

BLOW UP OF SUBCRITICAL QUANTITIES AT THE FIRST SINGULAR TIME OF THE MEAN CURVATURE FLOW

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ABSTRACT. Consider a family of smooth immersions $F(\cdot, t) : M^n \rightarrow \mathbb{R}^{n+1}$ of closed hypersurfaces in \mathbb{R}^{n+1} moving by the mean curvature flow $\frac{\partial F(p,t)}{\partial t} = -H(p,t) \cdot \nu(p,t)$, for $t \in [0, T)$. We show that at the first singular time of the mean curvature flow, certain subcritical quantities concerning the second fundamental form, for example $\int_0^t \int_{M_s} \frac{|A|^{n+2}}{\log(2+|A|)} d\mu ds$, blow up. Our result is a log improvement of recent results of Le-Sesum, Xu-Ye-Zhao where the scaling invariant quantities were considered.

1. INTRODUCTION

Let M^n be a compact n -dimensional hypersurface without boundary, and let $F_0 : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth immersion of M^n into \mathbb{R}^{n+1} . Consider a smooth one-parameter family of immersions

$$F(\cdot, t) : M^n \rightarrow \mathbb{R}^{n+1}$$

satisfying

$$F(\cdot, 0) = F_0(\cdot)$$

and

$$(1.1) \quad \frac{\partial F(p,t)}{\partial t} = -H(p,t)\nu(p,t) \quad \forall (p,t) \in M \times [0, T).$$

Here $H(p,t)$ and $\nu(p,t)$ denote the mean curvature and a choice of unit normal for the hypersurface $M_t = F(M^n, t)$ at $F(p,t)$. We will sometimes also write $x(p,t) = F(p,t)$ and refer to (1.1) as to the mean curvature flow equation. For any compact n -dimensional hypersurface M^n which is smoothly embedded in \mathbb{R}^{n+1} by $F : M^n \rightarrow \mathbb{R}^{n+1}$, let us denote by $g = (g_{ij})$ the induced metric, $A = (h_{ij})$ the second fundamental form, $d\mu = \sqrt{\det(g_{ij})} dx$ the volume form, ∇ the induced Levi-Civita connection and Δ the induced Laplacian. Then the mean curvature of M^n is given by $H = g^{ij}h_{ij}$. We will use the following notation throughout the whole paper,

$$\|v\|_{L^{p,q}(M \times [0,T))} := \left(\int_0^T \left(\int_{M_t} |v|^p d\mu \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}},$$

for a function $v(\cdot, t)$ defined on $M \times [0, T)$.

Without any special assumptions on M_0 , the mean curvature flow (1.1) will in general

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develop singularities in finite time, characterized by a blow up of the second fundamental form $A(\cdot, t)$.

Theorem 1.1 (Huisken [4]). *Suppose $T < \infty$ is the first singularity time for a compact mean curvature flow. Then $\sup_{M_t} |A|(\cdot, t) \rightarrow \infty$ as $t \rightarrow T$.*

In Le-Sesum [5, 6], Xu-Ye-Zhao [8], it was proved that at the first singularity time of the mean curvature flow, certain scaling invariant quantities blow-up. Specifically,

Theorem 1.2. *Suppose $T < \infty$ is the first singularity time for a compact mean curvature flow. Let p and q be positive numbers satisfying $\frac{n}{p} + \frac{2}{q} = 1$. Then $\|A\|_{L^{p,q}(M \times [0,t])} \rightarrow \infty$ as $t \rightarrow T$. In particular, for $p = q = n + 2$, one has $\int_0^t \int_{M_s} |A|^{n+2} d\mu ds \rightarrow \infty$ as $t \rightarrow T$.*

The proof in [6, 8] used a blow-up argument combined with a compactness property of the mean curvature flow [2]. The proof in [5] used a blow-up argument and Moser iteration.

