

ON THE EXTENSION OF THE MEAN CURVATURE FLOW

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ABSTRACT. Consider a family of smooth immersions $F(\cdot, t) : M^n \rightarrow \mathbb{R}^{n+1}$ of closed hypersurfaces in \mathbb{R}^{n+1} moving by the mean curvature flow $\frac{\partial F(p, t)}{\partial t} = -H(p, t) \cdot \nu(p, t)$, for $t \in [0, T)$. In [3] Cooper has recently proved that the mean curvature blows up at the singular time T . We show that if the second fundamental form stays bounded from below all the way to T , then the scaling invariant mean curvature integral bound is enough to extend the flow past time T , and this integral bound is optimal in some sense explained below.

1. INTRODUCTION

Let M^n be a compact n -dimensional hypersurface without boundary, and let $F_0 : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth immersion of M^n into \mathbb{R}^{n+1} . Consider a smooth one-parameter family of immersions

$$F(\cdot, t) : M^n \rightarrow \mathbb{R}^{n+1}$$

satisfying

$$F(\cdot, 0) = F_0(\cdot)$$

and

$$(1.1) \quad \frac{\partial F(p, t)}{\partial t} = -H(p, t)\nu(p, t) \quad \forall (p, t) \in M \times [0, T).$$

Here $H(p, t)$ and $\nu(p, t)$ denote the mean curvature and a choice of unit normal for the hypersurface $M_t = F(M^n, t)$ at $F(p, t)$. We will sometimes also write $x(p, t) = F(p, t)$ and refer to (1.1) as to the mean curvature flow equation.

Without any special assumptions on M_0 , the mean curvature flow (1.1) will in general develop singularities in finite time, characterized by a blow up of the second fundamental form $A(\cdot, t)$.

Theorem 1.1 (Huisken [6]). *Suppose $T < \infty$ is the first singularity time for a compact mean curvature flow. Then $\sup_{M_t} |A|(\cdot, t) \rightarrow \infty$ as $t \rightarrow T$.*

By the work of Huisken and Sinestrari [7] the blow up of H near a singularity is known for mean convex hypersurfaces. They also established lower bounds on the principal curvatures in this mean-convex setting. In [3], by a blowup argument, Cooper shows that the mean curvature being uniformly bounded up to $T < \infty$ is enough to extend the flow (1.1) past time T . All those results motivate a natural question: what are the optimal conditions that will guarantee the existence of a smooth solution to the mean curvature

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flow (1.1)?

We will use the following notation throughout the whole paper,

$$\|v\|_{L^p(M \times [0, T])} := \left(\int_0^T \int_{M_t} |v|^p d\mu dt \right)^{\frac{1}{p}},$$

for a function $v(\cdot, t)$ defined on $M \times [0, T)$.

In this paper, we prove the following

Theorem 1.2. *Assume that for the mean curvature flow (1.1), we have*

(i) *A lower bound on the second fundamental form*

$$(1.2) \quad h_{ij} \geq -B g_{ij}$$

where B is a nonnegative number.

(ii) *An integral bound on the mean curvature*

$$(1.3) \quad \|H\|_{L^\alpha(M \times [0, T])} < \infty$$

for some $\alpha \geq n + 2$.

Then the flow can be extended past time T .

In section 2 we will show the integral bound assumption (1.3) is optimal in certain sense.

In [11] Wang, extending a result of the second author [9], proved the analogous result for the Ricci flow, namely that if the Ricci curvature is bounded from below, a uniform integral scalar curvature bound is enough to extend the Ricci flow past some finite time.

As can be seen in the proof of Theorem 1.2 in Section 8, the actual conditions we need in lieu of (1.2) are the following

(iii) *A lower bound for the mean curvature*

$$H \geq -l \text{ for some } l > 0$$

and

(iv) *An upper bound for the squared second fundamental form in terms of a linear function of the squared mean curvature*

$$|A|^2 \leq C_* H^2 + b \text{ for some } C_*, b > 0.$$

These conditions can be verified in many situations, e.g, for mean convex initial hypersurfaces M^n (see Huisken and Sinestrari [7]) or more generally, all starshaped hypersurfaces and manifolds that can be obtained by buiding in small, concave dents into mean convex hypersurfaces (see Smoczyk [10]).

As a corollary we obtain the following result.

Corollary 1.1. *Let M^n be a mean convex or a starshaped hypersurface in \mathbb{R}^{n+1} . Assume we have*

$$\|H\|_{L^\alpha(M \times [0, T])} < \infty,$$

for some $\alpha \geq n + 2$, along the flow (1.1). The flow can be extended past time T .

The proof of Theorem 1.2 is based on a blow-up argument, and the Moser iteration using the Michael-Simon inequality. By their inequality there is a uniform constant c_n , depending only on n , such that for any nonnegative, C^1 function f on a hypersurface $M \subset \mathbb{R}^{n+1}$, the following holds

$$(1.4) \quad \left(\int_M f^{\frac{n}{n-1}} d\mu \right)^{\frac{n-1}{n}} \leq c_n \int_M (|\nabla f| + |H|f) d\mu.$$

The paper is organized as follows. In Section 2, we introduce basic notations concerning evolving hypersurfaces and provide an example showing that the integral bound (1.3) in Theorem 1.2 is optimal to some extent. In Section 3, we establish a modified Michael-Simon inequality and Sobolev type inequalities for the mean curvature flow that can be of independent interest. Section 4 is devoted to a reverse Holder inequality for a subsolution to a parabolic equation changing during mean curvature flow. It turns out to be the key estimate for the Moser iteration process carried out in Section 5 (for the supercritical case) and Section 6 (for the critical case with a smallness condition). Then we bound uniformly the mean curvature in terms of its integral bounds in Section 7. In the final Section 8, we give the proof of the Main Theorem using a blow up argument.

2. PRELIMINARIES

For any compact n -dimensional hypersurface M^n which is smoothly embedded in \mathbb{R}^{n+1} by $F : M^n \rightarrow \mathbb{R}^{n+1}$, let us denote by $g = (g_{ij})$ the induced metric, $A = (h_{ij})$ the second fundamental form, $d\mu = \sqrt{\det(g_{ij})} dx$ the volume form, ∇ the induced Levi-Civita connection and Δ the induced Laplacian. Then the mean curvature of M^n is given by

$$H = g^{ij} h_{ij}.$$

In [6] it has been computed that

$$\frac{\partial}{\partial t} d\mu = -H^2 d\mu,$$

$$\frac{\partial}{\partial t} H = \Delta H + |A|^2 H.$$

To some extent, the constant $\alpha = n+2$ appearing in Theorem 1.2 is optimal as illustrated by the following example.

Example 2.1. *Let M be the standard sphere S^n which is immersed into \mathbb{R}^{n+1} by F_0 . Then the mean curvature flow with initial data M has a simple formula: $F(\cdot, t) = r(t)F_0(\cdot)$ where $r(t) = \sqrt{1 - 2nt}$. Therefore $T = \frac{1}{2n}$ is the extinction time of this mean curvature flow. We have*

$$r(t) = \sqrt{2n(T - t)}, \quad H(t) = \frac{n}{r(t)}.$$

Let us denote by w_n the area of S^n . Compute,

$$\begin{aligned} \|H\|_{L^\alpha(M \times [0, T])} &= \left(\int_0^T \frac{n^\alpha}{(r(t))^\alpha} [r(t)]^n w_n dt \right)^{\frac{1}{\alpha}} \\ &= \frac{nw_n^{\frac{1}{\alpha}}}{(2n)^{\frac{\alpha-n}{2\alpha}}} \left(\int_0^T \frac{dt}{(T-t)^{\frac{\alpha-n}{2}}} \right)^{\frac{1}{\alpha}} = \begin{cases} < \infty & \text{if } \alpha < n+2 \\ = \infty & \text{if } \alpha \geq n+2. \end{cases} \end{aligned}$$

Thus, the constant α in (1.3) cannot be smaller than $n+2$.

