L^\alpha (\mathcal{M} \times [0, T])} \leq C_0$$

then the flow can be extended past time $T$.

The proofs of Theorems 1.2 and 1.3 are based on blow-up arguments using Huisken’s monotonicity formula, the classification of self-shrinkers and White’s local regularity theorem for mean curvature flow.

**Remark 1.1.** To some extent, the condition $\alpha \geq n+2$ appearing in Theorem 1.3 is optimal as illustrated by the mean curvature flow of the standard sphere $S^n$.

Our Theorems 1.2 and 1.3 left open the question on the possible blow up of the mean curvature at the first singular time $T$ for mean curvature flows with singularities other than Type I. This seems to be a difficult question. However, assuming the validity of Multiplicity One Conjecture (see page 7 of [14] and the precise statement in Conjecture 3.1 of the present article), we prove the following

**Theorem 1.4.** Let $M^2$ be a compact, smooth and embedded 2-dimensional manifold in $\mathbb{R}^3$. If

$$\max_{\mathcal{M}_t} |H|^2 (\cdot, t) \leq C_0$$

then the flow can be extended past time $T$.

The next result is independent of the Multiplicity One Conjecture. It is in some sense a refinement of White’s local regularity theorem [24]. White gives uniform curvature bounds in regions of spacetime where the Gaussian density is close to one. We prove the following.
Theorem 1.5. Let \( M^2 \) be a compact, smooth and embedded 2-dimensional manifold in \( \mathbb{R}^3 \). Suppose that (1.5) holds. Let \( y_0 \in \mathbb{R}^3 \) be a point reached by the mean curvature flow (1.1) at time \( T \). If

\[
\lim_{t \nearrow T} \int_{\mathcal{M}_t} \rho_{y_0,T} d\mu_t := \lim_{t \nearrow T} \int \frac{1}{[4\pi(T-t)]^{n/2}} \exp\left(-\frac{|y-y_0|^2}{4(T-t)}\right) d\mu_t < 2.
\]

then \((y_0,T)\) is a regular point of the mean curvature flow (1.1).

Remark 1.2. Our theorem says that for mean curvature flow of surfaces with Gaussian density \( \lim_{t \nearrow T} \int \rho_{y_0,T} d\mu_t \) below 2, for every \( y_0 \) reached by the flow at time \( T \), the mean curvature must blow up at the first singular time. In [22], Stone calculated the Gaussian density on spheres and cylinders. On spheres, the density is \( \frac{4}{e} \approx 1.47 \) and on cylinders it is \( \sqrt{\frac{2}{\pi e}} \approx 1.52 \).

We also give the following characterization of a finite time singularity of (1.1) that works in all dimensions \( n \geq 2 \).

Theorem 1.6. Assume that for the mean curvature flow (1.1), we have the following integral bound on the second fundamental form

\[
\|A\|_{L^p,q(M \times [0,T])} := \left( \int_0^T \left( \int_{\mathcal{M}_t} |A|^q d\mu \right)^{p/q} dt \right)^{1/p} < \infty
\]

where \( p,q \in (0,\infty) \) satisfy

\[
\frac{n+2}{q} = 1.
\]

Then the flow can be extended past time \( T \).

The previously mentioned results were all global characterizations ensuring that the flow can not develop any singularities as long as some global quantities are bounded uniformly in time. We also give a result regarding the local regularity theory.

Theorem 1.7. Suppose \( \mathcal{M} = (\mathcal{M}_t) \) is a smooth, properly embedded solution of the mean curvature flow in \( B(x_0,\rho) \times (t_0 - \rho^2, t_0) \) which reaches \( x_0 \) at time \( t_0 \). There exists \( \varepsilon_0 = \varepsilon_0(M_0) > 0 \) such that if \( 0 < \sigma \leq \rho \) and

\[
\int_{t_0-\sigma^2}^{t_0} \int_{\mathcal{M}_t \cap B(x_0,\sigma)} |A|^{n+2} d\mu dt < \varepsilon_0
\]

then

\[
\max_{0 \leq \delta \leq \sigma/2} \sup_{t \in [t_0 - (\sigma - \delta)^2, t_0]} \sup_{x \in B(x_0,\sigma - \delta) \cap \mathcal{M}_t} \delta^2 |A|^2 (x,t) < \varepsilon_0^{n+2} \left( \int_{t_0-\sigma^2}^{t_0} \int_{\mathcal{M}_t \cap B(x_0,\sigma)} |A|^{n+2} d\mu dt \right)^{\frac{2}{n+2}}.
\]
Our theorem is a parabolic version of Choi-Schoen estimate [3] for minimal surfaces. Related results can be found in Ecker [7]. The precise estimate of the form (1.9) for the case of minimal submanifolds can be found in Shen-Zhu [19], Proposition 2.2 (see also [4]). Moreover, in [17, 25, 26], the authors showed that if the $L^{n+2}$ norm in space-time of the second fundamental form (or the mean curvature but under various convexity assumptions) is finite then it is possible to extend the mean curvature flow beyond the time interval under consideration. Our theorem can be viewed as a local version of these results without imposing any convexity assumptions. It turns out that the conclusion of Theorem 1.7 also holds in the case when the ambient space is a complete Riemannian manifold with bounded geometry. We show that in Corollary 5.1.

The organization of the paper is as follows. In section 2 we give the proofs of Theorems 1.2 and 1.3. In section 3 we prove Theorems 1.4 and 1.5. The proof of Theorem 1.6 will be given in section 4. We conclude the paper with section 5 in which we prove Theorem 1.7 and give some applications to it.

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2. Characterization of type I singularities

Proof of Theorem 1.2. Without loss of generality, assume that $M^n \subset B_1(0) \subset \mathbb{R}^{n+1}$. Let $y_0 \in \mathbb{R}^{n+1}$ be a point reached by the mean curvature flow (1.1) at time $T$, that is, there exists a sequence $(y_j, t_j)$ with $t_j \nearrow T$ so that $y_j \in M_{t_j}$ and $y_j \to y_0$. We show that $(y_0, T)$ is a regular point of (1.1).

Note that the distance estimate ([8], Corollary 3.6) gives

(2.1) \[
\text{dist}(M_t, y_0) \leq \sqrt{2n(T-t)}, \text{ for } t < T.
\]

Consider the parabolic dilation $D_\lambda : \mathbb{R}^{n+1} \times [0, T) \to \mathbb{R}^{n+1} \times [-\lambda^2 T, 0)$ of scale $\lambda > 0$ at $(y_0, T)$ defined by

(2.2) \[
D_\lambda(y, t) = (\lambda(y - y_0), \lambda^2(t - T)).
\]

Denote the new time parameter by $s$. Then $t = T + \frac{s}{\lambda^2}$. Let

$$M^\lambda_s \equiv M^{(y_0, T)}_s = D_\lambda(M_t) = \lambda(M_{T + \frac{s}{\lambda^2}} - y_0).$$

Then $(M^\lambda_s)$ is a solution of the mean curvature flow in $B_\lambda(0)$ for $s \in [-\lambda^2 T, 0)$. Denote by $d\mu^\lambda_s$ the induced volume form on $M^\lambda_s$. Let $\rho_{y_0, T} : \mathbb{R}^{n+1} \times (-\infty, T) \to \mathbb{R}$ be the backward heat kernel at $(y_0, T)$, i.e,

(2.3) \[
\rho_{y_0, T}(y, t) = \frac{1}{(4\pi(T-t))^{n/2}} \exp\left(-\frac{|y - y_0|^2}{4(T-t)}\right).
\]
The monotonicity formula of Huisken [11] says that

\[
\frac{d}{dt} \int_{M_t} \rho_{y_0,T} d\mu_t = - \int_{M_t} \rho_{y_0,T} \left| H + \frac{F_{\perp}}{2(T-t)} \right|^2 d\mu_t,
\]

from which it follows that the limit \( \lim_{t \to T} \int_{M_t} \rho_{y_0,T} d\mu_t \) exists. Here \( F_{\perp}(\cdot, t) \) is the normal component of the position vector \( F(\cdot, t) \in \mathbb{R}^{n+1} \) in the normal space of \( M_t \) in \( \mathbb{R}^{n+1} \). Via the parabolic dilation, (2.4) becomes

\[
\frac{d}{ds} \int_{M_\lambda} \rho_{0,0} d\mu_{\lambda_s} = - \int_{M_\lambda} \rho_{0,0} \left| H_{\lambda_s} - \frac{(F_{\lambda_s})_{\perp}}{2s} \right|^2 d\mu_{\lambda_s}.
\]

