Problem 1. Show that if $f$ is analytic on an open set $D$, then the function $g$ defined by
\[ g(z) = f(\overline{z}) \]
is analytic on the reflected domain $D^* = \{ \overline{z} : z \in D \}$.

Problem 2. All functions in this problem are infinitely differentiable. If $f(z) = u(x, y) + iv(x, y)$, we define $\Delta f = \Delta u + i\Delta v$ where we recall $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In class, we saw that if $f$ is analytic then $u$ and $v$ are harmonic, that is $\Delta u = \Delta v = 0$ and hence $f$ is harmonic.

(a) If $f$ is harmonic, is it also analytic? Justify your answer.
(b) Suppose that $f$ and $zf$ are harmonic, that is $\Delta f = \Delta(zf) = 0$. Show that $f$ is analytic.

Problem 3. (a) Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_0 = 0$ and $B_k = \sum_{n=1}^k b_n$ $(k \geq 1)$ be the partial sums of the series $\sum b_n$. Prove the summation by parts formula
\[ \sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n)B_n. \]

(b) (Abel Theorem) Suppose $\sum_{n=1}^{\infty} a_n$ converges. Prove that
\[ \lim_{r \to 1, r < 1} \sum_{n=1}^{\infty} r^n a_n = \sum_{n=1}^{\infty} a_n. \]

Hint: Sum by parts.

Problem 4. Consider the power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$.

(a) Find the radius of convergence $R$ of the above power series.
(b) What can you say about the convergence/divergence of the series when $|z| = R$.

Problem 5.

(a) Find all entire functions $f(z)$ of the form $f(z) = u(x) + iv(y)$.
(b) Find all solutions $z \in C$ of $e^z = 1 - i$. 