Remark 2.1. When $\alpha = n+2$, the quantity $\|H\|_{L^\alpha(M \times [0, T])}$ is invariant under the following rescaling of the mean curvature flow (1.1):

$$\tilde{F}(p, t) = QF(p, \frac{t}{Q^2}), \text{ for } Q > 0.$$

Remark 2.2. The characterization of the maximal time of existence of a solution to an evolution equation by the blow up of its scaling invariant quantities seems to be a ubiquitous phenomenon. Let us mention a couple of examples among many. In fluid dynamics, we have the celebrated Beale-Kato-Majda Theorem [1] which says that if the maximal time of existence of solutions to the incompressible Euler or Navier-Stokes equation is finite then necessarily the $L^1_{time} L^\infty_{\mathbb{R}^3}$ norm of the vorticity blows up. In the cases of the Ricci and the mean curvature flow, in addition to controlling the scaling invariant quantities, we also need lower bounds on the Ricci curvature and the second fundamental form, respectively. For the incompressible Euler or Navier-Stokes equations, the divergence-free property of the velocity vector field plays a crucial role and is in some sense analogous to those lower bounds on the Ricci curvature and the second fundamental form.

3. SOBOLEV INEQUALITIES FOR THE MEAN CURVATURE FLOW

In this section, we establish a version of Michael-Simon inequality, Lemma 3.1, that allows us to derive a Sobolev type inequality, Proposition 3.1, for the mean curvature flow. This Sobolev inequality will be crucial for the Moser iteration in the next sections. The key step in the Moser iteration is the inequality (4.14).

The following lemma consists of a slightly modified Michael-Simon inequality whose proof is based on the original Michael-Simon inequality (1.4) together with the interpolation inequalities.

Lemma 3.1. *Let M be a compact n -dimensional hypersurface without boundary, which is smoothly embedded in \mathbb{R}^{n+1} . Let*

$$(3.1) \quad Q = \begin{cases} \frac{n}{n-2} & \text{if } n > 2 \\ < \infty & \text{if } n = 2 \end{cases}$$

Then, for all Lipschitz functions v on M , we have

$$\|v\|_{L^{2Q}(M)}^2 \leq c_n \left(\|\nabla v\|_{L^2(M)}^2 + \|H\|_{L^{n+2}(M)} \|v\|_{L^2(M)}^2 \right)$$

where H is the mean curvature of M and c_n is a positive constant depending only on n .

Proof. We only need to prove the lemma for $v \geq 0$. Applying Michael-Simon's inequality (1.4)[8] to the function $w = v^{\frac{2(n-1)}{n-2}}$, we get

$$\left(\int_M v^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-1}{n}} \leq c_n \left(\int_M |\nabla v| v^{\frac{n}{n-2}} d\mu + \int_M |H| v^{\frac{2(n-1)}{n-2}} d\mu \right).$$

By Holder's inequality it follows that

$$\begin{aligned} \left(\int_M v^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} &\leq c_n^{\frac{n-2}{n-1}} \left(\int_M |\nabla v| v^{\frac{n}{n-2}} d\mu + \int_M |H| v^{\frac{2(n-1)}{n-2}} d\mu \right)^{\frac{n-2}{n-1}} \\ &\leq c_n \left(\|\nabla v\|_{L^2(M)} \|v\|_{L^{2Q}(M)}^{\frac{n}{n-2}} + \|H\|_{L^{n+2}(M)} \|v\|_{L^{2m}(M)}^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{n-1}} \\ &\leq c_n \left(\|\nabla v\|_{L^2(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2Q}(M)}^{\frac{n-1}{n-1}} + \|H\|_{L^{n+2}(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2m}(M)}^2 \right). \end{aligned}$$

where

$$m = \frac{(n-1)(n+2)}{(n-2)(n+1)}.$$

Thus

$$(3.2) \quad \|v\|_{L^{2Q}(M)}^2 \leq c_n \left(\|\nabla v\|_{L^2(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2Q}(M)}^{\frac{n}{n-1}} + \|H\|_{L^{n+2}(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2m}(M)}^2 \right).$$

By Young's inequality

$$(3.3) \quad ab = (\varepsilon^{1/p} a)(\varepsilon^{-1/p} b) \leq \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{-q/p} b^q}{q} \leq \varepsilon a^p + \varepsilon^{-q/p} b^q,$$

where $a, b, \varepsilon > 0$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If we apply it to (3.2), with

$$a = \|v\|_{L^{2Q}(M)}^{\frac{n}{n-1}}, \quad b = \|\nabla v\|_{L^2(M)}^{\frac{n-2}{n-1}},$$

and

$$\varepsilon = \frac{1}{2c_n}, \quad p = \frac{2(n-1)}{n}, \quad q = \frac{2(n-1)}{n-2},$$

we obtain

$$\|v\|_{L^{2Q}(M)}^2 \leq c_n \left(\frac{1}{2c_n} \|v\|_{L^{2Q}(M)}^2 + \left(\frac{1}{2c_n} \right)^{\frac{-n}{n-2}} \|\nabla v\|_{L^2(M)}^2 + \|H\|_{L^{n+2}(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2m}(M)}^2 \right).$$

Hence

$$(3.4) \quad \|v\|_{L^{2Q}(M)}^2 \leq c_n \left(\|\nabla v\|_{L^2(M)}^2 + \|H\|_{L^{n+2}(M)}^{\frac{n-2}{n-1}} \|v\|_{L^{2m}(M)}^2 \right).$$

Next, we will use the following interpolation inequality (see inequality (7.10) in [5])

$$(3.5) \quad \|u\|_{L^r} \leq \varepsilon \|u\|_{L^s} + \varepsilon^{-\mu} \|u\|_{L^t}$$

where $t < r < s$ and

$$\mu = \left(\frac{1}{t} - \frac{1}{r}\right) / \left(\frac{1}{r} - \frac{1}{s}\right).$$

Note that, in our case

$$1 < m < Q,$$

and therefore, by (3.5)

$$(3.6) \quad \|v\|_{L^{2m}(M)} \leq \varepsilon \|v\|_{L^{2Q}(M)} + \varepsilon^{-\alpha} \|v\|_{L^2(M)}$$

where $\varepsilon > 0$ and

$$(3.7) \quad \alpha = \frac{Q(m-1)}{Q-m} = \frac{n^2}{n-2}.$$

Plugging (3.6) into the right hand side of (3.4), we deduce that

$$(3.8) \quad \begin{aligned} \|v\|_{L^{2Q}(M)}^2 &\leq c_n \|\nabla v\|_{L^2(M)}^2 + c_n \|H\|_{L^{n+2}(M)}^{\frac{n-2}{n-1}} \left(\varepsilon \|v\|_{L^{2Q}(M)} + \varepsilon^{-\alpha} \|v\|_{L^2(M)} \right)^2 \\ &\leq c_n \|\nabla v\|_{L^2(M)}^2 + c_n \|H\|_{L^{n+2}(M)}^{\frac{n-2}{n-1}} \left(\varepsilon^2 \|v\|_{L^{2Q}(M)}^2 + \varepsilon^{-2\alpha} \|v\|_{L^2(M)}^2 \right). \end{aligned}$$

Now, we can absorb the term involving $\|v\|_{L^{2Q}(M)}^2$ into the left hand side of (3.8) by choosing

$$\varepsilon^2 = \frac{1}{2c_n} \|H\|_{L^{n+2}(M)}^{-\frac{n-2}{n-1}}.$$

Since $\frac{n-2}{n-1}(1+\alpha) = n+2$, we obtain the desired inequality

$$\|v\|_{L^{2Q}(M)}^2 \leq c_n \|\nabla v\|_{L^2(M)}^2 + c_n \|H\|_{L^{n+2}(M)}^{n+2} \|v\|_{L^2(M)}^2.$$

□

Our Sobolev type inequality for the mean curvature flow is stated in the following proposition.