Fix \( s_0 < 0 \). Integrating both sides of (2.5) from \( s_0 - \tau \) to \( s_0 \) for \( \tau > 0 \), we get

\[
\int_{s_0}^{s_0 - \tau} \int_{M_\lambda} \rho_{0,0} \left| H_{\lambda_s} - \frac{(F_{\lambda_s})_{\perp}}{2s} \right|^2 d\mu_{\lambda_s} ds = \int_{M_{s_0 - \tau}} \rho_{0,0} d\mu_{\lambda_{s_0 - \tau}} - \int_{M_{s_0}} \rho_{0,0} d\mu_{\lambda_{s_0}}.
\]

Let \( t_1 = T + \frac{s_0}{\lambda^2} \). Then, by the invariance of \( \int_{M_t} \rho_{y_0,T} d\mu_t \) under the parabolic scaling,

\[
\int_{M_{t_1}} \rho_{y_0,T} d\mu_{t_1} = \int_{M_{s_0}} \rho_{0,0} d\mu_{\lambda_{s_0}}.
\]

Letting \( \lambda \to \infty \), one has \( t_1 \to T \) and

\[
\lim_{\lambda \to \infty} \int_{M_{\lambda_{s_0}}} \rho_{0,0} d\mu_{\lambda_{s_0}} = \lim_{t \to T} \int_{M_t} \rho_{y_0,T} d\mu_t.
\]

Similarly,

\[
\lim_{\lambda \to \infty} \int_{M_{\lambda_{s_0 - \tau}}} \rho_{0,0} d\mu_{\lambda_{s_0 - \tau}} = \lim_{t \to T} \int_{M_t} \rho_{y_0,T} d\mu_t.
\]

Therefore, by (2.6),

\[
\lim_{\lambda \to \infty} \int_{s_0 - \tau}^{s_0} \int_{M_\lambda} \rho_{0,0} \left| H_{\lambda_s} - \frac{(F_{\lambda_s})_{\perp}}{2s} \right|^2 d\mu_{\lambda_s} ds = 0.
\]

On the other hand, the second fundamental form of \( M_\lambda \) satisfies

\[
\max |A|^2(\cdot, s)(M_\lambda) = \frac{1}{\lambda^2} \max |A|^2(\cdot, t)(M_t) = -\frac{1}{s}(T-t) \max |A|^2(\cdot, t)(M_t).
\]

and thus, by (1.2),

\[
\max |A|^2(\cdot, s)(M_\lambda) \leq \frac{C_0}{s}, \quad \forall s \in [-\lambda^2 T, 0).
\]

In particular, for fixed \( \delta \in (0, 1/2) \), the inequality

\[
|A(y)|^2 \leq \frac{C_0}{\delta^2}
\]
holds for \( y \in M_s^\lambda \cap B_\lambda \) and \( s \in [-\lambda^2 T, -\delta^2] \) and therefore for \( y \in M_s^{1/\delta} \cap B_{1/\delta} \) and \( s \in [-1/\delta^2, -\delta^2] \) for \( \lambda \) sufficiently large depending on \( \delta \), say \( \lambda \geq \lambda_\delta \). By the interior estimate [9], one has for all \( m \geq 0 \)

\[
|\nabla^m A(y)|^2 \leq C(C_0, m, n, \frac{1}{\delta^2(m+1)})
\]

for \( y \in M_s^{1/\delta} \cap B_{1/\delta} \) and \( s \in [-1/4\delta^2, -\delta^2] \). Moreover, by (2.1),

\[
dist(0, M_s^\lambda) = \lambda \cdot \dist(y_0, M_{s+\frac{1}{\lambda^2}}) \leq \lambda \sqrt{2n\left(-\frac{s}{\lambda^2}\right)} = \sqrt{-2n}s
\]

for the above times \( s \) and \( \lambda \geq \lambda_\delta \). By Arzela-Ascoli theorem combined with a diagonal sequence argument when letting \( \delta \searrow 0 \) for local graph representations of \( (M_s^\lambda) \), we can find a subsequence \( \lambda_i \to \infty \) such that \( (M_s^{\lambda_i}) \) converges smoothly on compact subsets of \( \mathbb{R}^{n+1} \times (-\infty, 0) \) to a smooth solution \( (M_s^\infty)_{s < 0} \) of mean curvature flow. From (2.7), one sees that \( H = \frac{1}{2s} F^\perp \) on \( M_s^\infty \) for \( s \in (s_0 - \tau, s_0) \). Take \( s_0 \to 0 \) and \( \tau \to \infty \) to see that \( H = \frac{1}{2s} F^\perp \) on \( M_s^\infty \) for \( -\infty < s < 0 \). In other words, \( (M_s^\infty) \) is a self-shrinking mean curvature flow. Moreover, one deduces from (1.3) and \( |H^\lambda_s| = \frac{|H_0|}{\lambda} \) that \( H = 0 \) on \( M_s^\infty \). Thus \( M_s^\infty \) is a minimal cone for each \( s < 0 \); see Corollary 2.8 in [6]. Because \( M_s^\infty \) is smooth, it is a hyperplane. Now, fix \( s_0 < 0 \). One has, as \( i \to \infty \), \( M_{s_0}^{\lambda_i} \to M_s^\infty \cong \mathbb{R}^n \) and \( d\mu_{s_0} \to dx^n \). Thus

\[
\lim_{i \to \infty} \int_{M_{s_0}^{\lambda_i}} \rho_{0,0} d\mu_{s_0}^{\lambda_i} = \int_{M_s^\infty} \rho_{0,0} dx^n = 1.
\]

This implies that, for \( t_i = T + \frac{s_0}{\lambda_i^2} \)

\[
\lim_{t_i \to T} \int_{M_{t_i}} \rho_{y_0,T} d\mu_{t_i} = 1,
\]

and therefore

\[
\lim_{t \to T} \int_{M_t} \rho_{y_0,T} d\mu_t = 1.
\]

By White’s regularity theorem [24], the second fundamental form \( |A|(\cdot, t) \) of \( M_t \) is bounded as \( t \to T \) and \( (y_0, T) \) is a regular point. Thus, the flow can be extended past time \( T \). \( \square \)

**Proof of Theorem 1.3.** We will split the proof of Theorem 1.3 in the following two lemmas. \( \square \)

**Lemma 2.1.** Theorem 1.3 holds for \( \alpha > n + 2 \).

**Proof.** We use the same notations as in the proof of Theorem 1.2. Note that under the parabolic dilatons \( D_{\lambda_i} \), the inequality (1.4) becomes

\[
C_0^\alpha \geq \int_0^T \int_{M_t} |H^\alpha| d\mu_t dt = \lambda_i^{\alpha-(n+2)} \int_{-\lambda_i^2T}^0 \int_{M_{s_i}^{\lambda_i}} |H^\lambda_s|^{\alpha} d\mu_s^{\lambda_i} ds.
\]
Thus
\begin{equation}
\int_{-\lambda^2T}^{0} \int_{M^\lambda_s} |H^\lambda_s|^\alpha d\mu^\lambda_s ds \leq \frac{C_0^\alpha}{\lambda^{\alpha-(n+2)}}.
\end{equation}

Now, letting $i \to \infty$ as in the proof of Theorem 1.2, we get a self-shrinking mean curvature flow $(M^\infty)_s$ with the property that
\begin{equation}
\int_{-\infty}^{0} \int_{M^\infty_s} |H|^\alpha d\mu^\infty_s ds = 0
\end{equation}
because $\alpha > n + 2$. Therefore $H = 0$ on $M^\infty_s$. Now we can argue similarly as in the proof of Theorem 1.2. \hfill \Box