In this paper, we give a logarithmic improvement of the above results by showing that a family of subcritical quantities concerning the second fundamental form blows up at the first singular time of the mean curvature flow. Our proof covers a large class of such subcritical quantities including $\int_0^t \int_{M_s} \frac{|A|^{n+2}}{\log(1+|A|)} d\mu ds$. For clarity, we will focus on this quantity. Equivalently, we prove the following

Theorem 1.3. *Assume that for the mean curvature flow (1.1), we have*

$$(1.2) \quad \int_0^T \int_{M_t} \frac{|A|^{n+2}}{\log(1+|A|)} d\mu dt < \infty.$$

Then the flow can be extended past time T .

Our result is inspired by a recent log improvement of the Prodi-Serrin criteria for Navier-Stokes equations by Chan-Vasseur [1]. The usual Prodi-Serrin criterion ensures global regularity of a weak Leray-Hopf solution u of the Navier-Stokes equation in dimension 3 provided that $|u|^5$ is integrable in space time variables. Chan-Vasseur's result shows that the global regularity holds under the condition that $|u|^5 / \log(1 + |u|)$ is integrable in space time variables.

Note that, however, the techniques used in [1] and in the present article are different. In [1], the authors used De Giorgi's technique while in our paper, we use Moser iteration.

Remark 1.1. *To our knowledge, Theorem 1.3 is the first result in geometric evolutions where the finiteness of a slightly subcritical quantity implies global existence. It would be interesting to obtain similar results in other settings such as the heat flow of harmonic maps or the Ricci flow.*

Let us comment briefly on ideas of the proof of Theorem 1.3. The key point in the proof of Theorem 1.3 is that for any time $t > 0$, the second fundamental form $A(\cdot, t)$ can be bounded in an affine way by $\int_0^t \int_{M_s} |A|^{n+3} d\mu ds$. More precisely, we have the following

Proposition 1.1. *For all $\lambda \in (0, 1]$ there is a constant c_λ such that for all $T \geq \lambda$*

$$(1.3) \quad \sup_{x \in M_T} |A(x, T)| \leq c_\lambda \left(1 + \int_0^T \int_{M_t} |A|^{n+3} d\mu dt \right).$$

Theorem 1.3 then follows from a Gronwall-type argument on $\sup_{x \in M_t} |A|(x, t)$.

The rest of the paper is organized as follows. In Section 2, we establish Sobolev inequalities for the mean curvature flow. We will use these inequalities to prove reverse Holder and Harnack inequalities in Section 3. In Section 4, we prove Proposition 1.1. The proof of Theorem 1.3 will be carried out in the final section, Section 5, of the paper.

2. SOBOLEV INEQUALITIES FOR THE MEAN CURVATURE FLOW

In this section, we establish a version of Michael-Simon inequality, Lemma 2.1, that allows us to derive a Sobolev type inequality, Proposition 2.1, for the mean curvature flow. This Sobolev inequality will be crucial for the reverse Holder and Harnack inequalities in the next section.

The following lemma consists of a slightly modified Michael-Simon inequality whose proof is based on the original Michael-Simon inequality [7] together with the interpolation inequalities. By their inequality there is a uniform constant c_n , depending only on n , such that for any nonnegative, C^1 function f on a hypersurface $M \subset \mathbb{R}^{n+1}$, the following holds

$$(2.1) \quad \left(\int_M f^{\frac{n}{n-1}} d\mu \right)^{\frac{n-1}{n}} \leq c_n \int_M (|\nabla f| + |H|f) d\mu.$$

Lemma 2.1. *Let M be a compact n -dimensional hypersurface without boundary, which is smoothly embedded in \mathbb{R}^{n+1} . Let*

$$(2.2) \quad Q = \begin{cases} \frac{n}{n-2} & \text{if } n > 2 \\ < \infty & \text{if } n = 2 \end{cases}$$

Then, for all Lipschitz functions v on M , we have

$$\|v\|_{L^{2Q}(M)}^2 \leq c_n \left(\|\nabla v\|_{L^2(M)}^2 + \|H\|_{L^{\frac{2(n+3)}{3}}(M)} \|v\|_{L^2(M)}^2 \right)$$

where H is the mean curvature of M and c_n is a positive constant depending only on n .

