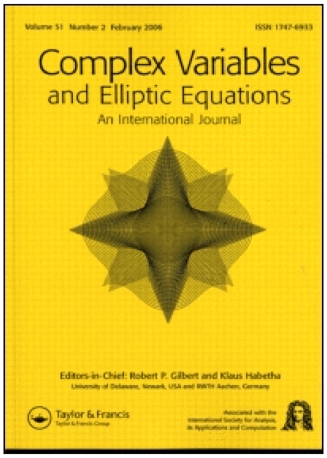

Complex Variables and Elliptic Equations

An International Journal

Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713455999

Reconstruction of $\mathbf{H}^{\mathbf{p}}$-Functions: Best Approximation, Regularization and Optimal Error Estimates

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To cite this Article: Trong, Dang Duc, Luc, Nguyen Le, Nam, Le Quang and Tuyen, Truong Trung (2004) 'Reconstruction of $\mathbf{H}^{\mathbf{p}}$-Functions: Best Approximation, Regularization and Optimal Error Estimates', Complex Variables and Elliptic Equations, 49:4, 285 - 301
To link to this article: DOI: 10.1080/0278107042000220267
URL: http://dx.doi.org/10.1080/02781070420002200267

Complex Variables, Vol. 49, No. 4, 15 March 2004, pp. 285–301

Taylor & Francis
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Reconstruction of H^p-Functions: Best Approximation, Regularization and Optimal Error Estimates

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Communicated by R.P. Gilbert

(Received 25 November 2003; In final form 24 March 2004)

Let U be the open unit disc of the complex plane. We explicitly construct the best pointwise approximation for determining a function in the Hardy space $H^p(U)$ from measured data at a countable set of points in U whenever the data is exact. A regularization scheme is given to deal with the case of nonexact data. Moreover, optimal error estimates are also studied.

Keywords: Hardy space; Ill-posed problem; Best pointwise approximation; Regularization;
Optimal error estimate; Blaschke condition

AMS Subject Classifications (2000): 30D55; 30E05; 30E10; 65J20

1 INTRODUCTION

Let $U = \{z \in \boldsymbol{C}\colon |z| < 1\}$ be the open unit disc of the complex plane $\boldsymbol{C}$ and, for $1 \le p \le \infty$, let $H^p(U)$ be the Hardy space on U (which will be denoted as H^p in what follows), i.e., the space of all analytic functions f on U with the norm

$$\|f\|_p = \lim_{r\uparrow 1}\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right\}^{1/p} < \infty.$$

We deal with the problem of reconstructing an H^p-function u from its values $\mu_1, \mu_2, \ldots$ at given distinct points $z_1, z_2, \ldots$ in U:

$$u(z_n) = \mu_n, \quad n = 1, 2, \ldots \tag{1}$$

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ISSN 0278-1077 print: ISSN 1563-5066 online © 2004 Taylor & Francis Ltd
DOI: 10.1080/0278107042000220267

We require that the points $z_1, z_2, \ldots$ satisfy the Blaschke condition

$$\sum_{n=1}^{\infty}(1 - |z_n|) = \infty. \tag{2}$$

By condition (2), problem (1) admits at most one solution in a Hardy space H^p $(1 \le p \le \infty)$. Despite uniqueness, the problem is still ill posed. The literature on the problem of approximation and interpolation of analytic functions from exact data is impressive (see, e.g., [3,4,6,7,9,14,15,16]). However, the ill-posedness of the problem, especially for H^p-functions, is not usually taken into account. In fact, the right-hand side of (1) is usually the result of experimental measurements and is subject to error, i.e., nonexact data (see [1,5] and references therein). Hence, a solution corresponding to the data does not always exist and moreover, solutions, even though they exist, do not depend continuously on the given data. This, of course, makes a numerical treatment impossible. Thus, one has to resort to a regularization. We shall consider the more general problem of finding an H^p-function u satisfying