**Lemma 2.2.** Theorem 1.3 holds for $\alpha = n + 2$.

**Proof.** We use the same notation as in the proof of Theorem 1.2. Under the parabolic dilations $D_{\lambda^i}$, the inequality (1.4) becomes
\begin{equation}
C_0 \geq \int_0^T \int_{M_t} |H|^{n+2} d\mu_t dt = \int_{-\lambda^2T}^{0} \int_{M^\lambda_s} |H^\lambda_s|^{n+2} d\mu^\lambda_s ds.
\end{equation}

Letting $i \to \infty$ as before we get a complete and smooth self-shrinker $M^\infty_s$ in the limit with the property that
\begin{equation}
\int_{-\infty}^{0} \int_{M^\infty_s} |H|^{n+2} d\mu^\infty_s ds \leq C_0 < \infty.
\end{equation}

Our self-shrinker satisfies
\begin{equation}
H = \frac{\langle x, \nu \rangle}{(-2s)},
\end{equation}
which is equivalent to saying that $M_s = \sqrt{-s}M_{-1}$, where $M_s$ satisfies the mean curvature flow. Notice that
\begin{align*}
\int_{M_s} |H|^{n+2}(\cdot, s) d\mu_s &= \int_{M_{-1}} \left( \frac{|H|(\cdot, -1)}{\sqrt{-s}} \right)^{n+2} \cdot (-s)^{\frac{n}{2}} d\mu_{-1} \\
&= \frac{1}{(-s)} \cdot \int_{M_{-1}} |H|^{n+2}(\cdot, -1) d\mu_{-1} \\
&= \frac{a}{(-s)},
\end{align*}
where $a := \int_{M_{-1}} |H|^{n+2}(\cdot, -1) d\mu_{-1}$. If $a > 0$ then
\begin{equation}
\int_{-\infty}^{0} \int_{M^\infty_s} |H|^{n+2} d\mu^\infty_s ds = a \cdot \int_{-\infty}^{0} \frac{ds}{(-s)} = \infty,
\end{equation}
which contradicts (2.13). Therefore $a = 0$, which implies $H(\cdot, -1) = 0$ on $M^\infty_s$. Similar argument shows that $H(\cdot, s) = 0$ on $M^\infty_s$ for every $s < 0$. To prove that $(y_0, T)$ is a regular point of the flow we argue as in the proof of Theorem 1.2. \hfill \Box
3. Extension results for surfaces

In [14] Ilmanen proposed the following conjecture.

**Conjecture 3.1** (Multiplicity One Conjecture). If $M^0$ is embedded in $\mathbb{R}^3$, then for any family of rescalings $\lambda_j (M^0_{s+T} - y_0)$ with $\lambda_j \to \infty$, there is a subsequence smoothly converging and with multiplicity one to the blowup $N_t$, that is, there are no concentration points or multiple layers in the limit.

Theorem 1.4 assumes that the Conjecture above holds and its proof is given below.

**Proof of Theorem 1.4.** In this proof, $n = 2$. Without loss of generality, assume that $M^n \subset B_1(0) \subset \mathbb{R}^{n+1}$. Let $y_0 \in \mathbb{R}^{n+1}$ be a point reached by the mean curvature flow (1.1) at time $T$, that is, there exists a sequence $(y_j, t_j)$ with $t_j \to T$ so that $y_j \in M_{t_j}$ and $y_j \to y_0$. We show that $(y_0, T)$ is a regular point of (1.1).

As in the proof of Theorem 1.2, let $t = T + s/\lambda$ and $M^\lambda_s \equiv M_s^{(y_0,T),\lambda} = \lambda (M_{T+s} - y_0)$.

Then $(M^\lambda_s)$ is a solution of the mean curvature flow in $B_1(0)$ for $s \in [-\lambda^2 T, 0)$. For any set $A \subset \mathbb{R}^{n+1}$, let us define the parabolically rescaled measures at $(y_0, T)$:

$$\mu^\lambda_s(A) = \lambda^{-n} H^n_s (M^\lambda_s (\lambda \cdot A)).$$

Let $\rho_{y_0,T} : \mathbb{R}^{n+1} \times (-\infty, T) \to \mathbb{R}$ be the backward heat kernel at $(y_0, T)$ as defined in (2.3). Then, a result on weak existence of blow ups of Ilmanen and White (see Lemma 8, page 14 of [14] and also [23]) says that: there exists a subsequence $\lambda_j$ and a limiting Brakke flow $\{\nu_s\}_{s<0}$ (also known as a tangent flow) such that $\mu^\lambda_s \rightharpoonup \nu_s$ in the sense of Radon measures for all $s < 0$ and the following statements hold:

(a) (self-similarity) $\nu_s(A) = \nu^\lambda_s(A) \equiv \lambda^{-n} \nu_{s\lambda_2s} (\lambda \cdot A)$, for all $s < 0$ and for all $\lambda > 0$

(b) (tangent flow is a self-shrinker) $\nu_{-1}$ satisfies

$$\overrightarrow{H}(x) + \frac{S(x) \cdot x}{2} = 0, \nu_{-1} \text{ a.e. } x$$

(c) Furthermore, Huisken’s integral converges

$$\int \rho_{0,0}(x,-1) d\nu_s(x) = \lim_{t \nearrow T} \int \rho_{y_0,T} d\mu_t, s < 0.$$ 

Equivalently, a subsequence of rescaled solutions $M^\lambda_s$ converges weakly to a limiting flow $X_s$ that is called a tangent flow at $(y_0, T)$. We know $X_s$ is a self shrinker. Ilmanen showed in [14] that it has to be smooth. Our proof will rely on this fact and the validity of Multiplicity One Conjecture. Let us briefly explain the notations used in (b).

For a locally $n$-rectifiable Radon measure $\mu$, we define its $n$-dimensional approximate tangent plane $T_x\mu$ (which exists $\mu$-a.e $x$) by

$$T_x\mu(A) = \lim_{\lambda \to 0} \lambda^{-n}\mu(x + \lambda \cdot A).$$
The tangent plane $T_x\mu$ is a positive multiple of $\mathcal{H}^n|P$ for some $n$-dimensional plane $P$. Let $S : \mathbb{R}^{n+1} \rightarrow G(n + 1, n)$ denotes the $\mu-$ measureable function that maps $x$ to the geometric tangent plane, denoted by $P$ above. An important quantity is the first variation of $\mu$, denoted by $\int \nabla S(x) X(x) d\mu(x)$ for $X \in C^\infty_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. Here $\nabla S X = \sum_{i=1}^n D_{e_i} X e_i$ where $e_1, \ldots, e_n$ is any orthonormal basis of $S$. We also denote by $S$ the orthogonal projection onto $S$ and thus $\nabla S X$ can be written as $S : DX$. Now, under suitable assumptions, we can define the generalized mean curvature vector $\overrightarrow{H} = \overrightarrow{H}_\mu \in L^{1}_{voc}(\mu)$ of $\mu$ as follows

$$\int \nabla S X d\mu = \int -\overrightarrow{H} \cdot X d\mu$$

for all $X \in C^\infty_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$. Note that when $\mu$ is the surface measure of a smooth $n$-dimensional manifold $M$, the generalized mean curvature vector $\overrightarrow{H}$ of $\mu$ is also the classical mean curvature vector of $M$; see Corollary 4.3 in [18]. From (3.3) and the definition of $\mu_s^\lambda$, one sees that the mean curvature vector $\overrightarrow{H}_s^\lambda$ of $\mu_s^\lambda$ is $\frac{\overrightarrow{H}_t}{\lambda}$ where $\overrightarrow{H}_t$ is the mean curvature vector of $M_t$ where $t = T + \frac{s}{\lambda}$. The lower semicontinuity of $\int |H| d\mu$ asserts that

$$\int |\overrightarrow{H}_s^\lambda| d\nu_s \leq \liminf_{\lambda \to \infty} \int |\overrightarrow{H}_s^\lambda| d\mu_s^\lambda \leq \limsup_{\lambda \to \infty} \int \frac{C_0}{\lambda} d\mu_s^\lambda = 0.$$ 

Thus $\overrightarrow{H}_s = 0$ for all $s < 0$. Now, because $X_s$ is smooth for all $s < 0$, the weak mean curvature vector $\overrightarrow{H}_s$ coincides with the mean curvature vector in classical sense. Thus we have a smooth solution $X_s$ that is a self-shrinker with $H = 0$ and therefore by the result in [6] it has to be a hyperplane. Furthermore $\nu_s$ represents the surface measure of the plane $X_s$ with multiplicity one by the validity of the Multiplicity One Conjecture. Using the convergence of Huisken’s integral (3.2), we see that

$$\lim_{t \to T} \int \rho_{y_0, t} d\mu_t = 1.$$ 

By White’s regularity theorem [24], the second fundamental form $|A| (\cdot, t)$ of $M_t$ is bounded as $t \to T$ and $(y_0, T)$ is a regular point. Thus, the flow can be extended past time $T$. 