Proposition 3.1. *For all nonnegative Lipschitz functions v , one has*

$$(3.9) \quad \|v\|_{L^\beta(M \times [0, T])}^\beta \leq c_n \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^{4/n} \left(\|\nabla v\|_{L^2(M \times [0, T])}^2 + \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^2 \|H\|_{L^{n+2}(M \times [0, T])}^{n+2} \right),$$

where $\beta := \frac{2(n+2)}{n}$.

Proof. By Holder's inequality, we have

$$\begin{aligned} \int_0^T \int_{M_t} v^{\frac{2(n+2)}{n}} d\mu dt &= \int_0^T dt \int_{M_t} v^2 v^{4/n} d\mu \\ &\leq \int_0^T dt \left(\int_{M_t} v^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} \left(\int_{M_t} v^2 d\mu \right)^{\frac{2}{n}} \\ &\leq \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^{4/n} \int_0^T \|v(\cdot, t)\|_{L^{2Q}(M_t)}^2 dt. \end{aligned}$$

Now, applying Lemma 3.1, we get

$$\begin{aligned} & \|v\|_{L^\beta(M \times [0, T])}^\beta \\ & \leq c_n \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^{4/n} \left(\int_0^T \int_{M_t} |\nabla v|^2 d\mu dt + \int_0^T \left(\int_{M_t} |H|^{n+2} d\mu \right) \|v(\cdot, t)\|_{L^2(M_t)}^2 dt \right) \\ & \leq c_n \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^{4/n} \left(\|\nabla v\|_{L^2(M \times [0, T])}^2 + \max_{0 \leq t \leq T} \|v\|_{L^2(M_t)}^2 \|H\|_{L^{n+2}(M \times [0, T])}^{n+2} \right). \end{aligned}$$

□

4. A REVERSE HOLDER INEQUALITY

In this section, we establish a soft version of reverse Holder inequality for parabolic inequality during the mean curvature flow. Because of the blow up argument that we will use in the end, we should keep track of our constants in deriving the estimates, since we do not want them to blow up after taking the limit of the rescaled flow. Moreover, since we also need certain smallness conditions to carry out the Moser iteration, we do not want constants quantifying these smallness conditions to vanish after taking the limit of the rescaled flow. Here is the convention that we will use.

Constants such as C_0, C_1, C_2, \dots will be defined. The constants with alphabetical subscripts C_a, C_b, \dots depend on other constants with numerical subscripts C_0, C_1, \dots in a controlled way, the former are increasing functions of the latter. The δ -constants with numerical subscripts, such as $\delta_1, \delta_2, \dots$ depend on the constants with numerical subscripts C_0, C_1, \dots , the former are decreasing functions of the latter. We will use those facts in the blow up argument in section 8.

We start with the differential inequality

$$(4.1) \quad \left(\frac{\partial}{\partial t} - \Delta \right) v \leq f v, \quad v \geq 0$$

where the function f has bounded $L^q(M \times [0, T])$ -norm with $q \geq \frac{n+2}{2}$. Let $\eta(t, x)$ be a smooth function with the property that $\eta(0, x) = 0$ for all x .

Lemma 4.1. *Let*

$$(4.2) \quad C_0 \equiv C_0(q) = \|f\|_{L^q(M \times [0, T])}, \quad C_1 = (1 + \|H\|_{L^{n+2}(M \times [0, T])}^{n+2})^{\frac{n}{n+2}},$$

$\beta > 1$ be a fixed number and $q > \frac{n+2}{2}$. Then there exists a positive constant $C_a = C_a(n, q, C_0, C_1)$ such that

$$(4.3) \quad \|\eta^2 v^\beta\|_{L^{(n+2)/n}(M \times [0, T])} \leq C_a \Lambda(\beta)^{1+\nu} \|v^\beta \left(\eta^2 + |\nabla \eta|^2 + 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| \right)\|_{L^1(M \times [0, T])},$$

where

$$(4.4) \quad \nu = \frac{n+2}{2q - (n+2)},$$

and $\Lambda(\beta)$ is a positive constant depending on β such that $\Lambda(\beta) \geq 1$ if $\beta \geq 2$ (e.g. we can choose $\Lambda(\beta) = 100\beta$).

Remark 4.1. As will be seen later, we can choose

$$(4.5) \quad C_a(n, q, C_0, C_1) = (2c_n C_0 C_1)^{1+\nu}.$$

Proof. We use $\eta^2 v^{\beta-1}$ as a test function in the inequality

$$-\Delta v + \frac{\partial v}{\partial t} \leq f v.$$

It follows that, for any $s \in (0, T]$, we have

$$(4.6) \quad \int_0^s \int_{M_t} (-\Delta v) \eta^2 v^{\beta-1} d\mu dt + \int_0^s \int_{M_t} \frac{\partial v}{\partial t} \eta^2 v^{\beta-1} d\mu dt \leq \int_0^s \int_{M_t} |f| \eta^2 v^\beta d\mu dt.$$

Note that, by integrating by parts

$$(4.7) \quad \int_{M_t} (-\Delta v) \eta^2 v^{\beta-1} d\mu = \int_{M_t} 2 \langle \nabla v, \nabla \eta \rangle \eta v^{\beta-1} d\mu + (\beta - 1) \int_{M_t} \eta^2 v^{\beta-2} |\nabla v|^2 d\mu.$$

Using the evolution of the volume form

$$\partial_t d\mu = -H^2 d\mu$$

and recalling the properties of η , we get

$$(4.8) \quad \begin{aligned} \int_0^s \int_{M_t} \frac{\partial v}{\partial t} \eta^2 v^{\beta-1} d\mu dt &= \frac{1}{\beta} \int_0^s \int_{M_t} \frac{\partial(v^\beta)}{\partial t} \eta^2 d\mu dt \\ &= \frac{1}{\beta} \int_{M_t} v^\beta \eta^2 d\mu \Big|_0^s - \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta \partial_t (\eta^2 d\mu) dt \\ &= \frac{1}{\beta} \int_{M_s} v^\beta \eta^2 d\mu - \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta \left[2\eta \frac{\partial \eta}{\partial t} - H^2 \right] d\mu dt. \end{aligned}$$

Therefore, we deduce from (4.6)-(4.8) the following inequality

$$(4.9) \quad \begin{aligned} &\int_0^s \int_{M_t} (2 \langle \nabla v, \nabla \eta \rangle \eta v^{\beta-1} + (\beta - 1) \eta^2 v^{\beta-2} |\nabla v|^2) d\mu dt + \frac{1}{\beta} \int_{M_s} v^\beta \eta^2 d\mu \\ &\leq \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta 2\eta \frac{\partial \eta}{\partial t} d\mu dt + \int_0^s \int_{M_t} |f| \eta^2 v^\beta d\mu dt. \end{aligned}$$

As will be seen later, because we can get good control of the quantity $(\frac{\partial}{\partial t} - \Delta)\eta$ for suitable choices of η , it is more convenient to make this term appear on the right hand side of (4.9).

