Regularity and Non-existence Results for Some Free-interface Problems Related to Ginzburg-Landau Vortices

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Abstract

We study regularity and non-existence properties for some free-interface problems arising in the study of limiting vorticities associated to the Ginzburg-Landau equations with magnetic field in two dimensions. Our results imply in particular that if these limiting vorticities concentrate on a smooth closed curve then they have a distinguished sign; moreover, if the domain is thin then solutions of the Ginzburg-Landau equations cannot have a number of vortices much larger than the applied magnetic field.

1 Introduction and Main Results

1.1 Presentation of the Problem

In this paper, we are interested in the following equation

$$\text{div } T_\mu = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $T_\mu \equiv ((T_\mu)_{ij})_{1\le i,j\le 2}$ defined by

$$T_\mu = \begin{pmatrix} \frac{1}{2}\left((\partial_2 h_\mu)^2 - (\partial_1 h_\mu)^2 + h_\mu^2\right) & -\partial_1 h_\mu \partial_2 h_\mu \\ -\partial_1 h_\mu \partial_2 h_\mu & \frac{1}{2}\left((\partial_1 h_\mu)^2 - (\partial_2 h_\mu)^2 + h_\mu^2\right) \end{pmatrix}$$

is the symmetric stress-energy tensor associated to the solution $h_\mu \in H^1(\Omega)$ of the equation

$$\begin{cases} -\Delta h + h = \mu & \text{in } \Omega \\ h = c & \text{on } \partial\Omega \ (c \equiv 0, \text{ or } c \equiv 1). \end{cases} \tag{1.2}$$

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Equation (1.1), understood in the sense of distributions as $\partial_1(T_\mu)_{i1} + \partial_2(T_\mu)_{i2} = 0$ for $i = 1, 2$, arises in the study of limiting vorticities for the Ginzburg-Landau equations in superconductivity which we will discuss in the next subsection. In the context of Ginzburg-Landau theory, (1.2) corresponds to the limit of the "London equation" and (1.1), obtained by passing to the limit in the stress-energy tensor associated to the Ginzburg-Landau energy G_ε (see (1.4)), is a criticality condition on the limiting measures μ of critical points of G_ε. In (1.1)-(1.2), the domain Ω is a smooth bounded domain of $\mathbb{R}^2$ corresponding to a section of the superconducting material and μ is a measure belonging to $H^{-1}(\Omega)$, the dual of the Sobolev space $H_0^1(\Omega)$. Finally, we note that (1.1) can be formally written as $\nabla h_\mu \mu = 0$ and thus it is (a weak version of) stationary solution of the evolution problem

$$\frac{d}{dt}\mu(t) - div(\nabla h_{\mu(t)}\,|\mu(t)|) = 0 \quad \text{in } \Omega \tag{1.3}$$

which has been recently studied using a gradient flow approach by Ambrosio and Serfaty. See [1] and the references therein for previous studies of (1.3).

If $\mu \in L^p(\Omega)$ for some $p > 1$, then it was proved in [11], Theorem 13.1, that for $c \equiv 1$, the solution $h_\mu \in H^1(\Omega)$ to (1.1) -(1.2) satisfies $0 \le h_\mu \le 1$ and μ is in fact a nonnegative L^∞ function and that for $c \equiv 0$, (1.1)-(1.2) has no solutions except for the trivial measure $\mu = 0$.

In the present paper, we investigate equation (1.1) with measures μ more singular than those considered above: μ is only in $H^{-1}(\Omega)$. A typical example of $\mu \in H^{-1}(\Omega)$ is the case when μ is the measure of arclength along a smooth closed curve in Ω with some weight. It is precisely this case that we will consider in the paper. This consideration first seems to be restrictive; however, when μ is viewed as the limiting vorticity for the Ginzburg-Landau equations, this situation is quite natural as can be seen in a recent result of Aydi [2]. Thus, in the sequel, we consider measures $\mu \in H^{-1}(\Omega)$ having support supp $\mu = \Sigma$, a smooth closed curve in Ω and being absolutely continuous with respect to the arclength measure on Σ with nowhere-zero density.

We will be in particular interested in the existence and non-existence results for (1.1)-(1.2) depending on μ, the regularity of T_μ and h_μ, the sign of μ and its positivity depending on the spectral property of the domain Ω. These problems in the end have some similar flavor to the studies of free-boundaries and their regularity (see also Section 4); however we have not seen many results on similar questions.

1.2 Connections to Ginzburg-Landau Vortices

Our study of the equation (1.1) is motivated by two open problems (Problems 17 & 18) in the book of Sandier and Serfaty [11] about the limiting vorticities for the the critical points $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ of the Ginzburg-Landau energy in superconductivity

$$G_\varepsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla u - iAu|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 + |h - h_{ex}|^2\,. \tag{1.4}$$

Here u is a complex-valued function called the "order parameter" and its isolated zeros are called vortices; $h_{ex} > 0$ is the intensity of the applied magnetic field; $A : \Omega \to \mathbb{R}^2$ is the vector potential and $h = \text{curl } A = -\partial_2 A_1 + \partial_1 A_2$ is the induced magnetic field.

There has been much interest in studying the limiting vorticity measures associated to critical points of the Ginzburg-Landau energy; see the books by Bethuel, Brezis and Hélein [3] and Sandier and Serfaty [11] and the references therein for an abundant literature on the subject.

In the following, we summarize some results and recall some open problems concerning Ginzburg-Landau vortices in [11] (see also [10]) that are closely related to (1.1)-(1.2).

Let $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ be critical points of G_ε. Then, from the criticality of $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$, one has

$$-\nabla^\perp h_\varepsilon = (iu_\varepsilon, \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon) \text{ in } \Omega,\ \ h_\varepsilon = h_{ex} \text{ on } \partial\Omega. \tag{1.5}$$

Here $\nabla^\perp$ denotes the operator $(-\partial_2, \partial_1)$, $(\cdot,\cdot)$ the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$, i.e., $(a, b) = \frac{\bar{a}b+a\bar{b}}{2}$. The important quantity carrying the topological information on the vortices of u_ε is the vorticity μ_ε defined by

$$\mu_\varepsilon = \mu_\varepsilon(u_\varepsilon, A_\varepsilon) = \text{curl } (iu_\varepsilon, \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon) + \text{curl } A_\varepsilon.$$

Taking the curl of (1.5), one obtains $-\Delta h_\varepsilon + h_\varepsilon = \mu_\varepsilon$ with μ_ε is approximately $2\pi \sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}$ (see (1.6)). This is what is called the London equation in physics. The a_i^ε's are essentially the vortices with degrees d_i^ε.