We conclude this section by the proof of Theorem 1.5, which can be viewed as a local regularity result without a smallness condition.

**Proof of Theorem 1.5.** We will use the same notation as in the proof of Theorem 1.4. Note that, when $n = 2$, by the fact that $H_2^s$ is bounded (follows from the Gauss-Bonnet theorem for surfaces) and Allard’s Compactness Theorem [20], each Radon measure $\nu_s$ is integer 2-rectifiable, that is

$$d\nu_s = \theta_s(x) d\mathcal{H}^2 |X_s$$

where $X_s$ is an $\mathcal{H}^2$-measurable, 2-rectifiable set and $\theta_s$ is an $\mathcal{H}^2 |X_s$-integrable, integer valued “multiplicity function”.

Furthermore the mean curvature vector $\overrightarrow{H}_s$ of $\nu_s$ satisfies $\overrightarrow{H}_s \in L^{\infty}(\nu_s)$. Here is the only place we wish to use (1.5). The same argument as in the proof of Theorem 1.4 implies $X_s$.
We have

Denote by $\Omega$ the open set $\Omega \subset \mathbb{R}^{n+m}$, $M \subset \Omega$ a connected $C^1$-manifold, $\theta : M \to N_0$ be $\mathcal{H}^n$-measurable with weak mean curvature $\overrightarrow{H}_\mu \in L^1_{loc}(\mu)$, that is

$$\int \text{div}_M \eta d\mu = \int_M \text{div}_M \eta d\mathcal{H}^n = - \int < \overrightarrow{H}_M, \eta > d\mu \ \forall \eta \in C^1_0(\Omega, \mathbb{R}^{n+m}).$$

Then $\theta$ is a constant: $\theta \equiv \theta_0 \in N_0$. Here $N_0$ is the set of all nonnegative integers and $< \cdot >$ is the standard Euclidean inner product on $\mathbb{R}^{n+m}$.

Now $\theta_0$ is a constant and $X_0$ is a plane. Thus by the convergence of Huisken’s integral (3.2), we see that

$$\lim_{t \to T} \int_{\rho_{t,0}(x), -1} \rho_{t,0}(x) d\nu_s(x) = \int_{\rho_{t,0}(x), -1} \rho_{t,0}(x) d\mathcal{H}^2 |X_s = \theta_0|.$$

By (1.6) and Proposition 2.10 in [24], $1 \leq \theta_0 < 2$. It follows from the integrality of $\theta_0$ that $\theta_0 \equiv 1$. Now, our result follows from White’s local regularity theorem [24].

**Proof of Theorem 3.1.** The proof of this theorem can be found in [16], Theorem 4.1. We include here for the reader’s convenience. We consider locally $C^1$-vector fields $\nu^1, \ldots, \nu^m$ on $M$, which are an orthonormal basis of the orthogonal complement $TM^\perp$ of the tangent bundle $TM$ in $T \mathbb{R}^{n+m}$. For $x \in M$, we choose an orthonormal basis $\tau_1, \ldots, \tau_m$ of the tangent space $T_x M$ of $M$ at $x$. We decompose $\eta \in C^1_0(\Omega, \mathbb{R}^{n+m})$ into $\eta = \eta^{tan} + \eta^\perp$, where

$$\eta^{tan}(x) = \pi_{T_x M}(\eta(x)) \in T_x M, \quad \eta^\perp(x) = \pi_{T_x M^\perp}(\eta(x)) = \sum_{j=1}^m < \eta^j, \eta(x) > \nu^j \in T_x M^\perp.$$

Here, we have denoted $\pi_V$ the orthogonal projection operator on the subspace $V$ of $\mathbb{R}^{n+m}$. In particular, $\eta^{tan}, \eta^\perp \in C^1_0(\Omega)$. Then, we have $\text{div}_M \eta = \text{div}_M \eta^{tan} + \text{div}_M \eta^\perp$. Let $D$ be the standard differentiation operator on $\mathbb{R}^{n+m}$ and $A_M$ the second fundamental form of $M$. Denote by $\overrightarrow{H}_M$ the weak mean curvature of $M$. Then

$$\overrightarrow{H}_M = \sum_{i=1}^n A_M(\tau_i, \tau_i).$$

We have

$$\text{div}_M \eta^\perp = \sum_{i=1}^n < \tau_i, \nabla^M_{\tau_i} \eta^\perp > = \sum_{i=1}^n \sum_{j=1}^m < \tau_i, \partial_{\tau_i} < \nu^j, \eta(x) > \nu^j >$$

$$= \sum_{i=1}^n \sum_{j=1}^m < \nu^j, \eta > < \tau_i, \partial_{\tau_i} \nu^j > = - < \eta, \sum_{i=1}^n A_M(\tau_i, \tau_i) > = - < \eta, \overrightarrow{H}_M >.$$
From (3.4), we can calculate

\[ - \int < \vec{H}_\mu, \eta > d\mu = - \int_M < \vec{H}_\mu, \eta > d\mathcal{H}^n = \int_M \text{div}_M \eta \theta d\mathcal{H}^n \]

\[ = \int_M \text{div}_M \eta \tan \theta d\mathcal{H}^n + \int_M \text{div}_M \eta \perp \theta d\mathcal{H}^n \]

\[ = \int_M \text{div}_M \eta \tan \theta d\mathcal{H}^n - \int_M < \vec{H}_\mu, \eta > d\mathcal{H}^n. \]

Let us make some special choices of \( \eta \). First, for \( \eta = \eta \perp \in TM \perp \), we conclude that the projection \( \vec{H}_\mu \perp \) of \( \vec{H}_\mu \) on \( TM \perp \) satisfies \( \vec{H}_\mu \perp = \vec{H}_M \). Since \( \mu \) is integral, we get \( \vec{H}_\mu \perp T\mu = TM \) by Theorem 5.8 in Brakke [1] and conclude \( \vec{H}_\mu = \vec{H}_M \). Finally, if we choose \( \eta \) such that \( \eta = \eta \tan \in TM \) then

\[ \int_M \text{div}_M \eta \tan \theta d\mathcal{H}^n = 0. \]

Calculating in local coordinates, this yields \( \nabla_M \theta = 0 \) weakly. Hence \( \theta \equiv \theta_0 \) is constant, as \( M \) is connected. \( \square \)

4. Some Global Results on the Extension of (1.1)

In this section we give global conditions for extending a smooth solution to (1.1), which has been a subject of study in [17].