Remark 2.1. *The exponent $\frac{2}{3}$ (< 1) appearing in $\|H\|_{L^{\frac{2(n+3)}{3}}(M)}^{n+3}$ in the above inequality plays a crucial role in our paper. It allows us to bound C_1 in terms of C_0 (defined in (3.2)). See (4.3).*

Proof. The proof of this lemma is very similar to that of Lemma 3.1 in [5]. For reader's convenience, we include the proof. We only need to prove the lemma for $v \geq 0$. Applying Michael-Simon's inequality (2.1)[7] to the function $w = v^{\frac{2(n-1)}{n-2}}$, we get

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-1}{n}} \leq c_n \left(\int_M |\nabla v| v^{\frac{n}{n-2}} d\mu + \int_M |H| v^{\frac{2(n-1)}{n-2}} d\mu \right).$$

By Holder's inequality it follows that

$$\begin{aligned}
\left(\int_M v^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} &\leq c_n^{\frac{n-2}{n-1}} \left(\int_M |\nabla v| v^{\frac{n}{n-2}} d\mu + \int_M |H| v^{\frac{2(n-1)}{n-2}} d\mu \right)^{\frac{n-2}{n-1}} \\
&\leq c_n \left(\|\nabla v\|_{L^2(M)} \|v\|_{L^{2Q}(M)}^{\frac{n}{n-2}} + \|H\|_{L^{n+3}(M)} \|v\|_{L^{2m}(M)}^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{n-1}} \\
&\leq c_n \left(\|\nabla v\|_{L^2(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2Q}(M)}^{\frac{n}{n-1}} + \|H\|_{L^{n+3}(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2m}(M)}^2 \right).
\end{aligned}$$

where

$$m = \frac{(n-1)(n+3)}{(n-2)(n+2)}.$$

Thus

$$(2.3) \quad \|v\|_{L^{2Q}(M)}^2 \leq c_n \left(\|\nabla v\|_{L^2(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2Q}(M)}^{\frac{n}{n-1}} + \|H\|_{L^{n+3}(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2m}(M)}^2 \right).$$

By Young's inequality

$$(2.4) \quad ab = (\varepsilon^{1/p} a)(\varepsilon^{-1/p} b) \leq \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{-q/p} b^q}{q} \leq \varepsilon a^p + \varepsilon^{-q/p} b^q,$$

where $a, b, \varepsilon > 0$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If we apply it to (2.3), with

$$a = \|v\|_{L^{2Q}(M)}^{\frac{n}{n-1}}, \quad b = \|\nabla v\|_{L^2(M)}^{\frac{n-2}{n-1}},$$

and

$$\varepsilon = \frac{1}{2c_n}, \quad p = \frac{2(n-1)}{n}, \quad q = \frac{2(n-1)}{n-2},$$

we obtain

$$\|v\|_{L^{2Q}(M)}^2 \leq c_n \left(\frac{1}{2c_n} \|v\|_{L^{2Q}(M)}^2 + \left(\frac{1}{2c_n} \right)^{\frac{-n}{n-2}} \|\nabla v\|_{L^2(M)}^2 + \|H\|_{L^{n+3}(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2m}(M)}^2 \right).$$

Hence

$$(2.5) \quad \|v\|_{L^{2Q}(M)}^2 \leq c_n \left(\|\nabla v\|_{L^2(M)}^2 + \|H\|_{L^{n+3}(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2m}(M)}^2 \right).$$

Next, we will use the following interpolation inequality (see inequality (7.10) in [3])

$$(2.6) \quad \|u\|_{L^r} \leq \varepsilon \|u\|_{L^s} + \varepsilon^{-\mu} \|u\|_{L^t}$$

where $t < r < s$ and

$$\mu = \left(\frac{1}{t} - \frac{1}{r} \right) / \left(\frac{1}{r} - \frac{1}{s} \right).$$

