In this problem set, all closed curves are assumed to have counterclockwise orientation.

**Problem 1.** Let $\Omega$ be a smooth domain in $\mathbb{C}$ with boundary $\gamma$. We choose the counterclockwise orientation on $\gamma$.

(i) Show that $\int_{\gamma} \bar{z} dz = 2i \text{Area}(\Omega)$.

(ii) Show that if $f$ is entire then there is a point $z \in \gamma$ such that $|\bar{z} - f(z)| \geq 2 \frac{\text{Area}(D)}{\text{Length}(\gamma)}$.

**Problem 2.** The purpose of this problem is to show that if $a$ and $b$ are complex numbers in the left half-plane $\{ z \in \mathbb{C} : \text{Re}(z) \leq 0 \}$ then $|e^a - e^b| \leq |a - b|$. You can start as follows.

a. Compute $\int_a^b e^z dz$ where the integral is taken over any curve connecting $a$ and $b$ in the complex plane.

b. Show that if $a$ and $b$ are complex numbers in the left half-plane then $|e^a - e^b| \leq |a - b|$.

**Problem 3.** Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a complex polynomial of degree $n > 0$. Show that

$$\frac{1}{2\pi i} \int_{|z|=R} z^{n-1} |f(z)|^2 dz = a_0 \overline{a_n} R^{2n}.$$

**Problem 4.** Evaluate the following integrals

(i) $\int_{|z|=1} \frac{1}{z^2 + 2az + 1} dz$

where $a > 1$.

(ii) $\int_{|z|=4} \frac{1}{(z-1)(z-2)(z-3)} dz$.

**Problem 5.** (i) Show that for all $R > 1$ we have

$$\int_{|z|=1} \frac{z^{2011}}{2z^{2012} - 1} dz = \int_{|z|=R} \frac{z^{2011}}{2z^{2012} - 1} dz.$$

(ii) Use the above result and let $R \to \infty$ to show that

$$\int_{|z|=1} \frac{z^{2011}}{2z^{2012} - 1} dz = \pi i.$$