Proof of Theorem 1.6. We argue by contradiction. Suppose that \( T \) is the extinction time of the flow. Then, by Theorem 1.1, \( |A| \) is unbounded. Therefore, there exists a sequence of points \((x_i, t_i)\) with \( x_i \in M_{t_i} \) such that

\[ Q_i := |A|(x_i, t_i) = \max_{0 \leq t \leq t_i} \max_{x \in M_t} |A|(x, t) \to +\infty. \]

Consider the sequence \( \tilde{M}^i_t \) of rescaled solutions for \( t \in [0, 1] \) defined by

\[ \tilde{F}_i(\cdot, t) = Q_i(F(\cdot, t_i + t - 1/Q_i^2) - x_i). \]

The sequence of rescaled solutions \( \tilde{M}^i_t \) converges (see [2]) to a complete smooth solution to the mean curvature flow, call it \( \tilde{M}_t \) for \( t \in [0, 1] \) with the property that

\[ |\tilde{A}|(0, 1) = 1. \]

If \( g \) and \( A := \{h_{jk}\} \) are the induced metric, the mean curvature and the second fundamental form of \( M_t \), respectively, then the corresponding rescaled quantities are given by

\[ \tilde{g}_i = Q_i^2 g; \quad |\tilde{A}|^2 = \frac{|A|^2}{Q_i^2}. \]
We calculate
\[
(4.3) \quad \lim_{i \to \infty} \left\{ \int_0^1 \left( \int_{(\tilde{M}_i) \cap B(0,1)} \left| \tilde{A}_i \right|^q \, d\mu \right)^\frac{p}{q} \, dt \right\}^{\frac{1}{p}}
\]
\[
= \lim_{i \to \infty} \left\{ \int_{t_i - \frac{1}{Q_i^2}}^{t_i} \left( \int_{M_t \cap B(0, \frac{1}{Q_i})} |A|^q \, d\mu \right)^\frac{p}{q} \, dt \right\}^{\frac{1}{p}} \leq \lim_{i \to \infty} \left\{ \int_{t_i - \frac{1}{Q_i^2}}^{t_i} \left( \int_{M_t} |A|^q \, d\mu \right)^\frac{p}{q} \, dt \right\}^{\frac{1}{p}} = 0.
\]

The last step follows from the facts that
\[
\left( \int_0^T \left( \int_{M_t} |A|^q \, d\mu \right)^{p/q} \, dt \right)^{1/p} < \infty; \quad \lim_{i \to \infty} \frac{1}{Q_i^2} = 0.
\]

By Fatou’s lemma and (4.3) it follows that
\[
\int_0^1 \left( \int_{\tilde{M}_t \cap B(0,1)} |\tilde{A}|^q \right)^\frac{p}{q} = 0.
\]

By the smoothness of \(\tilde{M}_t\), this implies \(|\tilde{A}|(x, t) \equiv 0\) for all \(x \in \tilde{M}_t \cap B(0,1)\) and all \(t \in [0,1]\). This contradicts (4.2).

\[\square\]

5. Some local regularity results and applications

5.1. \(\varepsilon\)-regularity theorem for the mean curvature flow. In this section we prove Theorem 1.7, which is parabolic version of the epsilon regularity theorem for minimal surfaces proven by Choi and Schoen [3]. A version of the epsilon regularity theorem for mean curvature flow has been obtained by Ecker in [7]. He required smallness of the supremum over small time intervals of spatial \(L^n\) norms of \(|A|\) over small balls.

**Proof of Theorem 1.7.** We may assume without loss of generality that the flow \(\mathcal{M} = (M_t)_{t < t_0}\) is smooth up to and including time \(t_0\), because we can first prove the theorem with \(t_0\) replaced by \(t_0 - \alpha\) for fixed \(\alpha > 0\) and then (since the right hand side of our desired inequality is independent of \(\alpha > 0\)) let \(\alpha \searrow 0\) afterwards.

Let
\[
F(\delta) = \sup_{t \in [t_0 - (\sigma - \delta)^2, t_0]} \sup_{x \in B(x_0, \sigma - \delta) \cap M_t} \delta^2 |A|^2 (x, t).
\]

Since our flow is smooth up to time \(t_0\), \(F(0) = 0\). Thus, there exists \(\delta_* \in (0, \sigma/2]\) such that \(F(\delta_*) = \max_{0 \leq \delta \leq \sigma/2} F(\delta)\). It suffices to show that
\[
F(\delta_*) < \varepsilon_0^{-\frac{2}{n+2}} \left( \int_{t_0 - \sigma^2}^{t_0} \int_{M_t \cap B(x_0, \sigma)} |A|^{n+2} \, d\mu \, dt \right)^{\frac{2}{n+2}} \equiv (\varepsilon_0^{-1} \eta)^{\frac{2}{n+2}}.
\]

Suppose not, then
\[
F(\delta_*) \geq (\varepsilon_0^{-1} \eta)^{\frac{2}{n+2}}.
\]
Because the flow is defined up to and including time \( t_0 \), we can find \( t_* \in [t_0 - (\sigma - \delta_*)^2, t_0] \) and \( x_* \in \overline{B}(x_0, \sigma - \delta_*) \cap M_{t_0} \) such that

\[
\delta_*^2 |A|^2 (x_*, t_*) = F(\delta_*).
\]

It follows from \( \delta_* \in (0, \sigma/2] \) that

\[
B(x_*, \delta_/2) \subset B(x_0, \sigma - \delta_*/2); [t_* - \delta_*^2/4, t_*] \subset [t_0 - (\sigma - \delta_*/2)^2, t_0].
\]

By the choice of \( \delta_*, t_* \) and \( x_* \),

\[
\left( \frac{\delta_*}{2} \right)^2 \sup_{t \in [t_0 - (\sigma - \delta_*/2)^2, t_0]} \sup_{x \in B(x_0, \sigma - \delta_*/2) \cap M_t} |A|^2 (x, t) \leq F(\delta_*) = \delta_*^2 |A|^2 (x_*).
\]

Hence

\[
\sup_{t \in [t_0 - (\sigma - \delta_*/2)^2, t_0]} \sup_{x \in B(x_0, \sigma - \delta_*/2) \cap M_t} |A|^2 (x, t) \leq 4 |A|^2 (x_*, t_*).
\]

and thus, it follows from (5.4) that

\[
\sup_{t \in [t_* - \delta_*^2/4, t_*]} \sup_{x \in B(x_*, \delta_*/2) \cap M_t} |A|^2 (x, t) \leq 4 |A|^2 (x_*, t_*).
\]

We now rescale our mean curvature flow by setting

\[
\tilde{F}(\cdot, t) = QF(\cdot, t_* + \frac{t - t_*}{Q^2}), \tilde{M}_t = \tilde{F}(M^n, t)
\]

where

\[
Q = 2(\varepsilon_0\eta^{-1})^{\frac{1}{n+2}} |A| (x_*, t_*).
\]

Then we have a mean curvature flow \( \tilde{M}_t \) on \( B(x_*, Q\delta_*/2) \) for \( t \in [0, 1] \). Let \( \tilde{g} = Q^2g \) be the induced metric on \( \tilde{M}_t \) and let \( \tilde{B}(x_*, r) \) be the geodesic ball w. r. t. the metric \( \tilde{g} \) and centered at \( x_* \) with radius \( r \). By (5.2) and (5.3), we have

\[
\frac{Q\delta_*}{2} \geq 1.
\]

This combined with (5.5) gives

\[
\sup_{t \in [0,1]} \sup_{x \in B(x_*, 1) \cap \tilde{M}_t} \left| \tilde{A} \right|^2 (x, t) \leq 4 \left| \tilde{A} \right|^2 (x_*, 1) \leq 1.
\]

Note that the last inequality follows from the facts that

\[
4 \left| \tilde{A} \right|^2 (x_*, 1) = \frac{4 |A|^2 (x_*, t_*)}{Q^2} = (\varepsilon_0^{-1}\eta)^{\frac{2}{n+2}} \leq 1.
\]

To obtain a contradiction, we will use the inequality

\[
(\partial_t - \Delta) \left| \tilde{A} \right|^2 \leq 2 \left| \tilde{A} \right|^4.
\]
This is a differential inequality of the form \((\partial_t - \Delta)v \leq fv\) where \(v = |\tilde{A}|^2\) and \(f = 2|\tilde{A}|^2\). Furthermore \(f\) satisfies a smallness condition:

\[
\int_0^1 \int_{\tilde{M}_t \cap \tilde{B}(x,1)} f^{\frac{n+2}{n}} d\tilde{\mu} dt \leq 2^{n+2}\eta \leq 2^{n+2}\varepsilon_0.
\]