Now, we can recall the following

Theorem 1.1. *([11], Theorem 13.1) Let $\{(u_\varepsilon, A_\varepsilon)\}_{\varepsilon>0}$ be critical points of G_ε that satisfy the energy bound $G_\varepsilon(u_\varepsilon, A_\varepsilon) \le C_0\varepsilon^{\alpha-1}$ for any $\varepsilon > 0$ where $\alpha > 2/3$ is independent of ε. Then, the vorticity μ_ε can be approximated by a sum of Dirac masses in the sense that: for any $\varepsilon > 0$ there exists a measure ν_ε of the form $2\pi \sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}$ where the sum is taken over a finite number of indices i's, $a_i^\varepsilon \in \Omega$ and $d_i^\varepsilon \in \mathbb{Z}$ for every i such that*

$$\|\mu_\varepsilon - \nu_\varepsilon\|_{W^{-1,p}(\Omega)}\, \|\mu_\varepsilon - \nu_\varepsilon\|_{C^0(\Omega)^*} \to 0 \tag{1.6}$$

for some $p \in (1, 2)$.

Let $\{\nu_\varepsilon\}_{\varepsilon>0}$ be any measure of the form $2\pi \sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}$ satisfying (1.6) and let us denote $n_\varepsilon = \sum_i |d_i^\varepsilon|$; n_ε here stands for the total degree of the vortices. There are 3 regimes to distinguish according to the ratio of n_ε to the external field h_{ex}: possibly after extraction, one of the following holds.

0. $n_\varepsilon = 0$ for any ε small enough and then μ_ε tends to 0 in $W^{-1,p}(\Omega)$.

1. $n_\varepsilon = o(h_{ex})$ is nonzero for ε small enough, and then $\mu_\varepsilon/n_\varepsilon$ converges in $W^{-1,p}(\Omega)$ to a measure μ such that $\mu\nabla H_0 = 0$. Hence the support of μ is contained in the set of critical points of H_0, the solution to the London equation, i.e., (1.2) with $\mu \equiv 0$ and $c \equiv 1$.

2. $h_{ex} \sim \lambda n_\varepsilon$ with $\lambda > 0$ and then μ_ε/h_{ex} converges in $W^{-1,p}(\Omega)$ to a measure μ and h_ε/h_{ex} converges in $W^{1,p}_{loc}(\Omega)$ to a solution of (1.2) with $c \equiv 1$.

3. $h_{ex} = o(n_\varepsilon)$ and then $\mu_\varepsilon/n_\varepsilon$ converges in $W^{-1,p}(\Omega)$ to a measure μ and $h_\varepsilon/n_\varepsilon$ converges in $W^{1,p}_{loc}(\Omega)$ to the solution of (1.2) with $c \equiv 0$.

In cases 2) and 3), if $\mu \in H^{-1}(\Omega)$ then the solution $h_\mu \in H^1(\Omega)$ of (1.2) and its associated symmetric stress-energy tensor $T_\mu \equiv ((T_\mu)_{ij})_{1\le i,j\le 2}$ defined by

$$T_\mu = \begin{pmatrix} \frac{1}{2}\left((\partial_2 h_\mu)^2 - (\partial_1 h_\mu)^2 + h_\mu^2\right) & -\partial_1 h_\mu \partial_2 h_\mu \\ -\partial_1 h_\mu \partial_2 h_\mu & \frac{1}{2}\left((\partial_1 h_\mu)^2 - (\partial_2 h_\mu)^2 + h_\mu^2\right) \end{pmatrix}, \tag{1.7}$$

satisfy the following properties: T_μ is in $L^1_{loc}(\Omega)$,

$$div\ T_\mu = 0 \text{ in the sense of distributions and } |\nabla h_\mu|^2 \in W^{1,q}_{loc}(\Omega)\ \forall q \in [1,\infty). \tag{1.8}$$

The first property of (1.8) means that for $i = 1, 2$, we have $\partial_1(T_\mu)_{i1} + \partial_2(T_\mu)_{i2} = 0$ in the sense of distributions while the latter property implies that h_μ is locally Lipschitz.

Thus, cases 2) and 3) correspond to equation (1.1). These cases are most interesting but not very well-understood; especially when μ is not absolutely continuous with respect to the Lebesgue measure. The main question is to understand the nontriviality and the sign of μ (recall that μ is the limiting measure of $(2\pi/n_\varepsilon)\sum_i d_i^\varepsilon \delta_{a_i^\varepsilon}$) which in turn gives qualitative information on the behavior of vortices.

In the rest of the section, we will be more specific on this question. In Theorem 1.1, d_i^ε's are the degrees of the vortices a_i^ε's; typically $d_i^\varepsilon = \pm 1$ in stable configurations. On the ε-level, it is expected that d_i^ε's can have different signs and thus the approximating measures ν_ε are not necessarily of distinguished sign. When we do the space rescaling as in cases 2) and 3), close vortices of different signs may annihilate each other. This leads us to an open problem (Problem 17) of [11] about the possibility of having solutions with nonpositive/changing sign limiting measures. Our Corollary 1.1 (i) in the next section partially answers in the negative direction.

In the above theorem, if all vortices have uniformly bounded degrees (which is physically possible) then the quantity n_ε is basically the total number of vortices of the critical points of the Ginzburg-Landau energy. Case 3) only happens for critical points with a number of vortices n_ε much larger than the applied magnetic field h_{ex}. This leads us to another open problem (Problem 18) in [11] about the possibility of having critical points with a number of vortices much larger than the applied magnetic field h_{ex}. As we will see in Theorem 1.2 below, for thin domains, the answer to this open problem is negative.

Recall that if $\mu \in L^p(\Omega)$ where $p > 1$ then the answer to both problems 17 & 18 is negative; this fact was established in Theorem 13.1 of [11].