$$u(z_{nk}) = \mu_{nk}, \quad n = 1, 2, \ldots; \;\; k = 1, \ldots, n, \tag{3}$$

where $\{z_{nk}\}_{n=1,\,1\le k\le n}^{\infty}$ is a given point system in U $(z_{nj} \ne z_{nk}, j \ne k, \; 1 \le j, \; k \le n)$ and $\{\mu_{nk}\}_{n=1,\,1\le k\le n}^{\infty}$ is a given data. If $z_{nk} = z_k$, $\mu_{nk} = \mu_k$ for $n \ge k$ then we get (1). Condition (2) can be rewritten as

$$\lim_{n\to\infty}\sum_{k=1}^{n}(1 - |z_{nk}|) = \infty. \tag{4}$$

The main purpose of our article is to present a regularization of the general problems (3) and (4) based on finite best approximation. As known, the problem of constructing the best approximant of an analytic function from its finite values has been studied extensively over three decades (see, e.g., [2,6–8,12–15]). However, the problem of how these finite approximations, corresponding to nonexact data, are related to the exact solution of the problem is not considered. In our work, we shall give an explicit form of a sequence of functions (not necessarily analytic) that are shown to converge optimally to the solution whenever data is exact, and that represent approximations to the solution whenever the data is nonexact if we stop at an appropriate dimension n of the approximation. In the latter case, we shall find the dimension that makes the error of the approximation be as small as possible. It is worth noting that the present treatment differs from the earlier in that it is more elementary and that the best approximation is constructed very explicitly. The significance of the above discussion is made clearer in the following sections.

The remainder of our article consists of three sections. In Section 2, we consider some preliminary results in the case of exact data and construct best approximations to H^p-functions. In Section 3, to deal with the case of nonexact data, we propose a regularization scheme and estimate the error of the approximation. In the final section, we prove a result on optimal error estimates.

2 PRELIMINARY RESULT: THE CASE OF EXACT DATA

In this section, we consider the problem of reconstructing an H^p-function from the exact data on its values at a countable set of points. We shall construct best pointwise approximations for determining the unknown function (Theorem 1) and propose a distribution of the set of points at which the measurements are performed to get small errors of the approximation (Theorem 2). Both results are fundamental for the development of Sections 3 and 4 which deal with regularization. Note that the results of this section are also of independent interest.

Before stating and proving Theorem 1, several literature reviews and remarks are in order. Best approximation of functions in the disc algebra by rational functions was considered by Shen and Lou in [12]. The problem of optimal recovery of a holomorphic function in the unit ball B_n of C^n when its values at the intersection of B_n with an affine subspace of C^n are given was considered in great details by Osipenko and Stessin [7] (see also [8]). Here, explicit formulae for the intrinsic error, worst function and optimal algorithm were given. On the other hand, when there are explicit rational approximants, many authors just showed the convergence of these approximants or gave the upper bound for the error, not the precise error; see [10,13]. Recently, best approximation of functions in H_q^p $(p \geq 1,\ q > 1)$ by rational functions with fixed poles was investigated by Xing and Jin [15]. The upper bound for the error of the best approximation was also given. However, all these results must have undergone nontrivial changes when applied to solve the problems of regularization and optimal error estimates. Thus, Theorem 1 in our present article, though whose result is not completely new but has the explicit formulae of the recovery coefficients together with the precise intrinsic error of the approximation scheme, will help directly solve these problems.

Throughout this section, unless specified, p is in the interval $[1, \infty]$. For fixed n, let $B_n(z)$ and $B_{n,j}(z)$ $(1 \leq k \leq n)$ be products

$$B_n(z) = \prod_{k=1}^{n} \frac{z - z_{nk}}{1 - \overline{z_{nk}} z}, \qquad B_{n,j}(z) = \prod_{k=1, k \neq j}^{n} \frac{z - z_{nk}}{1 - \overline{z_{nk}} z}.$$

For each $z \in U$, the function $k_z(\zeta) = 1 - \bar{z}\zeta$ has no zero in U and since U is simply connected, there exists an analytic function $\varphi \in H(U)$ such that $e^{\varphi} = k_z$ [11, Theorem 13.18, pp. 262–263]. For each real number θ, by $k_z^{\theta}(\zeta) = (1 - \bar{z}\zeta)^{\theta}$ we mean the function $e^{\theta\varphi(\zeta)}$. We have the following best pointwise approximation of H^p-functions.