Thus, we can localize our estimates in Lemmas 5.1 and 6.1 in [17] to obtain the following inequality

\[(5.9) \quad |\tilde{A}|^2(x,1) \leq \sup_{t \in [1/2,1]} \sup_{x \in \tilde{B}(x,1/2)} |\tilde{A}|^2(x,t) \leq C(n) \left( \int_0^1 \int_{\tilde{M}_t \cap \tilde{B}(x,1)} |\tilde{A}|^{n+2} d\tilde{\mu} dt \right)^{\frac{2}{n+2}} \leq C(n)\eta^{\frac{n+2}{n}}.\]

There is a simple proof of this inequality. It goes as follows. Note that for \(t \in [0,1]\) and \(x \in \tilde{B}(x,1)\), (5.7) and (5.8) give

\[
(\partial_t - \Delta)|\tilde{A}|^2 \leq 2|\tilde{A}|^2
\]

or equivalently \((\partial_t - \Delta)(e^{-2t}|\tilde{A}|^2) \leq 0\). Now, we can apply Moser’s mean value inequality ([7], Proposition 1.6) for \(e^{-2t}|\tilde{A}|^2\) to obtain a constant \(C_1(n)\) depending only on \(n\) such that

\[(5.10) \quad |\tilde{A}|^2(x,1) \leq C_1(n) \int_0^1 \int_{\tilde{M}_t \cap \tilde{B}(x,1)} |\tilde{A}|^2 d\tilde{\mu} dt.
\]

By Hölder inequality

\[(5.11) \quad |\tilde{A}|^2(x,1) \leq C_1(n) \left( \int_0^1 \int_{\tilde{M}_t \cap \tilde{B}(x,1)} d\tilde{\mu} dt \right)^{\frac{n}{n+2}} \left( \int_0^1 \int_{\tilde{M}_t \cap \tilde{B}(x,1)} |\tilde{A}|^{n+2} d\tilde{\mu} dt \right)^{\frac{2}{n+2}}.
\]

By (5.7) and the Gauss equation

\[
\tilde{R}_{ik} = \tilde{H}\tilde{h}_{ik} - \tilde{h}_{il}\tilde{g}^{lj}\tilde{h}_{jk}
\]

one easily sees that the Ricci tensor satisfies \(\tilde{R}_{ik} \geq -(n-1)\). By the Bishop-Gromov volume comparison theorem, for each time \(t \in [0,1]\), one has \(\int_{\tilde{M}_t \cap \tilde{B}(x,1)} \tilde{\mu} \leq V(n)\) where \(V(n)\) denotes the volume of a unit geodesic ball in an n-dimensional space form of constant
curvature $-1$. Thus

$$\left(5.12\right) \quad \left|\bar{A}\right|^2(x_*, 1) \leq C_1(n)V(n)^{\frac{n}{n+2}}(\int_0^1 \int_{\hat{M}_t \cap \hat{B}(x_*, 1)} \left|\bar{A}\right|^{n+2} d\mu dt)^{\frac{2}{n+2}}$$

$$= C_1(n)V(n)^{\frac{n}{n+2}}(\int_0^1 \int_{M_t \cap B(x_*, \frac{1}{2})} \left|A\right|^{n+2} d\mu dt)^{\frac{2}{n+2}}$$

$$\leq C_1(n)V(n)^{\frac{n}{n+2}}(\int_{\sigma - \delta/2}^{\delta} \int_{M_t \cap B(x_*, \sigma/2)} \left|A\right|^{n+2} d\mu dt)^{\frac{2}{n+2}}$$

$$\leq C_1(n)V(n)^{\frac{n}{n+2}}(\int_{t_0 - \sigma^2}^t \int_{M_t \cap B(x_0, \sigma)} \left|A\right|^{n+2} d\mu dt)^{\frac{2}{n+2}} = C_1(n)V(n)^{\frac{n}{n+2}} \mu^{\frac{2}{n+2}}.$$  

Consequently,

$$C_1(n)V(n)^{\frac{n}{n+2}}(\mu)^{\frac{2}{n+2}} \geq \left|\bar{A}\right|^2(x_*, 1) = \frac{1}{4}(\varepsilon_0^{-1})^2 \mu^{\frac{2}{n+2}}.$$  

This is a contradiction if $\varepsilon_0$ is small. \hfill $\square$

**Remark 5.1.** In general, we can modify the proof of Theorem 1.7 to obtain the following result. Suppose $M = (M_t)$ is a smooth, properly embedded solution of the mean curvature flow in $B(x_0, \rho) \times (t_0 - \rho^2, t_0)$ which reaches $x_0$ at time $t_0$. Let $p$ and $q$ be positive numbers satisfying

$$\frac{n}{q} + \frac{2}{p} = 1.$$  

Then, there exists $\varepsilon_0 = \varepsilon_0(M_0, p, q) > 0$ such that if $0 < \sigma \leq \rho$ and

$$\left(5.13\right) \quad \int_{t_0 - \sigma^2}^{t_0} \left(\int_{M_t \cap B(x_0, \sigma)} |A|^q d\mu\right)^{p/q} dt < \varepsilon_0$$

then

$$\left(5.14\right) \quad \max_{0 \leq \delta \leq \sigma/2} \sup_{t \in [t_0 - (\sigma - \delta)^2, t_0]} \sup_{x \in B(x_0, \sigma - \delta) \cap M_t} \delta^2 |A|^2(x, t) < \varepsilon_0^{\frac{2}{p}} \left(\int_{t_0 - \sigma^2}^{t_0} \left(\int_{M_t \cap B(x_0, \sigma)} |A|^q d\mu\right)^{p/q} dt\right)^{\frac{2}{p} \delta^2}.$$  

Theorem 1.7 can be extended to the case when an ambient manifold is an arbitrary Riemannian manifold.

**Corollary 5.1.** Let $n \geq 2$ and $N^{n+1}$ be a smooth complete, locally symmetric Riemannian manifold with bounded geometry. Let $M_0$ be a compact connected hypersurface without boundary which is smoothly immersed in $B(x_0, \rho) \subset N$. Suppose that $M = (M_t)$ is a smooth, properly embedded solution of the mean curvature flow in $B(x_0, \rho) \times (t_0 - \rho^2, t_0)$ which reaches $x_0$ at time $t_0$. There exists $\varepsilon_0 = \varepsilon_0(M_0, N)$ such that if $0 < \sigma \leq \rho$ and

$$\left(5.15\right) \quad \int_{t_0 - \sigma^2}^{t_0} \int_{M_t \cap B(x_0, \sigma)} |A|^{n+2} d\mu dt < \varepsilon_0$$
then
\begin{equation}
\max_{0 \leq \delta \leq \sigma/2} \sup_{t \in [t_0 - (\sigma - \delta)^2, t_0)} \sup_{x \in B(x_0, \sigma - \delta) \cap M_t} \delta^2 |A|^2 (x, t) < \varepsilon_0^{-2} \left( \int_{t_0 - \sigma^2}^{t_0} \int_{M_t \cap B(x_0, \sigma)} |A|^{n+2} \, d\mu \, dt \right)^{\frac{2}{n+2}}.
\end{equation}

**Proof.** In the formulas that follow, if we mean the metric or the connection on \( N \), this will be indicated by a bar, for example \( \bar{g}_{\alpha\beta} \), etc. The Riemann curvature tensors on \( M \) and \( N \) will be denoted by \( R_{ijkl} \) and \( \bar{R}_{\alpha\beta\gamma\delta} \). Let \( \nu \) be the outer unit normal to \( M_t \).