We note that our assumption on μ in the context of Ginzburg-Landau vortices corresponds to the case where limiting vortices concentrate on a closed smooth curve in Ω. This type of limiting vorticity can happen. Indeed, recently Aydi [2] showed that when Ω is the unit disc, a nonzero vorticity μ which is supported in a finite union of concentric circles can actually arise as limit of the vorticities of some family of solutions.

1.3 Main Results and Methods of the Proofs

Our first result concerns the regularity of h_μ in terms of that of μ.

Proposition 1.1. *Suppose that μ is absolutely continuous with respect to the arclength measure on Σ with density $f \in W^{2,p}(\Sigma)$ for some $p > 1$, i.e., $\mu = f\mathcal{H}^1 \lfloor \Sigma$. Then, the solution h_μ of (1.2) is $C^{1,1-1/p}$ up to the boundary on each subdomain of Ω enclosed by Σ and/or $\partial\Omega$.*

Remark 1.1. *With μ as in the above proposition, we have $\mu \in W^{-1,q}(\Omega)$ for all $q > 1$.*

We note that (1.1)-(1.2) implies (1.8); see, e.g., [11]. In the sequel, we need (1.8) and the following condition on the solution h_μ of (1.2)

$$(A) \quad h_\mu \text{ is } C^1 \text{ up to the boundary on each side of } \Sigma.$$

Rather than imposing this condition on h_μ, we feel it is more natural to impose a condition on the measure μ. This is why we gave a sufficient condition in terms of μ in Proposition 1.1. By this proposition, (A) is satisfied if $\mu = f\mathcal{H}^1 \lfloor \Sigma$ with $f \in W^{2,p}(\Sigma)$ for some $p > 1$.

When μ is clear in the context, the subscript μ is dropped in h_μ, T_μ, etc.

Notation. We denote by $\lambda_1(\Omega)$ the first eigenvalue the Laplace operator $-\Delta$ on Ω with zero Dirichlet boundary condition.

By (1.8), T is a locally bounded tensor. With the additional assumption (A), its regularity is improved. This is the object of our next result which confirms the continuity of T and the nonsolvability of (1.1)-(1.2) for nonzero measures μ on a thin domain.

Theorem 1.2. *Assume that (A) is satisfied. Then*
(i) T is continuous in Ω and h is equal to a constant c_* on Σ.
(ii) If either $\lambda_1(\Omega) > 1$ or diam $(\Omega)\,|\Omega|^{1/2} \leq 2\pi^{1/2}$ then (1.1)-(1.2) with $c \equiv 0$ has no solution with a nonzero measure $\mu \in H^{-1}(\Omega)$.
(iii) If $\Omega = B_R$ then (1.1)-(1.2) with $c \equiv 0$ has no solution with a nonzero measure $\mu \in H^{-1}(\Omega)$ supported on a circle.

For the case $\mu \in L^p(\Omega)$ for some $p > 1$ in case 2) of Theorem 1.1, it can be shown that the measure μ is in fact given by a nonnegative L^∞ function. In contrast, in our case with a more singular vorticity measure μ, there is no gain in the regularity of μ, except its density with respect to the arclength measure on Σ. However, there is a gain in regularity for the stress-energy tensor T and on h. As a simple consequence of the constancy of h on Σ, there is also an improvement in the regularity of h on each side of Σ: h is C^∞ up to the boundary on each side of Σ. Furthermore, μ is absolutely continuous with respect to the arclength measure on Σ with smooth density $f \in C^\infty(\Sigma)$, see (3.6). In addition, we obtain further information on the sign of μ in the following.

Corollary 1.1. *Suppose that (1.1)-(1.2) and (A) are satisfied. Then*
(i) μ is a Radon measure with a distinguished sign.
(ii) If μ is a measure with constant density on Σ, then Σ is a circle.
(iii) If, in addition, $h = 1$ on $\partial\Omega$ and Ω is thin enough, i.e, $\lambda_1(\Omega) > 1$ or diam$(\Omega)\,|\Omega|^{1/2} \leq 2\pi^{1/2}$, then μ is a positive Radon measure.

Let us say a few words about the methods of the proofs. The proof of Proposition 1.1 is straightforward from potential-theoretic arguments. For the proof of part (i) of Theorem 1.2, the key idea is to express the stress-energy tensor T in the normal-tangential frame (ν, τ) associated with Σ. In this frame, due to the smoothness of Σ and assumption (A), we have the regularity of T in the tangential direction. What prevents T from being continuous is the jump in the normal derivative of h. At this point we use the divergence-free property of T to show that its normal components are preserved and thus continuous through Σ. The constancy of h on Σ is crucial in the proof of parts (ii) and (iii) of Theorem 1.2. It allows us to reflect part of the graph of function h in Ω^-, the region enclosed by Σ, over the horizontal plane $z = c_*$ in $\mathbb{R}^3$. By this reflection, we obtain equation (3.10) which is less singular than (1.2). Then we can use some type of Aleksandrov weak maximum principle and the modified Bessel functions to conclude.

The paper is organized as follows. In Section 2, we prove Proposition 1.1. The proof of Theorem 1.2 and Corollary 1.1 will be presented in Section 3. Our study leaves as many problems open as it solves. We present a list of respective open questions and conjectures in Section 4.

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2 Proof of Proposition 1.1

Before proving this proposition, we make a few simplifications.

First, we can assume that $c = 0$. Indeed, if $c = 1$ then we can split $h = h_0 + h_1$ where h_0 and h_1 solve the following equations respectively

$$-\Delta h_0 + h_0 = \mu \text{ in } \Omega,\ h_0 = 0 \qquad \text{on } \partial\Omega$$

and

$$-\Delta h_1 + h_1 = 0 \text{ in } \Omega,\ h_1 = 1 \qquad \text{on } \partial\Omega.$$

Because $h_1 \in C^\infty(\overline{\Omega})$ by standard regularity theory, the regularity of h is that of h_0.