Note that, in our case $1 < m < Q$, and therefore, by (2.6)

$$(2.7) \quad \|v\|_{L^{2m}(M)} \leq \varepsilon \|v\|_{L^{2Q}(M)} + \varepsilon^{-\alpha} \|v\|_{L^2(M)}$$

where $\varepsilon > 0$ and

$$(2.8) \quad \alpha = \frac{Q(m-1)}{Q-m} = \frac{n(2n+1)}{3(n-2)}.$$

Plugging (2.7) into the right hand side of (2.5), we deduce that

$$(2.9) \quad \begin{aligned} \|v\|_{L^{2Q}(M)}^2 &\leq c_n \|\nabla v\|_{L^2(M)}^2 + c_n \|H\|_{L^{\frac{n-2}{n-1}}^{n+3}(M)} \left(\varepsilon \|v\|_{L^{2Q}(M)} + \varepsilon^{-\alpha} \|v\|_{L^2(M)} \right)^2 \\ &\leq c_n \|\nabla v\|_{L^2(M)}^2 + c_n \|H\|_{L^{\frac{n-2}{n-1}}^{n+3}(M)} \left(\varepsilon^2 \|v\|_{L^{2Q}(M)}^2 + \varepsilon^{-2\alpha} \|v\|_{L^2(M)}^2 \right). \end{aligned}$$

Now, we can absorb the term involving $\|v\|_{L^{2Q}(M)}^2$ into the left hand side of (2.9) by choosing $\varepsilon^2 = \frac{1}{2c_n} \|H\|_{L^{\frac{n-2}{n-1}}^{n+3}(M)}^{-\frac{n-2}{n-1}}$. Since $\frac{n-2}{n-1}(1+\alpha) = \frac{2(n+2)}{3}$, we obtain the desired inequality

$$\|v\|_{L^{2Q}(M)}^2 \leq c_n \|\nabla v\|_{L^2(M)}^2 + c_n \|H\|_{L^{\frac{2(n+3)}{3}}^{n+3}(M)} \|v\|_{L^2(M)}^2.$$

□

Our Sobolev type inequality for the mean curvature flow is stated in the following proposition.

Proposition 2.1. *For all nonnegative Lipschitz functions v , one has*

$$(2.10) \quad \begin{aligned} \|v\|_{L^\beta(M \times [0, T])}^\beta &\leq c_n \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^{4/n} \left(\|\nabla v\|_{L^2(M \times [0, T])}^2 + \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^2 \|H\|_{L^{\frac{2(n+3)}{3}}^{n+3}, \frac{2(n+3)}{3}}(M \times [0, T]) \right), \end{aligned}$$

where $\beta := \frac{2(n+2)}{n}$.

Proof. By Holder's inequality, we have

$$\begin{aligned} \int_0^T \int_{M_t} v^{\frac{2(n+2)}{n}} d\mu dt &= \int_0^T dt \int_{M_t} v^2 v^{4/n} d\mu \leq \int_0^T dt \left(\int_{M_t} v^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} \left(\int_{M_t} v^2 d\mu \right)^{\frac{2}{n}} \\ &\leq \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^{4/n} \int_0^T \|v(\cdot, t)\|_{L^{2Q}(M_t)}^2 dt. \end{aligned}$$

Now, applying Lemma 2.1, we get

$$\begin{aligned} \|v\|_{L^\beta(M \times [0, T])}^\beta &\leq c_n \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^{4/n} \left(\int_0^T \int_{M_t} |\nabla v|^2 d\mu dt + \int_0^T \left(\int_{M_t} |H|^{n+3} d\mu \right)^{\frac{2}{3}} \|v(\cdot, t)\|_{L^2(M_t)}^2 dt \right) \\ &\leq c_n \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^{4/n} \left(\|\nabla v\|_{L^2(M \times [0, T])}^2 + \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^2 \|H\|_{L^{\frac{2(n+3)}{3}}^{n+3}, \frac{2(n+3)}{3}}(M \times [0, T]) \right). \end{aligned}$$

□

3. REVERSE HOLDER AND HARNACK INEQUALITIES

In this section, we state a soft version of reverse Holder inequality (Lemma 3.1) and a Harnack inequality (Lemma 3.2) for parabolic inequality during the mean curvature flow. We start with the differential inequality

$$(3.1) \quad \left(\frac{\partial}{\partial t} - \Delta\right)v \leq fv, \quad v \geq 0$$

where the function f has bounded $L^q(M \times [0, T])$ -norm with $q > \frac{n+2}{2}$. Let $\eta(t, x)$ be a smooth function with the property that $\eta(0, x) = 0$ for all x .