Observe that, integrating by parts yields

$$\begin{aligned}
\frac{1}{\beta} \int_0^s \int_{M_t} v^\beta 2\eta \frac{\partial \eta}{\partial t} d\mu dt &= \frac{1}{\beta} \int_0^s \int_{M_t} \left(v^\beta 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta + v^\beta 2\eta \Delta \eta \right) d\mu dt \\
&= \frac{1}{\beta} \int_0^s \int_{M_t} \left(v^\beta 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta - 2\nabla(v^\beta \eta) \nabla \eta \right) d\mu dt \\
&= \frac{1}{\beta} \int_0^s \int_{M_t} \left(v^\beta 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta - 2v^\beta |\nabla \eta|^2 - 2\beta \langle \nabla v, \nabla \eta \rangle \eta v^{\beta-1} \right) d\mu dt \\
&\leq \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta d\mu dt - \int_0^s \int_{M_t} 2\eta \langle \nabla v, \nabla \eta \rangle v^{\beta-1} d\mu dt.
\end{aligned}$$

Then (4.9) implies

$$\begin{aligned}
(4.10) \quad \int_0^s \int_{M_t} (4 \langle \nabla v, \nabla \eta \rangle \eta v^{\beta-1} + (\beta-1) \eta^2 v^{\beta-2} |\nabla v|^2) d\mu dt + \frac{1}{\beta} \int_{M_s} v^\beta \eta^2 d\mu \\
\leq \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| d\mu dt + \int_0^s \int_{M_t} |f| \eta^2 v^\beta d\mu dt.
\end{aligned}$$

Using the Cauchy-Schwartz inequality

$$\int_0^s \int_{M_t} 4 \langle \nabla v, \nabla \eta \rangle \eta v^{\beta-1} d\mu dt \geq -2\varepsilon^2 \int_0^s \int_{M_t} \eta^2 v^{\beta-2} |\nabla v|^2 d\mu dt - \frac{2}{\varepsilon^2} \int_0^s \int_{M_t} v^\beta |\nabla \eta|^2 d\mu dt$$

we get from (4.10),

$$\begin{aligned}
(4.11) \quad \int_0^s \int_{M_t} (\beta-1-2\varepsilon^2) \eta^2 v^{\beta-2} |\nabla v|^2 d\mu dt + \frac{1}{\beta} \int_{M_s} v^\beta \eta^2 d\mu \\
\leq \frac{1}{\beta} \int_0^s \int_{M_t} v^\beta 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| d\mu dt + \int_0^s \int_{M_t} |f| \eta^2 v^\beta d\mu dt + \frac{2}{\varepsilon^2} \int_0^s \int_{M_t} v^\beta |\nabla \eta|^2 d\mu dt.
\end{aligned}$$

Choosing $\varepsilon^2 = \frac{\beta-1}{4}$ and observing that $|\nabla(v^{\beta/2})|^2 = \frac{\beta^2}{4} v^{\beta-2} |\nabla v|^2$ yield

$$\begin{aligned}
2\left(1 - \frac{1}{\beta}\right) \int_0^s \int_{M_t} \eta^2 |\nabla(v^{\beta/2})|^2 d\mu dt + \int_{M_s} v^\beta \eta^2 d\mu \\
\leq \int_0^s \int_{M_t} v^\beta 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| d\mu dt + \beta \int_0^s \int_{M_t} |f| \eta^2 v^\beta d\mu dt + \frac{8\beta}{\beta-1} \int_0^s \int_{M_t} v^\beta |\nabla \eta|^2 d\mu dt.
\end{aligned}$$

Combining the previous estimate with

$$|\nabla(\eta v^{\beta/2})|^2 \leq 2\eta^2 |\nabla(v^{\beta/2})|^2 + 2v^\beta |\nabla \eta|^2$$

implies

$$\begin{aligned} & (1 - \frac{1}{\beta}) \int_0^s \int_{M_t} |\nabla(\eta v^{\beta/2})|^2 d\mu dt + \int_{M_s} v^\beta \eta^2 d\mu \\ & \leq \int_0^s \int_{M_t} v^\beta 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| d\mu dt + \beta \int_0^s \int_{M_t} |f| \eta^2 v^\beta d\mu dt + 8 \left(\frac{\beta}{\beta-1} + \frac{\beta-1}{\beta} \right) \int_0^s \int_{M_t} v^\beta |\nabla \eta|^2 d\mu dt. \end{aligned}$$

It follows that, for some $\Lambda(\beta) \geq 1$ (say $\Lambda(\beta) = 100\beta$ if $\beta \geq 2$),

$$\begin{aligned} & \int_0^s \int_{M_t} |\nabla(\eta v^{\beta/2})|^2 d\mu dt + \int_{M_s} v^\beta \eta^2 d\mu \\ & \leq \Lambda(\beta) \left(\int_0^s \int_{M_t} v^\beta \left\{ 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| + |\nabla \eta|^2 \right\} d\mu dt + \int_0^s \int_{M_t} |f| \eta^2 v^\beta d\mu dt \right) \\ & \leq \Lambda(\beta) \left(\int_0^s \int_{M_t} v^\beta \left\{ 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| + |\nabla \eta|^2 \right\} d\mu dt + \|f\|_{L^q(M \times [0, T])} \|\eta^2 v^\beta\|_{L^{\frac{q}{q-1}}(M \times [0, T])} \right) \\ & = \Lambda(\beta) \left(\int_0^s \int_{M_t} v^\beta \left\{ 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| + |\nabla \eta|^2 \right\} d\mu dt + C_0 \|\eta^2 v^\beta\|_{L^{\frac{q}{q-1}}(M \times [0, T])} \right) =: A. \end{aligned}$$

Consequently,

$$(4.12) \quad \max_{0 \leq s \leq T} \int_{M_s} \eta^2 v^\beta d\mu \leq A$$

and

$$(4.13) \quad \int_0^T \int_{M_t} |\nabla(\eta v^{\beta/2})|^2 d\mu dt \leq A.$$

We are now in a position to apply Proposition 3.1 to $\eta v^{\beta/2}$ and get the following estimates

$$\begin{aligned} & \|\eta^2 v^\beta\|_{L^{\frac{(n+2)/n}{(n+2)/n}}(M \times [0, T])}^{(n+2)/n} = \|\eta v^{\beta/2}\|_{L^{2(n+2)/n}(M \times [0, T])}^{2(n+2)/n} \\ & \leq c_n \max_{0 \leq t \leq T} \|\eta v^{\beta/2}\|_{L^2(M_t)}^{4/n} \left(\|\nabla(\eta v^{\beta/2})\|_{L^2(M \times [0, T])}^2 + \max_{0 \leq t \leq T} \|\eta v^{\beta/2}\|_{L^2(M_t)}^2 \|H\|_{L^{n+2}(M \times [0, T])}^{n+2} \right) \\ & \leq c_n A^{2/n} \left(A + A \|H\|_{L^{n+2}(M \times [0, T])}^{n+2} \right) = c_n A^{\frac{n+2}{n}} (1 + \|H\|_{L^{n+2}(M \times [0, T])}^{n+2}). \end{aligned}$$

Let $S := M \times [0, T)$ and let the norm $\|\cdot\|_{L^p(M \times [0, T])}$ be shortly denoted by $\|\cdot\|_{L^p(S)}$. Then the previous estimate, using a definition of A , can be rewritten as

$$\begin{aligned} & \|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} \leq c_n A (1 + \|H\|_{L^{n+2}(S)}^{n+2})^{\frac{n}{n+2}} \\ (4.14) \quad & = c_n C_1 \Lambda(\beta) \left(\int_0^T \int_{M_t} v^\beta \left\{ 2\eta \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right| + |\nabla \eta|^2 \right\} d\mu dt + C_0 \|\eta^2 v^\beta\|_{L^{\frac{q}{q-1}}(S)} \right). \end{aligned}$$

Since $1 < \frac{q}{q-1} < \frac{n+2}{n}$, by using the interpolation inequality

$$\|\eta^2 v^\beta\|_{L^{\frac{q}{q-1}}(S)} \leq \varepsilon \|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} + \varepsilon^{-\nu} \|\eta^2 v^\beta\|_{L^1(S)}$$

in (4.14), for $\nu = \frac{n+2}{2q-(n+2)}$, one gets

$$\begin{aligned} & [1 - c_n \Lambda(\beta) C_0 C_1 \varepsilon] \|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} \\ & \leq c_n C_1 \Lambda(\beta) \left[C_0 \varepsilon^{-\nu} \|\eta^2 v^\beta\|_{L^1(S)} + \left\| v^\beta (|\nabla \eta|^2 + 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta) \right\|_{L^1(S)} \right]. \end{aligned}$$