Theorem 1 *Let $z_{n1}, \ldots, z_{nn}$ be n distinct points in U. For each $f \in H^p$, put*

$$T_n(f) = (f(z_{n1}), \ldots, f(z_{nn})).$$

Then for any $z \in U$, we have

$$|f(z) - R_n(T_n(f), z)| \leq \frac{\|f\|_p}{(1 - |z|^2)^{1/p}} |B_n(z)|, \tag{5}$$

where

$$R_n(\nu, z) = \sum_{k=1}^{n} c_{p,n,k}(z)\nu_k \quad \textit{for } \nu = (\nu_1, \ldots, \nu_n),$$

and

$$c_{p,n,k}(z) = \frac{1-|z_{nk}|^2}{z-z_{nk}}\left(\frac{1-\bar{z}z_{nk}}{1-|z|^2}\right)^{(2/p)-1}\frac{B_n(z)}{B_{n,k}(z_{nk})}.$$

Moreover, if $\rho > 0$ *and* ψ *is any mapping from* $\mathbf{C}^n \times U$ *to* $\mathbf{C}$ *then*

$$\sup_{\|w\|_p \le \rho} |w(z) - \psi(T_n(w), z)| \ge \sup_{\|w\|_p \le \rho} |w(z) - R_n(T_n(w), z)| = \frac{\rho}{(1-|z|^2)^{1/p}}|B_n(z)|. \tag{6}$$

Furthermore, if the sequence $\{z_{nk}\}$ *satisfies the condition* (4) *then*

$$\lim_{n\to\infty}\frac{\rho}{(1-|z|^2)^{1/p}}|B_n(z)| = 0 \quad \textit{for all } z \in U. \tag{7}$$

Remarks

1. The inequality (6) shows that $R_n(T_n(f), z)$ is the best approximation of the function $f \in H^p$ at z. The quantity $e(\rho, n) = (\rho/(1-|z|^2)^{1/p})|B_n(z)|$ expresses the precision of the best approximation.
2. If $p \neq 2$, then $R_n(T_n(f), z)$ is not an analytic function of z and hence it is not an element of H^p. This is a rather surprising fact (see also [6]).

Outline of the Proof of Theorem 1 Let $\partial U = \{e^{it}: 0 \le t < 2\pi\}$ be the boundary of U. Consider the contour integral

$$I_p(z) = \frac{1}{2\pi i}\int_{\partial U}\frac{f(\zeta)}{\zeta - z}\frac{B_n(z)}{B_n(\zeta)}\left(\frac{1-\bar{z}\zeta}{1-|z|^2}\right)^{(2/p)-1} d\zeta.$$

Using the Residue Theorem, we find that

$$I_p(z) = f(z) - \sum_{k=1}^{n} c_{p,n,k}(z) f(z_{nk}) = f(z) - R_n(T_n(f), z). \tag{8}$$

Now we prove (5). Let q be the conjugate of p, i.e., $(1/p) + (1/q) = 1$. In view of the fact that $|(1-\bar{z}\zeta)/(\zeta - z)| = 1$ for all $z \in U$ and $\zeta \in \partial U$ and by Holder's inequality, it implies that

$$\begin{aligned}
|I_p(z)| &\le (1-|z|^2)^{1-(2/p)}|B_n(z)|\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(e^{it})}{B_n(e^{it})}\right|^p dt\right\}^{1/p}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{|1-\bar{z}e^{it}|^{(2/p)-1}}{|e^{it}-z|}\right]^q dt\right\}^{1/q}\\
&= (1-|z|^2)^{1-(2/p)}|B_n(z)|\,\|f\|_p\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-\bar{z}e^{it}|^2}\,dt\right\}^{1-(1/p)}\\
&= (1-|z|^2)^{-1/p}\|f\|_p\,|B_n(z)|,
\end{aligned}$$

i.e., (5) holds.