Lemma 3.1. *Let*

$$(3.2) \quad C_0 \equiv C_0(q) = \|f\|_{L^q(M \times [0, T])}, \quad C_1 = \left(1 + \|H\|_{L^{n+3, \frac{2(n+3)}{3}}(M \times [0, T])}\right)^{\frac{n}{n+2}},$$

$\beta > 1$ be a fixed number and $q > \frac{n+2}{2}$. Then there exists a positive constant $C_a = C_a(n, q, C_0, C_1)$ such that

$$(3.3) \quad \|\eta^2 v^\beta\|_{L^{(n+2)/n}(M \times [0, T])} \leq C_a \Lambda(\beta)^{1+\nu} \|v^\beta \left(\eta^2 + |\nabla \eta|^2 + 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta\right)\eta \right| \right)\|_{L^1(M \times [0, T])},$$

where

$$(3.4) \quad \nu = \frac{n+2}{2q - (n+2)},$$

and $\Lambda(\beta)$ is a positive constant depending on β such that $\Lambda(\beta) \geq 1$ if $\beta \geq 2$ (e.g. we can choose $\Lambda(\beta) = 100\beta$).

In fact, we can choose

$$(3.5) \quad C_a(n, q, C_0, C_1) = (2c_n C_0 C_1)^{1+\nu}.$$

This lemma can be proved similarly as in the proof of Lemma 4.1 in [5], using the Sobolev type inequality for the mean curvature flow established in Proposition 2.1.

Next, we show that an L^∞ -norm of v over a smaller set can be bounded by an L^β -norm of v on a bigger set, where $\beta \geq 2$. Fix $x_0 \in \mathbb{R}^{n+1}$. Consider the following sets in space and time,

$$D = \cup_{0 \leq t \leq 1} (B(x_0, 1) \cap M_t); \quad D' = \cup_{\frac{1}{12} \leq t \leq 1} (B(x_0, \frac{1}{2}) \cap M_t).$$

Then, we have the following Harnack inequality.

Lemma 3.2. *Consider the equation (3.1) with $T = 1$. Let us denote by $\lambda = \frac{n+2}{n}$, let $q > \frac{n+2}{2}$ and $\beta \geq 2$. Then, there exists a constant $C_b = C_b(n, q, \beta, C_0, C_1)$ such that*

$$(3.6) \quad \|v\|_{L^\infty(D')} \leq C_b(n, q, \beta, C_0, C_1) \|v\|_{L^\beta(D)}.$$

In the above inequalities, C_0 and C_1 are defined by (3.2).
In fact, we can choose

$$(3.7) \quad C_b(n, q, \beta, C_0, C_1) = (4\lambda^{1+\nu} C_z \beta^{1+\nu})^{\frac{n^2}{\beta}},$$

where

$$(3.8) \quad C_z(n, q, C_0, C_1) := 4^2 \times 100^{1+\nu} c_n C_a(n, q, C_0, C_1).$$

The proof of this lemma, using Lemma 3.1 and Moser iteration, is similar to that of Lemma 5. 2 in [5].

4. BOUNDING THE SECOND FUNDAMENTAL FORM

In this section, we prove Proposition 1.1. First, we establish the following rescaled version of Proposition 1.1.