Second, we only need to establish the regularity results for the simpler equation

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

To see this, recall that by Remark 1.1 we have $\mu \in W^{-1,2p}(\Omega)$ and therefore, by elliptic regularity, $h \in W^{1,2p}(\Omega)$. Let v solve the following equation $-\Delta v = -h$ in Ω with $v = 0$ on $\partial\Omega$. Then $v \in W^{3,2p}(\Omega)$ and thus $v \in C^{2,1-\frac{1}{p}}(\Omega)$. Let $H = h - v$. Then H is the solution to (2.1). Therefore, the regularity (in C^1) of h is the same as that of H. From now on, by h we mean the solution to equation (2.1).

Third, we can assume that the smooth curve Σ, the support of μ, is a circle. This follows from the fact that the regularity of an elliptic equation is unaltered through a conformal change of variables. Let Ω^- be the subdomain of Ω enclosed by Σ. We can find a smooth map $(x,y) \in B_r \longmapsto (f(x,y), g(x,y)) \equiv (X,Y) \in \Omega^-$ with $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$, $\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$. In B_r, let $\tilde{h}(x,y) = h(X,Y) = h(f(x,y), g(x,y))$. Then, a simple calculation gives

$$-\Delta \tilde{h}(x,y) = \Delta h(X,Y)\,|\nabla f|^2 = \mu(X,Y)\,|\nabla f|^2 = \tilde{\mu}(x,y).$$

Note that the smoothness of $\frac{d\mu}{d\mathcal{H}^1 \lfloor \Sigma}$ and $\frac{d\tilde{\mu}}{d\mathcal{H}^1 \lfloor \partial B_r}$ are identical.

Finally, we can assume that Ω and B_r are concentric discs. Indeed, let $B_R \Supset \Omega$ and let u solves the following equation $-\Delta u = \mu$ in B_R, $u = 0$ on ∂B_R. Because supp $\mu \Subset \Omega \Subset B_R$, we find that u is C^∞ in $B_R \backslash \overline{\Omega}$ up to its boundary. Thus $u = g \in C^\infty(\partial\Omega)$ on $\partial\Omega$. We have

$$\begin{cases} -\Delta(u-h) = 0 & \text{in } \Omega \\ u - h = g & \text{on } \partial\Omega. \end{cases}$$

By elliptic regularity, $u - h \in C^\infty(\overline{\Omega})$. Thus, the regularity of h is the same as that of u. Consequently, by scaling, we assume that Ω is the unit disc, i.e., $\Omega = B_1$.

In conclusion, we only need to establish the regularity results for the simplest equation

$$\begin{cases} -\Delta u = \mu & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases} \tag{2.2}$$

with μ supported on a circle, say, ∂B_r $(0 < r < 1)$.

As a preparation for the proof of Proposition 1.1, we prove the following

Lemma 2.1. *If $d\mu(y) = f(y)d\mathcal{H}^1 \lfloor \partial B_r$ $(0 < r < 1)$ where $f \in W^{2,p}(\partial B_r)$ $(p > 1)$ then the function k defined by $k(x) = \int log\,|x-y|\,d\mu(y)$ belongs to $W^{2,p}(\partial B_r)$ when restricted to ∂B_r.*

Proof. For $x \in \partial B_r$, we write $x = re^{it}$ $(0 \le t < 2\pi)$ and denote

$$k(t) = k(x) = \int log\left|re^{it} - y\right| d\mu(y).$$

Then, writing $y = re^{i\varphi}$ $(0 \le \varphi < 2\pi)$ for $y \in \partial B_r$, we have

$$\left|re^{it} - y\right|^2 = \left|re^{it} - re^{i\varphi}\right|^2 = r^2\left|1 - e^{i(t-\varphi)}\right|^2 = 2r^2(1 - cos(t-\varphi))$$

and therefore

$$\begin{aligned} k(t) &= \frac{1}{2}\int log\,|x-y|^2\,d\mu(y) = \frac{1}{2}\int_{\partial B_r} log\,|x-y|^2\,d\mu(y) \\ &= \frac{1}{2}\int log(2r^2(1-cos(t-\varphi)))f(\varphi)rd\varphi \\ &= \frac{1}{2}\int log(1-cos(t-\varphi))f(\varphi)rd\varphi + \frac{1}{2}\int log(2r^2)f(\varphi)rd\varphi. \end{aligned}$$

It follows that the regularity of k is that of $g(t) = \int log(1 - cos(t - \varphi))f(\varphi)d\varphi$. We can rewrite g as $g(t) = w * f(t)$ where $w(s) = log(1 - cos(s)) \in L^1(0, 2\pi)$. Now, if $f \in W^{2,p}(\partial B_r)$ then $g \in W^{2,p}(0, 2\pi)$ and thus, when restricted to ∂B_r, $k \in W^{2,p}(\partial B_r)$ as claimed. □

Now, after simplifications, the proof of Proposition 1.1 is reduced to that of the following

Proposition 2.1. *Let $d\mu(y) = f(y)d\mathcal{H}^1 \lfloor \partial B_r$ $(0 < r < 1)$ where $f \in W^{2,p}(\partial B_r)$ $(p > 1)$. Then the solution to the equation (2.2) is $C^{1,1-1/p}$ up to the boundary on each subdomain of Ω enclosed by ∂B_r and/or ∂B_1.*

Proof. We have $u(x) = \int G(x, y)d\mu(y)$ where G is the Green function associated with B_1:

$$-\Delta_y G = \delta_x \text{ in } B_1, \quad G = 0 \text{ on } \partial B_1.$$

An explicit formula for $G(x, y)$ (see, e.g., [7]) is given by

$$G(x, y) = -\frac{1}{2\pi} log |x - y| + \frac{1}{2\pi} log \left| |x| \left(y - \frac{x}{|x|^2}\right) \right| = -\frac{1}{2\pi} log |x - y| + \frac{1}{2\pi} log \sqrt{|x|^2 |y|^2 - 2x \cdot y + 1}.$$

Thus

$$\begin{aligned} u(x) &= -\frac{1}{2\pi} \int log |x - y| \, d\mu(y) + \frac{1}{4\pi} \int log(|x|^2 |y|^2 - 2x \cdot y + 1) d\mu(y) \\ &= -\frac{1}{2\pi} k(x) + \text{ smooth function in } x. \end{aligned}$$