For a fixed time \( t \), we choose a local field of frame \( e_0, e_1, \ldots, e_n \) in \( N \) such that when restricted to \( M_t \), we have \( e_0 = \nu, e_i = \partial F/\partial x^i \). The relations between \( A = (h_{ij}) \), \( R_{ijkl} \) and \( \bar{R}_{\alpha\beta\gamma\delta} \) are given by the equations of Gauss and Codazzi:
\begin{equation}
R_{ijkl} = \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk},
\end{equation}
\begin{equation}
\nabla_k h_{ij} - \nabla_j h_{ik} = \bar{R}_{ijkl}.
\end{equation}

Observe that we have the following evolution equation:
\begin{equation}
\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + \bar{\text{Ric}}(\nu, \nu))
- 4(h^{ij}h^m_{lj} \bar{R}_{mli} - h_{ij}h^{lm} \bar{R}_{mitj}) - 2h^{ij}(\nabla_j \bar{R}_{0li} + \nabla_l \bar{R}_{0ij}),
\end{equation}
whose derivation can be found in [12]. Since \( N \) is, by our assumption, locally symmetric, we have \( \bar{\nabla} \bar{R}_{mli} = 0 \) and therefore the previous equation just reads as
\begin{equation}
\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + \bar{\text{Ric}}(\nu, \nu))
- 4(h^{ij}h^m_{lj} \bar{R}_{mli} - h_{ij}h^{lm} \bar{R}_{mitj}).
\end{equation}

Using the evolution equation (5.19) and the bounds on the geometry of \( N \) we obtain
\[ \frac{\partial}{\partial t} |A|^2 \leq \Delta |A|^2 + 2|A|^4 + C|A|^2. \]

After rescaling our solution and using (5.7), we obtain
\[ \left( \frac{\partial}{\partial t} - \Delta \right) |\tilde{A}|^2 \leq C|\tilde{A}|^2. \]

If \( f := e^{-C \cdot t} |\tilde{A}|^2 \) then we have
\[ \left( \frac{\partial}{\partial t} - \Delta \right) f \leq 0. \]

If we take a trace of (5.17) in \( jl \) we obtain
\[ R_{ik} = g^{il} \bar{R}_{ijkl} + h_{ik} \bar{H} - h_{il}h_{jk}g^{il} \geq -C. \]

Applying the Moser mean value inequality to \( f \) will lead to a contradiction in the same way as in the proof of Theorem 1.7. \qed
5.2. Some applications of Theorem 1.7. In this section, we give three applications of the local regularity results obtained in section 5.1.

The first application is a simple consequence of the Remark 5.1. It gives a sufficient integral condition for (1.1) to have a type I singularity. This will be achieved by showing that any type-I control on the $L^s$-norm ($s > n$) of the second fundamental form for all time slice $t$ gives a type-I control on the second fundamental form. Precisely, we prove the following.

**Corollary 5.2.** Let $s \in (n, \infty)$. Suppose there is a constant $C_s > 0$ such that for any $T/2 \leq t < T$, we have

\[
\|A\|_{L^s(M_t)} \leq \frac{C_s}{(T-t)^{\frac{s-n}{2s}}}.
\]

Then (1.2) holds.

**Remark 5.2.** We say that the $L^s$-control on the second fundamental form given by (5.20) is of type I. Notice that for the shrinking spheres $S^n$ we have the equality in (5.20).

**Proof of Corollary 5.2.** For $q = s > n$ there exists a positive number $p$ such that

\[
\frac{n}{s} + \frac{2}{p} = 1.
\]

Let $(x_0, t_0)$ be arbitrary, where $0 < t_0 < T$. Let $\sigma \in (0, \sqrt{t_0})$. Then, for any $t \in (t_0 - \sigma^2, t_0)$ we have

\[
\int_{t_0 - \sigma^2}^{t_0} \left( \int_{B(x_0, \sigma) \cap M_t} |A|^s \right)^{\frac{2}{s}} \leq \int_{t_0 - \sigma^2}^{t_0} \frac{dt}{T-t} \leq \frac{C \sigma^2}{T-t_0} =: C\alpha,
\]

where $\alpha(T-t_0) = \sigma^2$. Fix $\alpha$ sufficiently small so that $C\alpha \leq \varepsilon_0(M_0, p, q)$ where $\varepsilon_0(M_0, p, q)$ is the small constant in Remark 5.1. For this choice of $\alpha$, the estimate (5.14), taking $\delta = \frac{\sigma}{2}$ gives

\[
|A|^2(x_0, t_0) \leq \frac{4}{\sigma^2} = \frac{C}{T-t_0},
\]

and this completes the proof of our corollary. $\square$

The second application is a lower bound on the $L^s$-norm ($s > n$) of the second fundamental form at each time slice. This lower bound can be viewed as a slight generalization of Husiken’s estimate [11] where the case $s = \infty$ was considered. Let $s \in (n, \infty)$. Suppose that $T$ is the first singular time of the mean curvature flow. We are interested in the following question:

Does there exist a constant $C'_s > 0$ such that for any $t < T$, we have

\[
\|A\|_{L^s(M_t)} \geq \frac{C'_s}{(T-t)^{\frac{s-n}{2s}}}.
\]

We prove a weaker version of the above inequality as follows
Corollary 5.3. For $t < T$, let $f(t) = \sup_{t \leq t} \|A\|_{L^1(M_t)}$. Then there exists a constant $C'_s > 0$ such that

$$f(t) \geq \frac{C'_s}{(T - t)^{\frac{s}{2s} - n}}.$$  

Proof of Corollary 5.3. For $q = s > n$ there exists a positive number $p$ such that

$$\frac{n}{s} + \frac{2}{p} = 1.$$  

Let $(x_0, t_0)$ be arbitrary, where $0 < t_0 < T$. Let $\sigma \in (0, \sqrt{t_0})$. Then, for any $t \in (t_0 - \sigma^2, t_0)$ we have

$$\int_{t_0 - \sigma^2}^{t_0} \left( \int_{B(x_0, \sigma)} \left| A \right|^s \right)^{\frac{2}{s-n}} \leq \int_{t_0 - \sigma^2}^{t_0} (f(t_0))^{\frac{2s}{s-n}} = \sigma^2 (f(t_0))^{\frac{2s}{s-n}}.$$  

Fix $\sigma$ so that $\sigma^2 (f(t_0))^{\frac{2s}{s-n}} = \varepsilon_0(M_0, p, q)$ where $\varepsilon_0(M_0, p, q)$ is the small constant in Remark 5.1. For this choice of $\alpha$, the estimate (5.14), taking $\delta = \frac{\sigma}{2}$ gives

$$\left| A \right|^2 (x_0, t_0) \leq \frac{4}{\sigma^2} = \frac{4(f(t_0))^{\frac{2s}{s-n}}}{\sigma^2 (f(t_0))^{\frac{2s}{s-n}}} = \frac{4(f(t_0))^{\frac{2s}{s-n}}}{\varepsilon_0(p, q)}.$$  

Taking the supremum of the left hand side with respect to $x_0$ and in view of Huisken’s estimate [11] on the lower bound of $\sup_{M_t_0} |A|^2$, we get

$$\frac{2}{T - t_0} \leq \sup_{M_0} |A|^2 \leq \frac{4(f(t_0))^{\frac{2s}{s-n}}}{\varepsilon_0(p, q)}.$$  

This gives the desired inequality. □

The third application is a regularity result without a smallness condition. We will use the curvature estimate in Theorem 1.7 to obtain other curvature estimates without any smallness condition for mean curvature flow of surfaces. Our result in this direction states

Corollary 5.4. Suppose $\mathcal{M} = (M_t)$ is a smooth, properly embedded solution of the mean curvature flow in $B(x_0, 4\sigma) \times (t_0 - (4\sigma)^2, t_0) \subset \mathbb{R}^3 \times (t_0 - (4\sigma)^2, t_0)$ which reaches $x_0$ and time $t_0$. Given a constant $C_I > 0$, there is a constant $C_P > 0$ so that if

$$\int_{t_0 - (4\sigma)^2}^{t_0} \int_{B(x_0, 4\sigma) \cap M_t} |A|^4 \, d\mu \, dt \leq C_I$$  

then

$$\sup_{t \in [t_0 - \sigma^2, t_0]} \sup_{x \in B(x_0, \sigma) \cap M_t} |A|^2 (x, t) \leq C_P \sigma^{-2}.$$  

Proof of Corollary 5.4. We start with the following claim.
Claim 5.1. Given (5.25) there is a constant $C$ so that

$$
\sup_{t \in [t_0 - ((2\sigma)^2, t_0)} \int_{B(x_0, 2\sigma) \cap M_t} |A|^2 \, dx \leq C.
$$