If we choose $\varepsilon = \frac{1}{2\Lambda(\beta)c_n C_0 C_1}$, then

$$\begin{aligned} & \|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} \\ & \leq 2c_n C_1 \Lambda(\beta) \left[C_0 (2\Lambda(\beta)c_n C_0 C_1)^\nu \|\eta^2 v^\beta\|_{L^1(S)} + \left\| v^\beta (|\nabla \eta|^2 + 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta) \right\|_{L^1(S)} \right] \\ & \leq C_a(n, q, C_0, C_1) \Lambda(\beta)^{1+\nu} \left\| v^\beta (\eta^2 + |\nabla \eta|^2 + 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta) \right\|_{L^1(S)}, \end{aligned}$$

where $C_a(n, q, C_0, C_1) = (2c_n C_0 C_1)^{1+\nu}$. In conclusion, we get a soft reverse Holder inequality

$$\|\eta^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} \leq C_a(n, q, C_0, C_1) \Lambda(\beta)^{1+\nu} \|v^\beta \left(\eta^2 + |\nabla \eta|^2 + 2\eta \left(\frac{\partial}{\partial t} - \Delta \right) \eta \right)\|_{L^1(S)}.$$

□

5. THE MOSER ITERATION PROCESS FOR THE SUPERCRITICAL CASE

We will use the notation from previous sections. Consider the function v , which is a solution to (4.1), where $f \in L^q(S)$. Assume $q > \frac{n+2}{n}$ which corresponds to a supercritical case. We will show in this case that an L^∞ -norm of v over a smaller set can be bounded by an L^β -norm of v on a bigger set, where $\beta \geq 2$. Fix $x_0 \in \mathbb{R}^{n+1}$. Consider the following sets in space and time,

$$D = \cup_{0 \leq t \leq 1} (B(x_0, 1) \cap M_t); \quad D' = \cup_{\frac{1}{12} \leq t \leq 1} (B(x_0, \frac{1}{2}) \cap M_t).$$

Let us denote by

$$D_k = \cup_{t_k \leq t \leq 1} (B(x_0, r_k) \cap M_t)$$

where

$$r_k = \frac{1}{2} + \frac{1}{2^{k+1}}; \quad t_k = \frac{1}{12} \left(1 - \frac{1}{4^k} \right).$$

Then,

$$\rho_k := r_{k-1} - r_k = \frac{1}{2^{k+1}}; \quad t_k - t_{k-1} = \rho_k^2.$$

Let us choose a test function $\eta_k = \eta_k(t, x)$, following Ecker [4], of the form

$$(5.1) \quad \eta_k(t, x) = \varphi_{\rho_k}(t) \times \psi_{\rho_k}(|x - x_0|^2).$$

In (5.1), the function φ_{ρ_k} satisfies

$$\varphi_{\rho_k}(t) = \begin{cases} 1 & \text{if } t_k \leq t \leq 1, \\ \in [0, 1] & \text{if } t_{k-1} \leq t \leq t_k, \\ 0 & \text{if } t \leq t_{k-1}. \end{cases}$$

and

$$\left| \varphi'_{\rho_k} \right| (t) \leq \frac{c_n}{\rho_k^2};$$

whereas in (5.1), the function $\psi_{\rho_k}(s)$ satisfies

$$\psi_{\rho_k}(s) = \begin{cases} 0 & \text{if } s \geq r_{k-1}^2, \\ \in [0, 1] & \text{if } r_k^2 \leq s \leq r_{k-1}^2, \\ 1 & \text{if } s \leq r_k^2. \end{cases}$$

and

$$\left| \psi'_{\rho_k} \right| (s) \leq \frac{c_n}{\rho_k^2}.$$

We have

$$(5.2) \quad 0 \leq \eta_k \leq 1; \eta_k \equiv 1 \text{ in } D_k; \eta_k \equiv 0 \text{ outside } D_{k-1}.$$

Using the following identity for the mean curvature flow derived in Brakke [2]

$$(5.3) \quad \left(\frac{d}{dt} - \Delta \right) |x - x_0|^2 = -2n \quad \forall x \in M_t,$$

we can verify the following

Lemma 5.1.

$$(5.4) \quad \sup_{t \in [0,1]} \sup_{x \in M_t} \left(\eta_k^2(t, x) + |\nabla \eta_k(t, x)|^2 + 2\eta_k(t, x) \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta_k(t, x) \right| \right) \leq \frac{c_n}{\rho_k^2} = c_n 4^{k+1}.$$

The main result of this section is the following Harnack inequality in the supercritical case.

Lemma 5.2. *Consider the equation (4.1) with $T \geq 1$. Let us denote by $\lambda = \frac{n+2}{n}$, let $q > \frac{n+2}{2}$ and $\beta \geq 2$. Then, there exists a constant $C_b = C_b(n, q, \beta, C_0, C_1)$ such that*

$$(5.5) \quad \|v\|_{L^\infty(D')} \leq C_b(n, q, \beta, C_0, C_1) \|v\|_{L^\beta(D)},$$

and

$$(5.6) \quad \|v\|_{L^\infty(D')} \leq C_b(n, q, \beta, C_0, C_1) \|v\|_{L^{\beta\lambda^k}(D_k)} \quad \forall k \geq 1.$$

In the above inequalities, C_0 and C_1 are defined by (4.2).

Proof. If $\beta \geq 2$, then let $\Lambda(\beta) = 100\beta$. Note that $\eta_k \equiv 1$ on D_k and $\eta_k \equiv 0$ outside D_{k-1} by (5.2). Recall that $S := M \times [0, T]$. We have

$$\begin{aligned} \|v^\beta\|_{L^{\frac{n+2}{n}}(D_k)} &\leq \|\eta_k^2 v^\beta\|_{L^{\frac{n+2}{n}}(S)} \\ &\leq C_a(n, q, C_0, C_1) \Lambda(\beta)^{1+\nu} \int_0^T \int_{M_t} v^\beta \left(\eta_k^2 + |\nabla \eta_k|^2 + 2\eta_k \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta_k \right| \right) d\mu dt \\ &\leq c_n 4^{k+1} C_a(n, q, C_0, C_1) \Lambda(\beta)^{1+\nu} \int_{D_{k-1}} v^\beta d\mu dt \\ &= C_z(n, q, C_0, C_1) 4^{k-1} \beta^{1+\nu} \|v^\beta\|_{L^1(D_{k-1})}, \end{aligned}$$

where $C_z(n, q, C_0, C_1) := 4^2 \times 100^{1+\nu} c_n C_a(n, q, C_0, C_1)$. In the above chain of inequalities, the second line follows from Lemma 4.1, the third line results from Lemma 5.1 and the definition of ρ_k . For simplicity, let us denote by $C_z = C_z(n, q, C_0, C_1)$. Then the previous estimate can be written in the following form

$$(5.7) \quad \|v\|_{L^{\frac{n+2}{n}\beta}(D_k)} \leq C_z^{\frac{1}{\beta}} 4^{\frac{k-1}{\beta}} \beta^{\frac{1+\nu}{\beta}} \|v\|_{L^\beta(D_{k-1})}.$$

This form of the reverse Holder inequality is the key estimate for our Moser iteration process. Let $\lambda = \frac{n+2}{n}$. Then, replacing β by $\lambda^{k-1}\beta$ in (5.7), one obtains

$$(5.8) \quad \|v\|_{L^{\beta\lambda^k}(D_k)} \leq C_z^{\frac{1}{\beta\lambda^{k-1}}} 4^{\frac{k-1}{\beta\lambda^{k-1}}} (\lambda^{k-1}\beta)^{\frac{1+\nu}{\beta\lambda^{k-1}}} \|v\|_{L^{\beta\lambda^{k-1}}(D_{k-1})}.$$

It follows by iteration that for all $k_0 \geq 0$

$$(5.9) \quad \begin{aligned} \|v\|_{L^{\beta\lambda^k}(D_k)} &\leq \left(C_z^{\frac{1}{\beta}} \beta^{\frac{1+\nu}{\beta}} \right)^{\sum_{j=k_0}^{k-1} \frac{1}{\lambda^j}} \left(4^{\frac{1}{\beta}} \cdot \lambda^{\frac{1+\nu}{\beta}} \right)^{\sum_{j=k_0}^{k-1} \frac{j}{\lambda^j}} \|v\|_{L^{\beta\lambda^{k_0}}(D_{k_0})} \\ &\leq C_b(n, q, \beta, C_0, C_1) \|v\|_{L^{\beta\lambda^{k_0}}(D_{k_0})}, \end{aligned}$$

where we choose

$$(5.10) \quad C_b(n, q, \beta, C_0, C_1) = (4\lambda^{1+\nu} C_z \beta^{1+\nu})^{\frac{n^2}{\beta}},$$

since

$$\sum_{j=0}^{\infty} \frac{j}{\lambda^j} = \frac{\lambda}{(\lambda-1)^2} = \frac{n(n+2)}{4} \leq n^2.$$