Proposition 4.1. *There is a universal constant c_0 depending only on n such that if*

$$(4.1) \quad \int_0^1 \int_{M_t} |A|^{n+3} d\mu dt \leq c_0$$

then

$$(4.2) \quad \sup_{\frac{1}{2} \leq t \leq 1} \sup_{x \in M_t} |A(x, t)| \leq 1.$$

Proof of Proposition 4.1. Using the evolution

$$\left(\frac{\partial}{\partial t} - \Delta\right) |A|^2 = -2|\nabla A|^2 + 2|A|^4$$

derived in [4], we obtain for $v = |A|^2$

$$\left(\frac{\partial}{\partial t} - \Delta\right) v \leq f v$$

where $f = 2v$. Our proposition is now an easy consequence of Lemma 3.2 where $q = \frac{n+3}{2}$ and $\beta = n + 3$. In fact, from (3.4) and (3.5), one has $C_a = c_n(C_0 C_1)^{n+3}$. From (3.8) and (3.7), one has $C_b = c_n C_z^{\frac{n^2}{n+3}} = c_n(C_0 C_1)^{n^2}$. By Holder's inequality

$$(4.3) \quad C_1 \leq (1 + C_0^{n+3})^{\frac{n}{n+2}}.$$

Now, by (3.6), one has

$$\|v\|_{L^\infty(D')} \leq C_b \|v\|_{L^\beta(D)} \leq c_n (C_0 (1 + C_0^{n+3})^{\frac{n}{n+2}})^{n^2} \|v\|_{L^\beta(D)} \leq c_n (c_0 (1 + c_0^{n+3})^{\frac{n}{n+2}}) c_0^{\frac{1}{n+3}} \leq 1$$

if c_0 is small, universal. \square

Proof of Proposition 1.1. We first consider the special case $\lambda = 1$ and $T \geq 1$. There are two cases.

Case 1. This is the case when

$$\int_0^T \int_{M_t} |A|^{n+3} d\mu dt \leq c_0.$$

In this case, we consider a new one-parameter family of immersions \tilde{F} defined by $\tilde{F}(x, t) = F(x, T - 1 + t)$. Then

$$\int_0^1 \int_{\tilde{M}_t} |\tilde{A}|^{n+3} d\mu dt = \int_{T-1}^T \int_{M_t} |A|^{n+3} d\mu dt \leq \int_0^T \int_{M_t} |A|^{n+3} d\mu dt \leq c_0.$$

By Proposition 4.1, one has

$$\sup_{x \in \tilde{M}_1} |\tilde{A}(x, 1)| \leq 1.$$

Hence

$$(4.4) \quad \sup_{x \in M_T} |A(x, T)| \leq 1.$$

Case 2. This is the case when

$$\int_0^T \int_{M_t} |A|^{n+3} d\mu dt \geq c_0.$$

In this case, we consider a new one-parameter family of immersions \tilde{F} defined by $\tilde{F}(x, t) = QF(x, \frac{t}{Q^2})$. We find that

$$\int_0^{Q^2T} \int_{\tilde{M}_t} |\tilde{A}|^{n+3} d\mu dt = \frac{1}{Q} \int_0^T \int_{M_t} |A|^{n+3} d\mu dt = c_0$$

if we choose

$$Q = \frac{1}{c_0} \int_0^T \int_{M_t} |A|^{n+3} d\mu dt \geq 1.$$

Now, we are back in **Case 1** and thus can conclude

$$\sup_{x \in \tilde{M}_{Q^2T}} |\tilde{A}|(x, Q^2T) \leq 1.$$

This gives

$$\sup_{x \in M_T} |A(x, T)| = Q \sup_{x \in \tilde{M}_{Q^2T}} |\tilde{A}|(x, Q^2T) \leq Q = \frac{1}{c_0} \int_0^T \int_{M_t} |A|^{n+3} d\mu dt.$$

Combining the above two cases, we find that for $T \geq 1$, one has

$$(4.5) \quad \sup_{x \in M_T} |A(x, T)| \leq Q = \left(1 + \frac{1}{c_0}\right) \left(1 + \int_0^T \int_{M_t} |A|^{n+3} d\mu dt\right).$$

