Honors Complex Variables- HW2

Instructor: Nam Q. Le
due September 17, 2012 at 6pm

Problem 1. Simplify the following expressions

(a) $1 + \cos \varphi + \cos(2\varphi) + \cdots + \cos(n\varphi)$.
(b) $\sin \varphi + \sin(2\varphi) + \cdots + \sin(n\varphi)$.

Problem 2. Let $z_k = \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})$, $k = 0, 1, \ldots, n-1$, be all $n$th roots of unity. Show that

$$\sum_{j=0}^{n-1} z_j^m = \begin{cases} 0, & \text{if } 1 \leq m \leq n-1, \\ n, & \text{if } m = n. \end{cases}$$

Problem 3. In class, we showed that if $|z| < 1$ then

$$\sum_{k=1}^{\infty} z^k = \frac{z}{1-z}.$$

Show that if $|z| < 1$ then

$$\sum_{k=0}^{\infty} \frac{z^{2k}}{1-z^{2k+1}} = \frac{z}{1-z}. $$

Problem 4. In class, we showed that the function $f$ defined on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ by

$$f(z) = \sum_{k=1}^{\infty} kz^k$$

is continuous in $D$. Another way to show the continuity of $f$ in $D$ is to find the explicit formula for $f$. This problem asks you to find such a formula.

Problem 5. Let $f = u + iv$ be differentiable at all $z \in \mathbb{C}$.

(a) Show that the gradient vectors $\nabla u$ and $\nabla v$ are orthogonal.
(b) Show that in polar coordinates $(r, \theta)$, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}. $$