Note that $D_0 = D$ and $D' \subset D_k$ for all positive integer k . Thus

$$(5.11) \quad \|v\|_{L^{\beta\lambda^k}(D')} \leq C_b(n, q, \beta, C_0, C_1) \|v\|_{L^\beta(D_0)} = C_b(n, q, \beta, C_0, C_1) \|v\|_{L^\beta(D)}.$$

Letting $k \rightarrow \infty$, we find that

$$(5.12) \quad \|v\|_{L^\infty(D')} \leq C_b(n, q, \beta, C_0, C_1) \|v\|_{L^\beta(D)}.$$

If we fix a $k \geq 1$, doing the above iteration process for $l \geq k$, letting $l \rightarrow \infty$ we obtain (5.6). \square

6. THE MOSER ITERATION PROCESS FOR THE CRITICAL CASE

In this section we will deal with the differential inequality

$$(6.1) \quad \left(\frac{\partial}{\partial t} - \Delta\right)v \leq fv, \quad v \geq 0,$$

where the function f is bounded in the $L^{\frac{n+2}{2}}(S)$ norm. In this case $\frac{q}{q-1} = \frac{n+2}{n}$. Thus, we cannot absorb the term $c_n C_1 \Lambda(\beta) C_0 \|\eta^2 v^\beta\|_{L^{\frac{q}{q-1}}(S)}$ appearing on the right hand side of estimate (4.14) into the left hand side of the same equation, which was the crucial estimate in obtaining the reverse Hölder inequality (5.7). This inequality is the key estimate for performing the Moser iteration in the supercritical case. However if we assume a smallness condition on $C_0 = C_0(\frac{n+2}{2})$, we have the following lemma.

Lemma 6.1. *Let β be a constant greater than 1. Then there exist two constants $\delta_1(n, \beta, C_1)$ and $C_c(n, \beta, C_1)$ such that if*

$$(6.2) \quad \|f\|_{L^{\frac{n+2}{2}}(D)} \leq \delta_1(n, \beta, C_1)$$

then

$$(6.3) \quad \|v\|_{L^{\frac{n+2}{2}\beta}(D_1)} \leq C_c(n, \beta, C_1) \|v\|_{L^\beta(D)}.$$

Proof. We will use (4.14) with $\eta = \eta_1$, keeping in mind that the constant C_0 appearing on the right hand side of (4.14) can be chosen to be $\|f\|_{L^q(D)}$, where $q = \frac{n+2}{2}$. Since $\eta_1 \equiv 0$ outside D , we see that the term $\int_0^s \int_{M_t} |f| \eta_1^2 v^\beta d\mu dt$ can be bounded from above by

$$\int_D |f| \eta_1^2 v^\beta d\mu dt \leq \|f\|_{L^q(D)} \|\eta_1^2 v^\beta\|_{L^{\frac{q}{q-1}}(D)}.$$

Thus, for $\eta = \eta_1$, the term $C_0 \|\eta_1^2 v^\beta\|_{L^{\frac{q}{q-1}}(M \times [0, T])}$ in A (defined in (4.14)) can be replaced by $\|f\|_{L^q(D)} \|\eta_1^2 v^\beta\|_{L^{\frac{q}{q-1}}(D)}$.

Consequently, as $q = \frac{n+2}{2}$ and $\frac{q}{q-1} = \frac{n+2}{n}$, we have

$$(6.4) \quad \|\eta_1^2 v^\beta\|_{L^{\frac{n+2}{n}}(D)} \leq c_n C_1 \Lambda(\beta) \left(\int_0^T \int_{M_t} v^\beta \left\{ 2\eta_1 \left| \left(\frac{\partial}{\partial t} - \Delta\right)\eta_1 \right| + |\nabla \eta_1|^2 \right\} d\mu dt + \|f\|_{L^{\frac{n+2}{2}}(D)} \|\eta_1^2 v^\beta\|_{L^{\frac{n+2}{n}}(D)} \right).$$

If we choose

$$(6.5) \quad \delta_1(n, \beta, C_1) = \frac{1}{2c_n C_1 \Lambda(\beta)},$$

then from (6.2) and (6.4) we get

$$\|\eta_1^2 v^\beta\|_{L^{\frac{n+2}{n}}(D)} \leq c_n C_1 \Lambda(\beta) \left(\int_0^T \int_{M_t} v^\beta \left\{ 2\eta_1 \left| \left(\frac{\partial}{\partial t} - \Delta\right)\eta_1 \right| + |\nabla \eta_1|^2 \right\} d\mu dt \right).$$

By Lemma 5.1,

$$(6.6) \quad \sup \left(\eta_1^2 + |\nabla \eta_1|^2 + 2\eta_1 \left| \left(\frac{\partial}{\partial t} - \Delta \right) \eta_1 \right| \right) \leq \frac{c_n}{\rho_1^2} = c_n.$$

Hence

$$(6.7) \quad \|\eta_1^2 v^\beta\|_{L^{\frac{n+2}{n}}(D)} \leq c_n C_1 \Lambda(\beta) \|v^\beta\|_{L^1(D)}.$$

Now, recalling that $\eta_1 \equiv 1$ on D_1 , we have

$$\|v\|_{L^{\frac{n+2}{n}\beta}(D_1)}^\beta = \|v^\beta\|_{L^{\frac{n+2}{n}}(D_1)} \leq \|\eta_1^2 v^\beta\|_{L^{\frac{n+2}{n}}(D)} \leq c_n C_1 \Lambda(\beta) \|v^\beta\|_{L^1(D)} = c_n C_1 \Lambda(\beta) \|v\|_{L^\beta(D)}^\beta.$$

Thus

$$\|v\|_{L^{\frac{n+2}{2}\beta}(D_1)} \leq C_c(n, \beta, C_1) \|u\|_{L^\beta(D)},$$

where

$$(6.8) \quad C_c(n, \beta, C_1) := (c_n C_1 \Lambda(\beta))^{\frac{1}{\beta}}.$$

□

7. BOUNDING THE MEAN CURVATURE

In this section we will bound the mean curvature along the mean curvature flow in terms of $\|H\|_{L^{n+2}(S)}$, having that the second fundamental form is uniformly bounded from below and an extra smallness assumption. First we derive a differential inequality for the modified mean curvature \hat{H} defined below.

Lemma 7.1. *Suppose that*

$$(7.1) \quad h_{ij} \geq -B g_{ij}.$$

Then, for

$$\hat{H} := H + nB \geq 0$$

one has

$$(7.2) \quad \partial_t \hat{H} \leq \Delta \hat{H} + \hat{H}^3 + nB^2 \hat{H}.$$

Proof. Recall that the evolution equation of the mean curvature H ([6]) is,

$$(7.3) \quad \partial_t H = \Delta H + |A|^2 H.$$

Let $\lambda_i (i = 1, \dots, n)$ be the principle curvatures. Then $\lambda_i \geq -B$ and

$$|A|^2 = \lambda_1^2 + \dots + \lambda_n^2.$$

Because $\lambda_i + B \geq 0$, one gets

$$(\lambda_1 + B)^2 + \dots + (\lambda_n + B)^2 \leq (\lambda_1 + B + \dots + \lambda_n + B)^2 = (H + nB)^2 = \hat{H}^2.$$

That is

$$|A|^2 + 2BH + nB^2 \leq \hat{H}^2$$

or, equivalently

$$(7.4) \quad |A|^2 \leq \hat{H}^2 - 2BH - nB^2 = \hat{H}^2 - 2B(\hat{H} - nB) - nB^2 = \hat{H}^2 - 2B\hat{H} + nB^2.$$

Now, by (7.3), we have

$$\begin{aligned} \partial_t \hat{H} &= \partial_t H = \Delta H + |A|^2 H = \Delta \hat{H} + (\hat{H} - nB) |A|^2 \\ &\leq \Delta \hat{H} + \hat{H}(\hat{H}^2 - 2B\hat{H} + nB^2) \\ &\leq \Delta \hat{H} + \hat{H}(\hat{H}^2 + nB^2). \end{aligned}$$

□

Let C_0, C_1 be as in (4.2) and C_c as in Lemma 6.1. Then, using Moser iteration, we can establish a Harnack type inequality for the mean curvature.