By Lemma 2.1, when restricted to ∂B_r, k is in $W^{2,p}$ on ∂B_r and so is u. Therefore $u \in C^{1,1-1/p}$ on ∂B_r. By a standard regularity result (see, e.g [6]), the proposition follows. □

3 Proofs of Theorem 1.2 and Corollary 1.1

Let Ω^- (resp. Ω^+) be the region in Ω enclosed by Σ (resp. by Σ and $\partial\Omega$). On Σ, we choose the unit normal vector ν pointing into Ω^+. In the normal-tangential frame (ν, τ), we have

$$T = \begin{pmatrix} T_{\nu\nu} & T_{\nu\tau} \\ T_{\tau\nu} & T_{\tau\tau} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left((\partial_\tau h)^2 - (\partial_\nu h)^2 + h^2\right) & -\partial_\nu h \partial_\tau h \\ -\partial_\nu h \partial_\tau h & \frac{1}{2}\left((\partial_\nu h)^2 - (\partial_\tau h)^2 + h^2\right) \end{pmatrix}.$$

Now we prove Theorem 1.2.

Proof. (i) By the assumption (A), the tangential derivative $\partial_\tau h$ exists and is continuous on Σ, and in Ω^- with the frame (ν, τ), the normal derivative $\partial_\nu h$ exists and is continuous up to its boundary Σ. Therefore, T is continuous on each side of Σ up to the curve Σ.

First, we prove the continuity of T through Σ. Indeed, since h is smooth off Σ, T has

Σ as its only curve of discontinuity. Denote by $T^+(\Sigma)$ (resp. $T^-(\Sigma)$) the trace of $T\mid_{\Omega^+}$ (resp. $T\mid_{\Omega^-}$) on Σ. By the divergence-free property of T, we have

$$\int_\Omega T \cdot \nabla\varphi = 0 \tag{3.1}$$

for all test functions $\varphi \in C_c^1(\Omega)$. We now consider test functions φ whose supports cross Σ. Denote by $[T] = T^+(\Sigma) - T^-(\Sigma)$ the jump of T across Σ. Then, integrating by parts the equation (3.1), we see that

$$0 = \int_\Omega T \cdot \nabla\varphi = -\int_{\Omega^+} (\text{div } T)\varphi - \int_{\Omega^-} (\text{div } T)\varphi - \int_\Sigma \varphi[T] \cdot \nu d\mathcal{H}^1, \tag{3.2}$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. On the right hand side, the first two terms are zero by the divergence-free property of T in the strong sense on each side of the curve Σ; hence the last term must vanish. Because φ is arbitrary this implies that $[T] \cdot \nu = 0$, meaning that there is no jump in the normal components of T. Therefore, the normal components of T, $T_{\nu\nu} = (\partial_\tau h)^2 - (\partial_\nu h)^2 + h^2$ and $T_{\tau\nu} = \partial_\nu h \partial_\tau h$, are continuous through Σ. Recall that (1.1)-(1.2) implies (1.8) and thus h is continuous through Σ. Hence, it follows that $(\partial_\tau h)^2 - (\partial_\nu h)^2$ is continuous through Σ, showing that T is continuous through Σ.

Next, we prove the constancy of h on Σ. Let $h^\pm$ be the restrictions of h on $\Omega^\pm$. Then, we can rewrite the equation $\mu = -\Delta h + h$ in terms of $h^\pm$ as follows

$$\mu = \left(\frac{\partial h^-}{\partial\nu} - \frac{\partial h^+}{\partial\nu}\right) d\mathcal{H}^1 \lfloor \Sigma. \tag{3.3}$$

By (1.8), $|\nabla h|^2$ is continuous and as proved above, $(\partial_\tau h)^2 - (\partial_\nu h)^2$ is continuous through Σ. Thus, $(\partial_\tau h)^2$ and $(\partial_\nu h)^2$ are continuous through Σ. Consequently, on Σ, we have

$$\frac{\partial h^-}{\partial\nu} = \pm\frac{\partial h^+}{\partial\nu}. \tag{3.4}$$

Because μ has a nowhere-zero density on Σ, we deduce from (3.3) and (3.4) that

$$\frac{\partial h^-}{\partial\nu} = -\frac{\partial h^+}{\partial\nu} \neq 0 \tag{3.5}$$

and that

$$\mu = 2\frac{\partial h^-}{\partial\nu} d\mathcal{H}^{n-1} \lfloor \Sigma. \tag{3.6}$$

Since $\partial_\nu h \partial_\tau h$ is continuous through Σ and $\partial_\nu h$ changes its sign through Σ, we must have $\partial_\tau h = 0$ on Σ. This allows us to conclude that h is equal to a constant c_* on Σ. $\square$

Using equality (3.6), we can prove parts (i) and (ii) of Corollary 1.1 as follows.

Proof. (i) We first prove the nonchanging sign character of μ. By (3.6), it suffices to prove that $\frac{\partial h^-}{\partial \nu}$ does not change sign but this is a consequence of the constancy of h^- on Σ and Hopf's Lemma.
(ii) Suppose that μ is a measure with constant density on Σ. Then, from (3.6), it follows that $\frac{\partial h^-}{\partial \nu}$ is a nonzero constant on Σ. Therefore, h^- solves the following overdetermined system

$$-\Delta h^- + h^- = 0 \text{ in } \Omega^-,\ h^- = \text{constant on } \partial\Omega^-,\ \frac{\partial h^-}{\partial \nu} = \text{constant} \neq 0 \text{ on } \partial\Omega^-.$$

By a celebrated theorem of Serrin [12], we conclude that Ω^- is a disc and thus Σ is a circle.