Now, we consider (6). Denote by B_ρ the ball in H^p of radius ρ centered at 0 in H^p. Since $\operatorname{Ker} T_n \cap B_\rho \subset B_\rho$ and since $w(z_{n1}) = \cdots = w(z_{nn}) = 0$ for $w \in \operatorname{Ker} T_n$, we have

$$\sup_{w\in B_\rho} |w(z) - \psi(T_n(w), z)| \geq \sup_{w\in \operatorname{Ker} T_n \cap B_\rho} |w(z) - \psi(0, z)|. \tag{9}$$

But $e^{i\theta} w \in \operatorname{Ker} T_n \cap B_\rho$, for $w \in \operatorname{Ker} T_n \cap B_\rho$. Hence

$$\begin{aligned}\sup_{w\in \operatorname{Ker} T_n\cap B_\rho} |w(z) - \psi(0,z)| &= \sup_{w\in \operatorname{Ker} T_n\cap B_\rho} \sup_{\theta\in R} |e^{i\theta} w(z) - \psi(0,z)| \\ &= \sup_{w\in \operatorname{Ker} T_n\cap B_\rho} (|w(z)| + |\psi(0,z)|) \\ &\geq \sup_{w\in \operatorname{Ker} T_n\cap B_\rho} |w(z)| \geq |g(z)| \\ &= \frac{\rho}{(1-|z|^2)^{1/p}} |B_n(z)|, \end{aligned} \tag{10}$$

where $g(\zeta) = (\rho(1-|z|^2)^{1/p}/(1-\bar{z}\zeta)^{2/p})\, B_n(\zeta)$ (note that $\|g\|_p = \rho$ and $g \in \operatorname{Ker} T_n$). From (9) and (10), we have, for all $\psi : \boldsymbol{C}^n \times U \to \boldsymbol{C}$,

$$\sup_{w\in B_\rho} |w(z) - \psi(T_n(w), z)| \geq \frac{\rho}{(1-|z|^2)^{1/p}} |B_n(z)|.$$

Especially, we have

$$\sup_{w\in B_\rho} |w(z) - R_n(T_n(w), z)| \geq \frac{\rho}{(1-|z|^2)^{1/p}} |B_n(z)|.$$

Combining the latter inequality with (5), we get the inequality in (6).

For the proof of (7), we estimate $|(z-\alpha)/(1-\bar{\alpha}z)|$ for $z, \alpha \in U$. We have

$$1 - \left|\frac{z-\alpha}{1-\bar{\alpha}z}\right|^2 = \frac{(1-|z|^2)(1-|\alpha|^2)}{|1-z\bar{\alpha}|^2} \geq \frac{(1-|z|^2)(1-|\alpha|^2)}{(1+|\alpha|)^2} \geq \frac{(1-|z|^2)(1-|\alpha|)}{2}.$$

It follows that

$$1 - \left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| \geq \frac{(1-|z|^2)(1-|\alpha|)}{2(1+(|(z-\alpha)/(1-\bar{\alpha}z)|))} \geq \frac{1}{4}(1-|z|^2)(1-|\alpha|),$$

and

$$\left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| \leq \exp\Big(\left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| - 1\Big) \leq \exp\left(-\frac{1-|z|^2}{4}(1-|\alpha|)\right).$$

Now, (7) follows from the inequality

$$|B_n(z)| \leq \exp\left(-\frac{1-|z|^2}{4}\sum_{k=1}^{n}(1-|z_{nk}|)\right), \tag{11}$$

and the condition (4). The proof of the theorem is completed. ■

The error estimate given by (5) is sharp for fixed $z \in U$. However, in reality, we are only given the points $z_{n1}, \ldots, z_{nn}$ at which the measurements are performed, so it is essential to consider the global error estimate

$$\sup_{|z|\leq R}\frac{\|f\|_p}{(1-|z|^2)^{1/p}}|B_n(z)| \equiv \frac{\|f\|_p}{(1-R^2)^{1/p}}\sup_{|z|=R}|B_n(z)|. \tag{12}$$