Lemma 7.2. *Suppose that*

$$(7.5) \quad h_{ij} \geq -Bg_{ij}.$$

Let

$$(7.6) \quad C_3 := 2C_c^2(n, n+2, C_1) \left(\|H\|_{L^{n+2}(D)}^2 + n^2 B^2 \|1\|_{L^{n+2}(D)}^2 + nB^2 \|1\|_{L^{\frac{(n+2)^2}{2n}}(D)} \right),$$

then there exist positive constants $\delta_2(n, C_1)$ and $C_d(n, C_1, C_3)$ such that if

$$(7.7) \quad \|H\|_{L^{n+2}(D)} + B \|1\|_{L^{n+2}(D)} \leq \delta_2(n, C_1)$$

then

$$(7.8) \quad \|H^+\|_{L^\infty(D')} \leq C_d(n, C_1, C_3) (\|H\|_{L^{n+2}(D)} + B \|1\|_{L^{n+2}(D)}).$$

Proof. Let $\hat{H} = H + nB \geq 0$. Then, by Lemma 7.1

$$(7.9) \quad \partial_t \hat{H} \leq \Delta \hat{H} + f \hat{H}, \quad \text{where} \quad f = \hat{H}^2 + nB^2.$$

We have

$$\begin{aligned} \|f\|_{L^{\frac{n+2}{2}}(D)} &= \int_0^1 \int_{M_t \cap B(x_0, 1)} \left(\hat{H}^2 + nB^2 \right)^{\frac{n+2}{2}} d\mu dt \\ &\leq c_n \int_0^1 \int_{M_t \cap B(x_0, 1)} (\hat{H}^{n+2} + B^{n+2}) d\mu dt \\ &\leq c_n \int_0^1 \int_{M_t \cap B(x_0, 1)} (|H|^{n+2} + B^{n+2}) d\mu dt \\ &= c_n \|H\|_{L^{n+2}(D)}^{n+2} + c_n B^{n+2} \|1\|_{L^{n+2}(D)}^{n+2} \\ &\leq c_n \left(\|H\|_{L^{n+2}(D)} + B \|1\|_{L^{n+2}(D)} \right)^{n+2}. \end{aligned}$$

It follows that

$$\|f\|_{L^{\frac{n+2}{2}}(D)} \leq c_n \left(\|H\|_{L^{n+2}(D)} + B \|1\|_{L^{n+2}(D)} \right)^2.$$

Let us choose

$$(7.10) \quad \delta_2(n, C_1) = \frac{\delta_1^{\frac{1}{2}}(n, n+2, C_1)}{c_n^{\frac{1}{2}}},$$

where $\delta_1(n, \beta, C_1)$ has been defined in the proof of Lemma 6.1. Then, if

$$\|H\|_{L^{n+2}(D)} + B \|1\|_{L^{n+2}(D)} \leq \delta_2(n, C_1),$$

one has

$$\|f\|_{L^{\frac{n+2}{2}}(D)} \leq \delta_1(n, n+2, C_1).$$

This means we are in the critical case ($q = \frac{n+2}{2}$) having a smallness assumption (6.2) satisfied. Hence, we can apply (6.3) with $\beta = n+2$ to obtain

$$(7.11) \quad \left\| \hat{H} \right\|_{L^{\left(\frac{n+2}{n}\right)(n+2)}(D_1)} \leq C_c(n, n+2, C_1) \left\| \hat{H} \right\|_{L^{n+2}(D)}.$$

This inequality brings us to the supercritical case for (7.9). In fact, let

$$q = \left(\frac{n+2}{n}\right) \cdot \left(\frac{n+2}{2}\right) > \frac{n+2}{2}.$$

Then we can bound f in $\|\cdot\|_{L^q(D_1)}$ by C_3 defined by (7.6). Indeed, using (7.11) we get

$$\begin{aligned} \|f\|_{L^q(D_1)} &= \left\| \hat{H}^2 + nB^2 \right\|_{L^q(D_1)} \\ &\leq \left\| \hat{H} \right\|_{L^{\left(\frac{n+2}{n}\right)(n+2)}(D_1)}^2 + nB^2 \|1\|_{L^{\frac{(n+2)^2}{2n}}(D_1)} \\ &\leq C_c^2(n, n+2, C_1) \left\| \hat{H} \right\|_{L^{n+2}(D)}^2 + nB^2 \|1\|_{L^{\frac{(n+2)^2}{2n}}(D)} \\ &\leq 2C_c^2(n, n+2, C_1) \left(\|H\|_{L^{n+2}(D)}^2 + n^2B^2 \|1\|_{L^{n+2}(D)}^2 + nB^2 \|1\|_{L^{\frac{(n+2)^2}{2n}}(D)} \right). \end{aligned}$$

Thus, we can use (5.6) with $\lambda = \frac{n+2}{n}$, $\beta = n+2$ and $k = 1$ to obtain

$$\begin{aligned} \left\| \hat{H} \right\|_{L^\infty(D')} &\leq C_b(n, q, n+2, C_3, C_1) \left\| \hat{H} \right\|_{L^{\frac{n+2}{n}(n+2)}(D_1)} \\ &\leq C_b(n, q, n+2, C_3, C_1) C_c(n, n+2, C_1) \left\| \hat{H} \right\|_{L^{n+2}(D)}. \end{aligned}$$

Noting that

$$\|H^+\|_{L^\infty(D')} \leq \left\| \hat{H} \right\|_{L^\infty(D')}$$

we finally obtain the desired estimate

$$\begin{aligned} \|H^+\|_{L^\infty(D')} &\leq C_b(n, q, n+2, C_3, C_1) C_c(n, n+2, C_1) \left\| \hat{H} \right\|_{L^{n+2}(D)} \\ &\leq nC_b(n, q, n+2, C_3, C_1) C_c(n, n+2, C_1) (\|H\|_{L^{n+2}(D)} + B \|1\|_{L^{n+2}(D)}) \\ &= C_d(n, C_1, C_3) (\|H\|_{L^{n+2}(D)} + B \|1\|_{L^{n+2}(D)}), \end{aligned}$$

where

$$(7.12) \quad C_d(n, C_1, C_3) := nC_b(n, q, n+2, C_3, C_1)C_c(n, n+2, C_1).$$

□

8. PROOF OF THE MAIN THEOREM

In this section, we give the proof of the main Theorem 1.2 stated in the introduction.

Proof of Theorem 1.2. Since $\|H\|_{L^\alpha(M \times [0, T])} < \infty$ implies that $\|H\|_{L^{n+2}(M \times [0, T])} < \infty$ if $\alpha > n+2$, we only need to prove the Theorem for $\alpha = n+2$.