Now, we prove parts (ii) and (iii) of Theorem 1.2. We argue by contradiction. Suppose that (1.1)-(1.2) with $c \equiv 0$ has a solution h_μ for a nonzero measure $\mu \in H^{-1}(\Omega)$. Recall by Corollary 1.1 (i) that μ is a Radon measure with a distinguished sign. Because the boundary condition of h in (1.2) is 0, we can assume without loss of generality that μ is a positive Radon measure. Thus, by the maximum principle,

$$0 < h^+ < c_* \text{ in } \Omega^+, \qquad 0 < h^- < c_* \text{ in } \Omega^-. \tag{3.7}$$

Our proof of parts (ii) and (iii) of Theorem 1.2 will be easier once we reflect part of the graph of function h in Ω^- over the horizontal plane $z = c_*$ in $I\!R^3$. More precisely, we consider the function v defined as follows

$$v = \begin{cases} 2c_* - h^- & \textit{in } \Omega^- \\ h^+ & \textit{in } \Omega^+. \end{cases} \tag{3.8}$$

Then, as in the proof of Theorem 1.2 (i), we have the following identity on Σ

$$-\Delta v + v = \left(\frac{\partial v^-}{\partial \nu} - \frac{\partial v^+}{\partial \nu}\right) d\mathcal{H}^{n-1}\lfloor\Sigma = \left(\frac{-\partial h^-}{\partial \nu} - \frac{\partial h^+}{\partial \nu}\right) d\mathcal{H}^{n-1}\lfloor\Sigma = 0. \tag{3.9}$$

The last equality follows from (3.5). Thus, by a simple calculation, we get

$$-\Delta v + v = 2c_*\chi_{\Omega^-}. \tag{3.10}$$

By (3.7), we have $0 < v < c_*$ in Ω^+, $v = c_*$ on Σ, $c_* < v < 2c_*$ in Ω^-. Therefore, $\Omega^- = \{v > c_*\}$. Dividing both sides of (3.10) by c_*, we can assume that $c_* = 1$ and the equation (3.10) can be rewitten as

$$-\Delta v + v = 2\chi_{\{v>1\}}. \tag{3.11}$$

The assertion of Theorem 1.2 (ii) with the thinness assumption is now a consequence of the following

Proposition 3.1. *If either* $\lambda_1(\Omega) > 1$ *or* $diam(\Omega)\,|\Omega|^{1/2} \le 2\pi^{1/2}$ *then the equation*

$$\begin{cases} -\Delta v + v = 2\chi_{\{v>1\}} & in\ \Omega \\ v = 0 & on\ \partial\Omega. \end{cases} \tag{3.12}$$

has no nontrivial solution.

Proof. Suppose by contradition that there exists such a nontrivial solution. Then the set $w = \{v > 1\}$ has positive Lebesgue measure. By the maximum principle, $v > 0$ in Ω.
Case 1: $\lambda_1(\Omega) > 1$. Let $\varphi > 0$ be an eigenvector associated with $\lambda := \lambda_1(\Omega)$. Then,

$$\begin{aligned} \int_w 2\varphi dx = \int_\Omega 2\chi_{\{v>1\}}\varphi dx &= \int_\Omega (-\Delta v + v)\varphi dx = \int_\Omega (-\Delta\varphi + \varphi)v dx = \int_\Omega (\lambda+1)\varphi v dx \\ &\ge \int_w (\lambda+1)\varphi v dx \ge (\lambda+1)\int_w \varphi dx > 2\int_w \varphi dx. \end{aligned}$$

This is impossible because $\varphi > 0$ and $|w| > 0$.
Case 2: $\mathrm{diam}(\Omega)\,|\Omega|^{1/2} \le 2\pi^{1/2}$. In this case v solves the equation $-\Delta v = f$ where $f = 2\chi_{\{v>1\}} - v$. Observe that $\|f\|_{L^\infty(\Omega)} \le 1$. Therefore, $v \in C^0(\overline{\Omega}) \cap W^{2,2}_{loc}(\Omega)$ with $\sup_\Omega v > 1$ by the nontriviality of v. We are going to use the following weak maximum principle of A. D. Aleksandrov to find a contradiction:

Lemma 3.1. *([7], Lemma 9.3) For* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, *we have*

$$\sup_\Omega u \le \sup_{\partial\Omega} u + \frac{diam(\Omega)}{2\pi^{1/2}} \|\Delta u\|_{L^2(\Gamma_u^+)}\,. \tag{3.13}$$

In the above lemma, Γ_u^+ is the *upper contact set* of the continuous function u. It is defined as the subset of Ω where the graph of u lies below a support hyperplane in $\mathbb{R}^3$, i.e.

$$\Gamma_u^+ = \{y \in \Omega \mid u(x) \le u(y) + p.(x-y) \text{ for all } x \in \Omega \text{ for some } p = p(y) \in R^2\}.$$

In view of (3.13) and the fact that $\Gamma_u^+ \subset \Omega$, one finds that for $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

$$\sup_\Omega u \le \sup_{\partial\Omega} u + \frac{\mathrm{diam}(\Omega)}{2\pi^{1/2}} \|\Delta u\|_{L^2(\Omega)}\,. \tag{3.14}$$

By an approximation argument, we can extend the above inequality to functions $u \in C^0(\overline{\Omega}) \cap W^{2,2}_{loc}(\Omega)$. Now applying (3.14) to our function v, we obtain

$$1 < \sup_\Omega v \le \sup_{\partial\Omega} v + \frac{\mathrm{diam}(\Omega)}{2\pi^{1/2}} \|\Delta v\|_{L^2(\Omega)} = \frac{\mathrm{diam}(\Omega)}{2\pi^{1/2}} \|f\|_{L^2(\Omega)} \le \frac{\mathrm{diam}(\Omega)}{2\pi^{1/2}} |\Omega|^{1/2} \le 1,$$

which is clearly a contradition and this completes the proof of Proposition 3.1. □

Remark 3.1. *For the unit disc* B_1, *we have* $\lambda_1(B_1) = (J_0(1))^2 > 5$ *where* $J_0(x)$ *is the Bessel function of the first kind.*

The assertion of Theorem 1.2 (iii) is confirmed by the following

Proposition 3.2. *For any disc domain $\Omega = B_R$, the equation (3.12) has no nontrivial solution such that $w = \{v > 1\}$ is a disc $B_r(x_0)$.*

Proof. If w is a disc, then from Theorem 5 in Sirakov [13], it follows that Ω and w are concentric. Therefore, $x_0 = 0$ and $w = B_r$. It is now convenient to switch back to the potential function h. Because h^- and h^+ are respectively the solutions to the equation $-\Delta h + h = 0$ in B_r and $B_R \backslash B_r$, they must be radial: $h^-(x) = g^-(|x|)$ and $h^+(x) = g^+(|x|)$. For radial functions $h(x) = g(|x|)$, the equation $-\Delta h + h = 0$ reads

$$-g^{''}(|x|) - \frac{g^{'}(|x|)}{|x|} + g(|x|) = 0.$$

This equation can be solved with the help of *the modified Bessel functions* as in Aydi [2]. We define I_0 and K_0 to be respectively the modified Bessel function of the first kind and of the second kind:

