Honors Complex Variables- HW 8

Instructor: Nam Q. Le
due November 12, 2012 at 6pm

Problem 1. (a) Find the Laurent series expansion of \( f(z) = \frac{1}{z(1+z^2)} \) in powers of \( z \) that is valid in the following domains

(i) \( 0 < |z| < 1 \)
(ii) \( |z| > 1 \).

(b) Find the Laurent series expansion for \( f(z) = \frac{1}{2z^2+3z-2} \) in powers of \( z \) that is valid in an annulus containing \( z = 1 \).

Problem 2. Suppose that \( f \) and \( g \) are entire functions with \( |f(z)| \leq |g(z)| \) for all \( z \in \mathbb{C} \). Prove that \( f(z) = cg(z) \) for some constant \( c \).

Problem 3. (i) Does there exist a function \( f \), analytic on \( \mathbb{C} \setminus \{0\} \), such that \( |f(z)| \geq \frac{1}{\sqrt{|z|}} \) \( \forall \ z \in \mathbb{C} \setminus \{0\} \)?

(ii) Find all functions \( f \), analytic on \( \mathbb{C} \setminus \{0\} \), such that \( |f(z)| \geq \frac{1}{|z|} \) \( \forall \ z \in \mathbb{C} \setminus \{0\} \).

Problem 4. Let \( f \) be a non-constant complex-valued function in the open unit disc \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) such that the functions \( g = f^2 \) and \( h = f^3 \) are both analytic.

(i) Suppose that \( z_0 \in D \) is a zero of both \( g \) and \( h \). Let \( j \) and \( k \) be the multiplicities of \( g \) and \( h \), respectively, at \( z_0 \). Prove that \( j < k \).

(ii) Prove that \( f \) is analytic in \( D \).

Problem 5.(a) Let \( g : [0, 2\pi] \to \mathbb{C} \) be a continuous function. Prove that

\[
\int_0^{2\pi} |g(x)|^2 \, dx \geq \frac{1}{2\pi} \left( \int_0^{2\pi} |g(x)| \, dx \right)^2.
\]

(Hint: Use the inequality \( \int_0^{2\pi} (|g(x)| - c)^2 \, dx \geq 0 \) for suitable real constant \( c \).)

(b) Let \( f \) be an analytic function in \( D(0, R) \setminus \{0\} = \{ z \in \mathbb{C} : 0 < |z| < R \} \). Suppose that for all \( r \in (0, R) \), we have

\[
r^4 \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < 10.
\]

Prove that \( 0 \) is either a removable singularity of \( f \) or a pole of \( f \) with order \( n \leq 2 \).