We argue by contradiction. Suppose that T is the extinction time of the flow. Then, by Theorem 1.1, $|A|$ is unbounded. It follows from (1.2) that we have (7.4),

$$|A|^2 \leq \hat{H}^2 - 2B\hat{H} + nB^2 \leq \hat{H}^2 + nB^2 = (H + nB)^2 + nB^2,$$

and thus $|H|$ is unbounded. Because $H \geq -nB$, we know that H^+ is unbounded. Therefore, there exists a sequence of points (x_i, t_i) with $x_i \in M_{t_i}$ such that

$$(8.1) \quad Q_i := H(x_i, t_i) = \max_{0 \leq t \leq t_i} \max_{x \in M_t} H(x, t) \rightarrow +\infty.$$

Consider the sequence \tilde{M}_t^i of rescaled solutions for $t \in [0, 1]$ defined by

$$\tilde{F}_i(\cdot, t) = Q_i F(\cdot, t_i + \frac{t-1}{Q_i^2}).$$

If g, H and $A := \{h_{jk}\}$ are the induced metric, the mean curvature and the second fundamental form of M_t , respectively, then the corresponding rescaled quantities are given by

$$\tilde{g}_i = Q_i^2 g; \quad \tilde{H}_i = \frac{H}{Q_i}; \quad |\tilde{A}_i|^2 = \frac{|A|^2}{Q_i^2}.$$

It follows from (8.1) and (1.2) that, for the rescaled solutions we have

$$\tilde{H}_i(x_i, 1) = 1$$

and

$$\tilde{A}_i \geq -\frac{B}{Q_i} \tilde{g}_i.$$

Consider the following sets in space and time

$$\tilde{D}^i = \cup_{0 \leq t \leq 1} (B(x_i, 1) \cap (\tilde{M}_i)_t); \quad (\tilde{D}^i)' = \cup_{\frac{1}{12} \leq t < 1} (B(x_i, \frac{1}{2}) \cap (\tilde{M}_i)_t).$$

Then, we can calculate

$$\begin{aligned}
(8.2) \quad & \lim_{i \rightarrow \infty} \left(\left\| \tilde{H}_i \right\|_{L^{n+2}(\tilde{D}^i)} + \frac{B}{Q_i} \|1\|_{L^{n+2}(\tilde{D}^i)} \right) \\
&= \lim_{i \rightarrow \infty} \left\{ \left(\int_{t_i - \frac{1}{Q_i^2}}^{t_i} \int_{M_t \cap B(x_i, \frac{1}{Q_i})} |H|^{n+2} d\mu dt \right)^{\frac{1}{n+2}} + B \left(\int_{t_i - \frac{1}{Q_i^2}}^{t_i} \int_{M_t \cap B(x_i, \frac{1}{Q_i})} d\mu dt \right)^{\frac{1}{n+2}} \right\} \\
&\leq \lim_{i \rightarrow \infty} \left\{ \left(\int_{t_i - \frac{1}{Q_i^2}}^{t_i} \int_{M_t} |H|^{n+2} d\mu dt \right)^{\frac{1}{n+2}} + B \left(\int_{t_i - \frac{1}{Q_i^2}}^{t_i} \int_{M_t} d\mu dt \right)^{\frac{1}{n+2}} \right\} = 0.
\end{aligned}$$

The last step follows from the facts that

$$\int_0^T \int_{M_t} |H|^{n+2} d\mu dt < \infty; \quad \int_0^T \int_{M_t} d\mu dt < \infty$$

and

$$\lim_{i \rightarrow \infty} \frac{1}{Q_i^2} = 0.$$

Consequently, there is a universal constant $C > 1$ such that, for our rescaled flows, the constants

$$\tilde{C}_1^i = \left(1 + \left\| \tilde{H}_i \right\|_{L^{n+2}(\tilde{M}_i \times [0,1])}^{n+2} \right)^{\frac{n}{n+2}} = \left(1 + \int_{t_i - \frac{1}{Q_i^2}}^{t_i} \int_{M_t} |H|^{n+2} d\mu dt \right)^{\frac{n}{n+2}}$$

satisfy

$$(8.3) \quad 1 \leq \tilde{C}_1^i \leq C.$$

By our choice of the constants $\delta_2(n, \tilde{C}_1^i)$, which are decreasing in the second variable (follows from (6.5) and (7.10)), we obtain

$$\delta_2(n, \tilde{C}_1^i) \geq \delta_2(n, C) > 0.$$

Hence, recalling (8.2), we have for i sufficiently large,

$$(8.4) \quad \left\| \tilde{H}_i \right\|_{L^{n+2}(\tilde{D}^i)} + \frac{B}{Q_i} \|1\|_{L^{n+2}(\tilde{D}^i)} \leq \delta_2(n, \tilde{C}_1^i).$$

Thus, by Lemma 7.2

$$(8.5) \quad \left\| \tilde{H}_i^+ \right\|_{L^\infty((\tilde{D}^i)')} \leq C_d(n, \tilde{C}_1^i, \tilde{C}_3^i) \left(\left\| \tilde{H}_i \right\|_{L^{n+2}(\tilde{D}^i)} + \frac{B}{Q_i} \|1\|_{L^{n+2}(\tilde{D}^i)} \right).$$

On the one hand, we can also check that, for our rescaled flows, the constants

$$\tilde{C}_3^i = 2C_c^2(n, n+2, \tilde{C}_1^i) \left(\left\| \tilde{H}_i \right\|_{L^{n+2}(\tilde{D}^i)}^2 + n^2 \frac{B^2}{Q_i^2} \|1\|_{L^{n+2}(\tilde{D}^i)}^2 + n \frac{B^2}{Q_i^2} \|1\|_{L^{\frac{(n+2)^2}{2n}}(\tilde{D}^i)} \right)$$

satisfy

$$(8.6) \quad \tilde{C}_3^i \leq C.$$

This easily follows from (6.8) and (8.3). On the other hand, by our choice of the constants $C_d(n, \cdot, \cdot)$, which are increasing in the second and third variables (by (7.12), (5.10), (6.8) and (7.6)), since we have (8.3) and (8.6) we obtain

$$C_d(n, \tilde{C}_1^i, \tilde{C}_3^i) \leq C_d(n, C, C) < \infty.$$

As a result, we deduce from (8.5) that

$$(8.7) \quad \left\| \tilde{H}_i^+ \right\|_{L^\infty((\tilde{D}^i)')} \leq C_d(n, C, C) \left(\left\| \tilde{H}_i \right\|_{L^{n+2}(\tilde{D}^i)} + \frac{B}{Q_i} \|1\|_{L^{n+2}(\tilde{D}^i)} \right).$$

Letting $i \rightarrow \infty$ in (8.7), and recalling (8.2), we find that

$$(8.8) \quad \lim_{i \rightarrow \infty} \left\| \tilde{H}_i^+ \right\|_{L^\infty((\tilde{D}^i)')} = 0.$$

This is a contradiction because $\left\| \tilde{H}_i^+ \right\|_{L^\infty((\tilde{D}^i)')} \geq \tilde{H}_i(x_i, 1) = 1$ for all i . The proof of our Main Theorem is complete. \square

We will give a proof of Corollary 1.1.

Proof of Corollary 1.1. In both cases, the mean convex case and the starshaped case we have

(i) A lower bound for the mean curvature

$$H \geq -l \text{ for some } l \geq 0$$

and

(ii) An upper bound for the squared second fundamental form in terms of a linear function of the squared mean curvature

$$|A|^2 \leq C_* H^2 + b \text{ for some } C_*, b > 0.$$

In the mean convex case (ii) follows from [7] and in the starshaped case, both (i) and (ii) follow from [10], for some uniform constants C_*, b, l . Choose k large enough so that $k > 2l$ and $(k - l)^2 \geq b$ which imply, for $\tilde{H} = H + k$,

$$\tilde{H} > l \implies (\tilde{H} - k)^2 \leq \tilde{H}^2,$$

$$\tilde{H}^2 \geq (k - l)^2 \geq b, \text{ and therefore,}$$

$$|A|^2 \leq C_* H^2 + b = C_*(\tilde{H} - k)^2 + b \leq \tilde{C}_* \tilde{H}^2,$$

for a uniform constant \tilde{C}_* . It easily follows that

$$\partial_t \tilde{H} \leq \Delta \tilde{H} + \tilde{C}_* \tilde{H}^3.$$

As can be seen from the proofs of Lemma 7.2 and Theorem 8.1, this differential inequality combined with the integral bound (1.3) of the mean curvature in Theorem 1.2 allows us to extend the mean curvature flow past time T . \square

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