Hence, we need to estimate the minimum of the quantity $|B(z;z_{n1},\ldots,z_{nn})| \equiv |B_n(z)|$ for points $z_{n1},\ldots,z_{nn}$ satisfying certain conditions. The following theorem suggests a model for such $z_{n1},\ldots,z_{nn}$.

THEOREM 2 *Let* $R, r, r_1, \ldots, r_n$ *be in* $(0,1)$. *Then*

$$\sup_{|z|=R,\ |z_{nj}|=r_j}|B_n(z;z_{n1},\ldots,z_{nn})| \geq \frac{1}{2}\prod_{j=1}^{n}\left\{\frac{R^n+r_j^n}{1+R^n r_j^n}\right\}^{1/n}. \tag{13}$$

If $\zeta_1,\ldots,\zeta_n$ *are vertices of a regular n-gon on the circle of radius r centered at the origin of the complex plane* $\boldsymbol{C}$ *then*

$$\sup_{|z|=R}|B_n(z;\zeta_1,\ldots,\zeta_n)| \leq 2\inf_{|z_{n1}|=\cdots=|z_{nn}|=r}\ \sup_{|z|=R}|B_n(z;z_{n1},\ldots,z_{nn})|. \tag{14}$$

Remark Inequality (14) implies that the distribution of $\zeta_1,\ldots,\zeta_n$ in U is near-optimal.

Proof of Theorem 2 First, we note that, for $|z| = R$, we have

$$|B_n(z;\zeta_1,\ldots,\zeta_n)| = \prod_{k=1}^{n}\left|\frac{z-\zeta_k}{1-\overline{\zeta_k}z}\right| = \left|\frac{z^n-r^n}{1-r^n z^n}\right| \leq \frac{|z|^n+r^n}{1+r^n|z^n|} = \frac{R^n+r^n}{1+r^n R^n}.$$

The equality occurs if z is an nth root of $(-R^n)$. Hence, the proof of the theorem will be completed once (13) is proved. In fact, for $z_{n1},\ldots,z_{nn}$ satisfying $|z_{n1}| = r_1,\ldots,|z_{nn}| = r_n$, put

$$a = \sup_{|z|=R}\prod_{j=1}^{n}\left|\frac{z-z_{nj}}{1-\overline{z_{nj}}z}\right|.$$

Assuming that $z_{nj} = r_j e^{i\theta_j}$ $(j = 1,\ldots,n)$, we put $k_{mj} = 2\pi\lfloor\theta_j m/2\pi\rfloor, z^*_{mj} = r_j e^{ik_{mj}/m}$, where $\lfloor a\rfloor$ denotes the greatest integer not exceeding a. Then $(z^*_{mj})^m = r_j^m$ and $z^*_{mj} \to z_{nj}$ as $m \to \infty$, for $j = 1,\ldots,n$. Put

$$a^*_m = \sup_{|z|=R}\prod_{j=1}^{n}\left|\frac{z-z^*_{mj}}{1-\overline{z^*_{mj}}z}\right|.$$

Letting ζ be $e^{2\pi i/m}$, for all $k = 1, \ldots, m$, we have

$$a_m^* = \sup_{|z|=R} \prod_{j=1}^{n} \left| \frac{z - \zeta^k z_{mj}^*}{1 - \overline{\zeta^k z_{mj}^*} z} \right|.$$