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2 2^{2n}}, \quad K_0(x) = -(log(\frac{x}{2}) + \gamma) I_0(x) + \sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2 2^{2n}} \phi(n),$$

where $\phi(n) = \sum_{k=1}^{n} \frac{1}{k}$ for $n \neq 0$, $\phi(0) = 0$, and $\gamma = \lim_{n \to +\infty} (\phi(n) - \log n)$. These functions are continuous solutions to the ODE

$$-y^{''} - \frac{y^{'}}{x} + y = 0 \text{ for } 0 \leq x < \infty$$

with a singularity at 0 for K_0. See Watson [14] for the alternate definitions and properties. Define I_1 and K_1 to be respectively the derivative of I_0 and $-K_0$.

Now, we can use the modified Bessel functions to find h^-, g^-, h^+ and g^+. Clearly, they have the forms

$$h^-(x) = g^-(|x|) = aI_0(|x|) \text{ in } B_r, \quad h^+(x) = g^+(|x|) = bI_0(|x|) + cK_0(|x|) \text{ in } B_R \backslash B_r$$

where a, b, c are constants to be determined. Let us use the compatibility conditions on $\Sigma = \partial B_r$ to find the relations between a, b, c. First, by the continuity of the function h on Σ, we require that $g^-(r) = g^+(r)$, which is equivalent to

$$aI_0(r) = bI_0(r) + cK_0(r).$$

Second, by the jump condition (3.5) of the normal derivative of the function h on Σ, we must have $\frac{\partial g^-}{\partial \nu}(r) = -\frac{\partial g^+}{\partial \nu}(r)$ which is equivalent to

$$aI_1(r) = -bI_1(r) + cK_1(r).$$

Solving a, b in terms of c from these equations, we obtain

$$a = \frac{c}{2}\left(\frac{K_1(r)}{I_1(r)} + \frac{K_0(r)}{I_0(r)}\right), \; b = \frac{c}{2}\left(\frac{K_1(r)}{I_1(r)} - \frac{K_0(r)}{I_0(r)}\right). \tag{3.15}$$

Observe the following property of Bessel functions

$$\frac{d}{dx}(-K_0(x)I_0(x)) = K_1(x)I_0(x) - I_1(x)K_0(x) > 0 \text{ for all } x > 0.$$

Thus, a, b, c have the same sign. Because $h^- > 0$, it follows that $a, b, c > 0$. Thus $h^+(R) > 0$. This is a contradiction with the zero Dirichlet boundary condition of h. □

Finally, we prove part (iii) of Corollary 1.1.
(iii) Suppose otherwise that μ is a negative Radon measure. Then we must have $c_* < 0$. Indeed, by (3.6) and the negativity of μ, we find that $\frac{\partial h^-}{\partial \nu} < 0$. Therefore, h^- achieves its maximum at a point x_0 inside Ω^-. Note also that h^- satisfies $-\Delta h^- + h^- = 0$ in Ω^-. Consequently, $c_* < h^-(x_0) = \Delta h^-(x_0) \leq 0$. Consider the function h^+. It satisfies $-\Delta h^+ + h^+ = 0$ in the domain Ω^+ and has boundary values $h^+ = c_* < 0$ on Σ and $h^+ = 1$ on $\partial\Omega$. Therefore, h^+ is smooth in $\overline{\Omega^+}$ and we can find a smooth closed curve $\Sigma' \subset \{x \in \Omega^+ \mid h^+(x) = 0\}$ in Ω^+. Let Ω' be the subdomain of Ω with boundary $\partial\Omega' = \Sigma'$. Then, by the thinness assumption on Ω, we also have the thinness property of Ω', i.e, $\lambda_1(\Omega') > 1$ or $\text{diam}(\Omega') \left|\Omega'\right|^{1/2} \leq 2\pi^{1/2}$. Now, consider the function $h' = h \mid_{\Omega'}$. It solves

$$\begin{cases} -\Delta h' + h' = \mu \neq 0 & \text{in } \Omega' \\ h' = 0 & \text{on } \partial\Omega \end{cases}$$

with div $T' = 0$ where T' is the stress-energy tensor associated with h' defined similarly as in (1.7). This is impossible by Theorem 1.2 (ii) proved above. Thus μ must be a postive Radon measure. The proof of Corollary 1.1 is now complete. □

4 Perspectives and Open Problems

4.1 Optimal Regularity and Maximum Principle

It was proved in [10] that if the limiting vorticity μ is in $L^p(\Omega)$ for some $p > 1$ then the function h solving (1.1)-(1.2) is in $C^{1,\alpha}(\Omega)$ for all $\alpha < 1$, and furthermore $0 \leq h \leq 1$ if its boundary value is 1. These conclusions are not true anymore if μ is only in $H^{-1}(\Omega)$. The Lipschitz continuity of h in this case is optimal as can be seen from the jump relation (3.5) in the normal derivatives of h on each side of Σ. However, the reflected graph of h is $C^{1,\alpha}(\Omega)$ for all $\alpha < 1$. Indeed, this reflected graph is just the graph of the function v defined in (3.8). Because v solves $-\Delta v + v = 2c_*\chi_{\Omega^-}$ in Ω with constant boundary value $v = c$ on $\partial\Omega$, by standard L^p-estimates, we have $v \in W^{2,p}(\Omega)$ for all $p > 1$. By Sobolev embedding, we conclude that $v \in C^{1,\alpha}(\Omega)$ for all $\alpha < 1$.