Proof. Let $\eta(x, t)$ be a cut off function compactly supported in $B(x_0, 4\sigma) \cap M_t \times [t_0 - (4\sigma)^2, t_0]$, identically equal to one on $B(x_0, 2\sigma) \times [t_0 - (3\sigma)^2, t_0]$ (the same one that Ecker used in [7]). Multiply the evolution equation of $|A|^2$

$$
\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4,
$$

by $\eta^2$ and integrate it over $M_t$. Using the evolution equation of the volume form $\frac{d}{dt} \mu = -H^2 d\mu$, we see that

$$
\frac{d}{dt} \int_{M_t} |A|^2 \eta^2 \leq \int_{M_t} \frac{d}{dt} |A|^2 \eta^2 + |A|^2 \frac{d}{dt} \eta^2
$$

$$
= \int_{M_t} (\Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4)\eta^2 + |A|^2 \frac{d}{dt} \eta^2
$$

(5.27)

$$
= \int_{M_t} |A|^2 2\eta(\frac{d}{dt} \Delta)\eta + 2|A|^4 \eta^2 + 2|A|^2 \eta \Delta \eta + \Delta |A|^2 \eta^2 - 2|\nabla A|^2 \eta^2.
$$

Integrating by parts gives

$$
\int_{M_t} 2|A|^2 \eta \Delta \eta + \Delta |A|^2 \eta^2 - 2|\nabla A|^2 \eta^2 = \int_{M_t} -2\nabla(|A|^2 \eta) \nabla \eta - \nabla |A|^2 \nabla \eta^2 - 2|\nabla A|^2 \eta^2
$$

(5.28)

$$
= \int_{M_t} -6|A|\nabla |A| \eta \nabla \eta - 2|A|^2 |\nabla \eta|^2 - 2|\nabla A|^2 \eta^2.
$$

Using Kato’s inequality $|\nabla |A|| \leq |\nabla A|$ and Cauchy-Schwarz’s inequality, one deduces from (5.28) that

(5.29)

$$
\int_{M_t} 2|A|^2 \eta \Delta \eta + \Delta |A|^2 \eta^2 - 2|\nabla A|^2 \eta^2 \leq \int_{M_t} 2|A|^2 |\nabla \eta|^2.
$$

Combining (5.27) and (5.29), we get

$$
\frac{d}{dt} \int_{M_t} |A|^2 \eta^2 \leq \int_{M_t} |A|^2 2\eta(\frac{d}{dt} \Delta)\eta + 2|A|^2 |\nabla \eta|^2 + 2|A|^4 \eta^2.
$$

Using that

$$
\sup_{M \times [t_0 - 1, t_0]} (\eta^2 + |\nabla \eta|^2 + 2\eta(\frac{d}{dt} \Delta)\eta) \leq \frac{c}{\sigma^2},
$$

and that $\text{vol}(B(x_0, 4\sigma) \cap M_t) \leq C\sigma^2$ (this can be proved using Huisken’s monotonicity formula [11]; see for example Lemma 1. 4 in [7]) we have

$$
\frac{d}{dt} \int_{B(x_0, 4\sigma)} |A|^2 \eta^2 \, d\mu \leq \frac{C}{\sigma^2} \int_{B(x_0, 4\sigma) \cap M_t} |A|^2 \, d\mu + C \int_{B(x_0, 4\sigma) \cap M_t} |A|^4 \, d\mu.
$$
Choose a cut off function $\psi(t)$ in time so that $\psi = 0$ for $t \in [0, t_0 - (4\sigma)^2]$, $\psi(t) = 1$ for $t \geq t_0 - (2\sigma)^2$ and in between grows linearly. Multiply the previous inequality by $\psi(t)$ and integrate it over $[t_0 - (4\sigma)^2, t]$, where $t \geq t_0 - (2\sigma)^2$. Then,

$$\int_{B(x_0,2\sigma)} |A|^2 \, d\mu \leq \frac{C}{\sigma^2} \int_{t_0 - (4\sigma)^2}^{t_0} \int_{B(x_0,4\sigma) \cap M_t} |A|^2 \, d\mu + \tilde{C}.$$ 

By Hölder inequality and the euclidean volume growth we have

$$\int_{B(x_0,2\sigma)} |A|^2 \, d\mu \leq \frac{C}{\sigma^2} \left( \int_{t_0 - (4\sigma)^2}^{t_0} \int_{B(x_0,4\sigma) \cap M_t} |A|^4 \, d\mu \right)^{\frac{1}{2}} \cdot \left( \int_{t_0 - (4\sigma)^2}^{t_0} \int_{B(x_0,4\sigma) \cap M_t} d\mu \right)^{\frac{1}{2}} + \tilde{C}$$

$$\leq \frac{CC_I}{\sigma^2} \cdot (16\sigma^2 \cdot x^2)^{\frac{1}{2}} + \tilde{C}$$

$$= \tilde{C},$$

where $\tilde{C}$ is a uniform constant, independent of $\sigma$. $\square$

Having (5.25) and Claim 5.1 we can continue as follows. Let $\varepsilon \in (0, \varepsilon_0)$ be a small number to be determined. Here $\varepsilon_0$ is as in Theorem 1.7. Let $N$ be an integer greater than $C_I/\varepsilon$. Given $x \in B(x_0, \sigma) \cap M_t$ where $t \in [t_0 - \sigma^2, t_0)$, there exists $1 \leq j \leq N$ with

$$\int_{t_0 - (2\sigma)^2}^{t_0} \int_{B(x,9^{-j}\sigma) \cap M_t} |A|^4 \, d\mu d\nu \leq C_I/N \leq \varepsilon \leq \varepsilon_0.$$

Note that, if $s = 9^{-j}\sigma$ then $B(x, 9s) \subset B(x_0, 2\sigma)$. Therefore

$$\int_{t_0 - (2\sigma)^2}^{t_0} \int_{B(x,9s) \cap M_t} |A|^4 \, d\mu dt \leq C_I.$$

From the estimate

$$\int_{t_0 - (2\sigma)^2}^{t_0} \int_{B(x,9s) \cap M_t} |A|^4 \, d\mu dt \leq \varepsilon \leq \varepsilon_0$$

we have, by the Choi-Schoen type estimate in Theorem 1.7

$$\sup_{t \in [t_0 - (2\sigma - s)^2, t_0]} \sup_{y \in B(x_0,2\sigma) \cap M_t} |A|^2(y, t)$$

$$\leq \varepsilon_0^{-1/2} s^{-2} \left( \int_{t_0 - (2\sigma)^2}^{t_0} \int_{B(x,9s) \cap M_t} |A|^4 \, d\mu dt \right)^{1/2} \leq \varepsilon_0^{-1/2} \varepsilon^{1/2} s^{-2}.$$

Moreover, inspecting the proof, we can replace extrinsic balls by intrinsic balls $B(x, s)$. Thus, for each time slice $t \in [t_0 - (2\sigma - s)^2, t_0)$, we have the following two estimates

$$\sup_{y \in B(x,9s) \cap M_t} |A|^2(y, t) \leq \varepsilon_0^{-1/2} \varepsilon^{1/2} s^{-2}$$

and

$$\int_{B(x,9s) \cap M_t} |A|^2 \, d\mu \leq \int_{B(x_0,2\sigma) \cap M_t} |A|^2 \, d\mu \leq C_I.$$
Now, arguing as in the proof of Colding-Minicozzi [5], Lemma 1.10, one can find a small number \( \varepsilon \) depending only on \( C_I \) and \( \varepsilon_0 \) such that (5.32) and (5.33) imply the following curvature estimate

\[
\sup_{B(x,s) \subset M_t} |A|^2 \leq s^{-2} = (9^{-j}\sigma)^2 \leq 9^{2N}\sigma^{-2}.
\]

Hence, for \( x \in B(x_0, \sigma) \cap M_t \) where \( t \in [t_0 - \sigma^2, t_0) \), the following estimate holds

\[
|A|^2(x,t) \leq 9^{2N}\sigma^{-2}.
\]

□

REFERENCES


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