Thus

$$a_m^* \geq \sup_{|z|=R} \sqrt[m]{\prod_{k=1}^{m} \prod_{j=1}^{n} \left| \frac{z - \zeta^k z_{mj}^*}{1 - \overline{\zeta^k z_{mj}^*} z} \right|} = \sup_{|z|=R} \sqrt[m]{\prod_{j=1}^{n} \prod_{k=1}^{m} \left| \frac{z - \zeta^k z_{mj}^*}{1 - \overline{\zeta^k z_{mj}^*} z} \right|}$$

$$= \sup_{|z|=R} \prod_{j=1}^{n} \left| \frac{z^m - r_j^m}{1 - z^m r_j^m} \right|^{1/m} = \prod_{j=1}^{n} \left\{ \frac{R^m + r_j^m}{1 + R^m r_j^m} \right\}^{1/m}$$

$$\geq \prod_{j=1}^{n} \max\{R, r_j\} \geq \frac{1}{2} \prod_{j=1}^{n} \left\{ \frac{R^n + r_j^n}{1 + R^n r_j^n} \right\}^{1/n}.$$

Letting $m \to \infty$, we obtain

$$a = \lim_{m \to \infty} a_m^* \geq \frac{1}{2} \prod_{j=1}^{n} \left\{ \frac{R^n + r_j^n}{1 + R^n r_j^n} \right\}^{1/n},$$

proving (13) and completing the proof. ∎

3 THE CASE OF NONEXACT DATA: REGULARIZATION AND ERROR ESTIMATE

In this section, we deal with the problems (3) and (4) in the case of $\mu = \{\mu_{nk}\}_{k=1}^{n}$ being the nonexact data, i.e., $\mu_{nk} \neq u(z_{nk})$ for some positive integers n and $1 \leq k \leq n$. We approximate the exact solution of (3) and (4) by $R_n(\mu, z)$. Then the approximating function $R_n(\mu, z)$ differs from the exact solution by a larger quantity illustrated in the following theorem.

THEOREM 3 *Let $\epsilon > 0$ and $z_{n1}, \ldots, z_{nn}$ be n distinct points in U. Let $\mu^0 = \{\mu_{nk}^0\}_{k=1}^{n}$ be a finite sequence of complex numbers and denote by S the set of all finite sequences μ of complex numbers $\mu = \{\mu_{nk}\}_{k=1}^{n}$ such that*

$$\|\mu - \mu^0\|_\infty := \max_{1 \leq k \leq n} |\mu_{nk} - \mu_{nk}^0| \leq \epsilon. \tag{15}$$

Then, for all r in $(0, 1)$ and $z \in U$, we have

$$\sup_{\mu \in S} |R_n(\mu, z) - u^0(z)| = \epsilon \sum_{k=1}^{n} |c_{p,n,k}(z)| + \frac{\|u^0\|_p}{(1 - |z|^2)^{1/p}} |B_n(z)|, \tag{16}$$

and

$$\sup_{|z|\le r,\,\mu\in S}|R_n(\mu,z)-u^0(z)| \le \epsilon \sup_{|z|\le r}\left(\sum_{k=1}^{n}|c_{p,n,k}(z)|\right)+\frac{\|u^0\|_p}{(1-r^2)^{1/p}}\sup_{|z|\le r}|B_n(z)|, \tag{17}$$

where $u^0 \in H^p$ *is the unique solution of* (3) *corresponding to* μ^0, *i.e.*,

$$u^0(z_{nk}) = \mu^0_{nk} \quad (1\le k\le n).$$

Proof of Theorem 3 In view of the relation $R_n(\mu,z)=R_n(\mu-\mu^0,z)+R_n(\mu^0,z)$ and the recovery of H^p-functions as in Theorem 1, we see that, for $z\in U$

$$\begin{aligned}|R_n(\mu,z)-u^0(z)| &\le |R_n(\mu-\mu^0,z)|+|R_n(\mu^0,z)-u^0(z)|\\ &= \left|\sum_{k=1}^{n}c_{p,n,k}(z)(\mu_{nk}-\mu^0_{nk})\right|+|R_n(\mu^0,z)-u^0(z)|\\ &\le \epsilon\sum_{k=1}^{n}|c_{p,n,k}(z)|+\frac{\|u^0\|_p}{(1-r^2)^{1/p}}|B_n(z)|.\end{aligned}$$