Now, we show by an example that the bounds $0 \leq h \leq 1$ fail for $\mu \in H^{-1}(\Omega)$. We prove the following

Proposition 4.1. *Let $\Omega = B_1$. Then there exists a largest number $M > 1$ such that for all $c_* \in (\frac{1}{I_0(1)}, M]$, we can find $r \in (0, 1)$ and a vorticity measure μ concentrated on ∂B_r with* *the following property: the function h_μ solving (1.1)- (1.2) with $c \equiv 1$ satisfies $h_\mu = c_*$ on ∂B_r.*

Proof. We use the same notations as in the proof of Proposition 3.2. We are looking for a function h of the form

$$h^-(x) := h(x) = aI_0(|x|) \text{ in } B_r, \quad h^+(x) := h(x) = bI_0(|x|) + cK_0(|x|) \text{ in } B_1\backslash B_r$$

where a, b, c, r are to be determined such that

$$aI_0(r) = bI_0(r) + cK_0(r) = c^*, \; \frac{\partial h^-}{\partial \nu}(r) = -\frac{\partial h^+}{\partial \nu}(r), \; bI_0(1) + cK_0(1) = 1. \tag{4.1}$$

As in (3.15), we now solve b, c in terms of a from the first two equations in (4.1) to obtain

$$\begin{cases} b = a\dfrac{K_1(r)I_0(r) - I_1(r)K_0(r)}{I_0(r)K_1(r) + K_0(r)I_1(r)} = \dfrac{c^*}{I_0(r)}\dfrac{K_1(r)I_0(r) - I_1(r)K_0(r)}{I_0(r)K_1(r) + K_0(r)I_1(r)}, \\ c = 2a\dfrac{I_0(r)I_1(r)}{I_0(r)K_1(r) + K_0(r)I_1(r)} = \dfrac{2c^*}{I_0(r)}\dfrac{I_0(r)I_1(r)}{I_0(r)K_1(r) + K_0(r)I_1(r)}. \end{cases}$$

Plugging these values of b and c into the last equation in (4.1), we get

$$q(r) := \frac{1}{I_0(r)}\frac{\{K_1(r)I_0(r) - I_1(r)K_0(r)\}\, I_0(1) + 2I_0(r)I_1(r)K_0(1)}{I_0(r)K_1(r) + K_0(r)I_1(r)} = \frac{1}{c^*}. \tag{4.2}$$

Note that for Modified Bessel functions (see [14], p.80), $I_0(r)K_1(r) + K_0(r)I_1(r) = \frac{1}{r}$. Thus

$$q(r) = r\frac{\{K_1(r)I_0(r) - I_1(r)K_0(r)\}\, I_0(1) + 2I_0(r)I_1(r)K_0(1)}{I_0(r)}.$$

From the graph of q (see Figure 1), the Proposition easily follows; observe that $\min_{0\le r\le 1} q(r) \le$

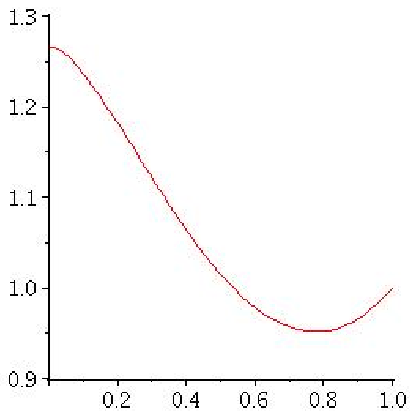

Figure 1: The graph of q

$q(0.8) < 1$ and thus we can choose $M = \frac{1}{\min_{0\le r\le 1} q(r)}$. □

Remark 4.1. *For $c_* \in (1, M)$, there are exactly two pairs (r, μ) where $r \in (0, 1)$ and the vorticity measure μ concentrated on ∂B_r with the property: the function h solving (1.1)-(1.2) with $c \equiv 1$ satisfies $h = c_*$ on ∂B_r. This is a rather unexpected fact.*

4.2 Links to the Euler Equation and Free Boundary Problems

Equation (1.1)-(1.2) serves as a stationary mean-field model (also called hydrodynamic limit) for super-conducting materials and it has many common features with the Euler equations for incompressible flow in fluid mechanics. Consider the 2D Euler equation:

$$\partial_t v + \text{div}\ (v \otimes v) + \nabla p = 0,\ \text{div}\ (v) = 0.$$

Introduce the stream function Ψ such that $v = \nabla^{\perp}\Psi$. Then the vorticity becomes

$$w = \text{curl}\ v = \text{curl}\ (\nabla^{\perp}\Psi) = \Delta\Psi.$$

In our equation, μ plays the role of w and h plays the role of Ψ. If $\mu \in L^p(\Omega)$ for some $p > 1$ then it was proved in [11], Theorem 13.1 that for $c \equiv 1$ we have $0 \leq h_\mu \leq 1$ and μ is a nonnegative L^∞ function: $\mu = h_\mu \chi_{\{|\nabla h_\mu|=0\}}$. Thus μ is constant on each "patch" in Ω. This is reminiscent of the vortex patch problem. In this case, equation (1.2) becomes

$$-\Delta h + h = h\chi_{\{|\nabla h|=0\}}$$

and its free-boundary regularity was investigated by Caffarelli-Salazar-Shahgholian [4].

When the limiting vorticity μ is supported on a curve then it has many common features with its counterpart in the Euler equations: the vortex sheet. For more information on this, we refer to Chapters 9-11 of Majda and Bertozzi [9].

4.3 Open Problems

Our study leaves several open problems.

1. Can we deduce assumption (A) just from the regularity of Σ and the divergence-free property of T?
2. If we assume that μ is supported on just a one-dimensional rectifiable curve, can we obtain similar results? Can we give the best regularity of the curve? Is there an improvement of regularity?
3. Can the support of μ have a noninteger fractional dimension?
4. To our knowledge, not much is known on equations of the form (3.12) although they seem to be interesting on their own. Natural questions that arise are:

- When the domain Ω is not thin, does (3.12) have nontrivial solution?
- If Ω is a ball, is the solution to (3.12) radially symmetric? If this is the case then Proposition 3.1 is still true without the thinness assumption on $\Omega = B_R$.

Recall that for an equation of the form $-\Delta u = f(u)$ in B_R with zero Dirichlet boundary condition, if either f has some suitable continuity (see, e.g. [5]) or f has a definite sign (see, e.g. [8]) then u is radially symmetric. In our problem, $f(u) = 2\chi_{\{u>1\}} - u$ is neither continuous nor of a definite sign.

- What is the optimal regularity of the free boundary $\{v = 1\}$?

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