Finally, we consider the general case $\lambda \in (0, 1]$ and $T \geq \lambda$. As usual, let us consider a new one-parameter family of immersions \tilde{F} defined by $\tilde{F}(x, t) = QF(x, \frac{t}{Q^2})$ where $Q = \frac{1}{T^{\frac{1}{2}}} \leq \frac{1}{\lambda^{\frac{1}{2}}}$. Then $Q^2T \geq 1$. Thus, from the estimate (4.5) in the special case, one has

$$\begin{aligned} \sup_{x \in \tilde{M}_{Q^2T}} |\tilde{A}|(x, Q^2T) &\leq (1 + \frac{1}{c_0})(1 + \int_0^{Q^2T} \int_{M_t} |\tilde{A}|^{n+3} d\mu dt) \\ &= (1 + \frac{1}{c_0})(1 + \frac{1}{Q} \int_0^T \int_{M_t} |A|^{n+3} d\mu dt). \end{aligned}$$

Consequently,

$$\begin{aligned} \sup_{x \in M_T} |A|(x, T) &= Q \sup_{x \in \tilde{M}_{Q^2T}} |\tilde{A}|(x, Q^2T) \leq Q(1 + \frac{1}{c_0})(1 + \frac{1}{Q} \int_0^T \int_{M_t} |A|^{n+3} d\mu dt) \\ &\leq \frac{1}{\lambda^{\frac{1}{2}}}(1 + \frac{1}{c_0})(1 + \int_0^T \int_{M_t} |A|^{n+3} d\mu dt). \end{aligned}$$

□

Remark 4.1. We can choose the constant c_λ in Proposition 1.1 as follows: $c_\lambda = \frac{1}{\lambda^{\frac{1}{2}}}(1 + \frac{1}{c_0})$.

5. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.3. Fix $\tau_1 < T$ such that $0 < \tau_1 < 1$. Then, by Proposition 1.1, for any $t \geq \tau_1$, there is a universal constant c depending only on τ_1 , such that

$$(5.1) \quad \sup_{x \in M_t} |A(x, t)| \leq c(1 + \int_0^t \int_{M_s} |A|^{n+3} d\mu ds).$$

Let $f(t) = \sup_{x \in M_t} |A(x, t)|$, $\Psi(s) = s \log(2 + s)$ and

$$G(s) = \int_{M_s} \frac{|A|^{n+2}}{\log(2 + |A|)} d\mu.$$

Then Ψ is an increasing function. Note that (5.1) gives

$$\begin{aligned} f(t) &\leq c(1 + \int_0^t \int_{M_s} \Psi(|A|) \frac{|A|^{n+2}}{\log(2 + |A|)} d\mu ds) \\ &\leq c(1 + \int_0^t \Psi(\sup_{x \in M_s} |A(x, s)|) \int_{M_s} \frac{|A|^{n+2}}{\log(2 + |A|)} d\mu ds) = c(1 + \int_0^t \Psi(f(s))G(s)ds). \end{aligned}$$

Let

$$h(t) = c(1 + \int_0^t \Psi(f(s))G(s)ds).$$

Then for $t \geq \tau_1$

$$f(t) \leq h(t)$$

and

$$h'(t) = c\Psi(f(t))G(t) \leq c\Psi(h(t))G(t).$$

Let $\tilde{\Psi}(y) = \int_c^y \frac{1}{\Psi(s)} ds$. Then for $t \geq \tau_1$

$$\tilde{\Psi}(h(t)) - \tilde{\Psi}(h(\tau_1)) \leq c \int_{\tau_1}^t G(s) ds \leq c \int_0^T G(s) ds < \infty.$$

Hence, since $h(\tau_1)$ is finite

$$\sup_{\tau_1 \leq t < T} \tilde{\Psi}(h(t)) \leq \tilde{\Psi}(h(\tau_1)) + c \int_0^T G(s) ds < \infty.$$

Since $\int_c^\infty \frac{1}{\Psi(s)} ds = \infty$, we deduce that $\sup_{\tau_1 \leq t < T} h(t) < \infty$. Hence $\sup_{\tau_1 \leq t < T} f(t) < \infty$. Therefore, the flow can be extended past T . \square

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