The above estimate combined with the sharpness of the inequality (5) and a suitable choice of μ such that $\|\mu-\mu^0\|_\infty\le\epsilon$ yields (17) and (16). ■

Since the problem (3) is ill posed, we cannot use all the experimental data to exactly recover the unknown function. As shown in the estimates (17) and (18), if n is large then so is the error. In fact, using the inequality

$$\left|\frac{\alpha-\beta}{1-\overline{\alpha}\beta}\right|\le\frac{|\alpha|+|\beta|}{1+|\alpha|\,|\beta|}\quad\text{for all } \alpha,\beta\in U,$$

estimating directly the right-hand side of (17), we get

$$\sup_{|z|\le r}|R_n(\mu,z)-u^0(z)|\le\epsilon\beta_n+\alpha_n, \tag{18}$$

where

$$\alpha_n=\frac{\|u^0\|_p}{(1-r^2)^{1/p}}\prod_{k=1}^{n}\frac{r+|z_{nk}|}{1+r|z_{nk}|}, \tag{19}$$

$$\beta_n=A(p,r)\sum_{k=1}^{n}\left\{(1-|z_{nk}|^2)(1-r|z_{nk}|)^{(2-2p)/p}\prod_{j=1,\,j\ne k}^{n}\left(\frac{r+|z_{nj}|}{1+r|z_{nj}|}\left|\frac{1-\overline{z_{nj}}z_{nk}}{z_{nk}-z_{nj}}\right|\right)\right\}, \tag{20}$$

$$A(p,r)=\begin{cases}1, & \text{if } p\ge 2\\ (1-r^2)^{(p-2)/p}, & \text{if } 1\le p<2.\end{cases}$$

We have to determine the dimension $m(\epsilon, r)$ of the approximation at which we stop so that it can be easily computed and the quantity $\epsilon\beta_n + \alpha_n$ is as small as possible for each fixed ϵ. The following theorem will be useful.

THEOREM 4 *Let* $\epsilon > 0, r \in (0,1)$ *and let* $\{z_{nk}\}$ *be as in Theorem* 3. *Let* $u^0 \in H^p$ *be the unique solution of* (3) *corresponding to* μ^0. *Suppose that* $\lim_{n\to\infty}\beta_n = +\infty$. *Put*

$$m = m(\epsilon, r) = \max\{n \in \mathbb{N} \mid \epsilon\beta_n \le \alpha_n\},$$

where α_n, β_n *are defined in* (19) *and* (20). *Then*

$$\frac{\alpha_{m+1}}{\beta_{m+1}} < \epsilon \le \frac{\alpha_m}{\beta_m}, \qquad \lim_{\epsilon\downarrow 0} m(\epsilon, r) = +\infty, \qquad \lim_{\epsilon\downarrow 0}\alpha_{m(\epsilon,r)} = 0,$$

and

$$\sup_{|z|\le r}|R_{m(\epsilon,r)}(\mu, z) - u^0(z)| \le 2\alpha_{m(\epsilon,r)}. \tag{21}$$

If we assume, in addition, that

$$\beta_1 < \beta_2 < \beta_3 \cdots \quad \textit{and} \quad \alpha_1 > \alpha_2 > \alpha_3 > \cdots \tag{22}$$

then, for $\epsilon > 0$ *sufficiently small, we have*

$$\sup_{|z|\le r}|R_{n(\epsilon,r)}(\mu, z) - u^0(z)| \le 2\inf_{n\in\mathbb{N}}(\epsilon\beta_n + \alpha_n), \tag{23}$$

where

$$n(\epsilon, r) = m(\epsilon, r) \qquad \textit{if } \frac{\alpha_m}{\beta_m} \ge \epsilon \ge \frac{\alpha_m}{\beta_{m+1}} \tag{24}$$

$$= m(\epsilon, r) + 1 \quad \textit{if } \frac{\alpha_m}{\beta_{m+1}} > \epsilon > \frac{\alpha_{m+1}}{\beta_{m+1}}. \tag{25}$$

