Problem 1. Let $f$ be an analytic function in the unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$. Consider a point $z_0 \in D$. Show that there must be a positive integer $n$ such that the $n$-th derivative of $f$ at $z_0$ satisfies $|f^{(n)}(z_0)| \leq n!n^n$.

Problem 2. Let $f$ be an analytic function in the unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$. Show that the diameter $d := \sup_{z,w \in D} |f(z) - f(w)|$ of the image of $f$ satisfies $2|f'(0)| \leq d$.

Hint: show that $2f'(0) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z) - f(-z)}{z^2} dz$ for all $0 < r < 1$.

Problem 3. Let $u$ be a real-valued harmonic function in the plane. We proved in class that $u$ satisfies the mean value property in the following form: for all $r > 0$

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$  

Deduce from this formula the following identities for all $r > 0$

(i) $$u(z_0) = \frac{1}{2\pi r} \int_{\partial D(z_0,r)} u(x,y) ds.$$  

(ii) $$u(z_0) = \frac{1}{\pi r^2} \iint_{D(z_0,r)} u(x,y) dxdy.$$  

Here $D(z_0;r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

Problem 4. The purpose of this problem is to prove that any bounded, real-valued harmonic function $u$ in the plane is a constant (Liouville’s Theorem). You can proceed as follows

(i) Prove that $u_x$ is also harmonic.

(ii) Use Problem 3(ii) to show that for all $z_0 = (x_0, y_0) \in \mathbb{R}^2$ and $r > 0$

$$u_x(z_0) = \frac{1}{\pi r^2} \iint_{D(z_0,r)} u_x(x, y) dxdy.$$  

Then use Green’s Theorem and let $r \to \infty$.

Problem 5. Suppose that $f$ and $g$ are both analytic in the closed unit disc $\overline{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. Show that $|f(z)| + |g(z)|$ attains its maximum value on the boundary of $\overline{D}$. 