Remarks

1. If $\{\beta_n\}$ is bounded then by (18), $\lim_{n\to\infty}\sup_{|z|\le r}|R_n(\mu, z) - u^0(z)| \le \epsilon\sup_n\beta_n$, i.e., we have the stability of the approximations.
2. If we consider problems (1) and (2) then we have $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$

Proof of Theorem 4 Using (11), we get

$$\alpha_n \le \frac{\|u^0\|_p}{(1-r^2)^{1/p}}\exp\left(-\frac{1-r^2}{4}\sum_{k=1}^{n}(1-|z_{nk}|)\right).$$

Combining the latter inequality with the condition (4) gives

$$\lim_{n\to\infty}\alpha_n = 0. \tag{26}$$

From the definition of $m = m(\epsilon, r)$ one has $(\alpha_{m+1})/(\beta_{m+1}) < \epsilon \leq \alpha_m/\beta_m$, and

$$\sup_{|z|\leq r} |R_{m(\epsilon,r)}(\mu, z) - u^0(z)| \leq \epsilon\beta_{m(\epsilon,r)} + \alpha_{m(\epsilon,r)} \leq 2\alpha_{m(\epsilon,r)}.$$

We claim that $m(\epsilon, r) \to \infty$ as $\epsilon \to 0$. For any $M > 0$, we choose $0 < \epsilon_0 < \min_{1\leq n\leq M}(\alpha_n/\beta_n)$. Then, for $0 < \epsilon < \epsilon_0$, we have $m(\epsilon, r) \geq M$. Hence

$$\lim_{\epsilon\downarrow 0} m(\epsilon, r) = +\infty. \tag{27}$$

From (26) and (27), we get $\lim_{\epsilon\downarrow 0} \alpha_{m(\epsilon,r)} = 0$. On the other hand,

$$E(\epsilon) \equiv \inf_n(\epsilon\beta_n + \alpha_n) \leq \epsilon\beta_{m(\epsilon,r)} + \alpha_{m(\epsilon,r)} \leq 2\alpha_{m(\epsilon,r)}.$$

Therefore $\lim_{\epsilon\downarrow 0} E(\epsilon) = 0$. Hence, for ϵ sufficiently small, we have

$$E(\epsilon) < \alpha_1. \tag{28}$$

Now, we prove (23). Fix $\epsilon > 0$ such that (28) holds. Since $\lim_{n\to\infty} \beta_n = +\infty$, there exists $n_0 \in \mathbb{N}$ such that

$$E(\epsilon) = E_{n_0}(\epsilon) \equiv \epsilon\beta_{n_0} + \alpha_{n_0}. \tag{29}$$

Put $K_1 = \{n \in \mathbb{N} \mid \alpha_n \leq E(\epsilon)\}$. By (29), we have $n_0 \in K_1$. Put $n_1 = \min K_1$. By (28), we have $n_1 > 1$ and therefore

$$\alpha_{n_1} \leq E(\epsilon) < \alpha_{n_1-1}. \tag{30}$$

Since $n_1 \leq n_0$ and since the sequence $\{\beta_n\}$ is strictly increasing, from (29) we obtain $\epsilon\beta_{n_1} \leq \epsilon\beta_{n_0} \leq E(\epsilon)$, and thus, by (30), we get

$$\epsilon\beta_{n_1} < \alpha_{n_1-1}. \tag{31}$$

Now, put $K_2 = \{n \in \mathbb{N} \mid \epsilon\beta_n < \alpha_{n-1}\}$. From (31) we have $K_2 \neq \emptyset$ since $n_1 \in K_2$. Put $n = n(\epsilon, r) = \max K_2$. Since the sequence $\{\alpha_k/\beta_{k+1}\}_{k=1}^{\infty}$ is strictly decreasing and converges to 0, n is just the unique positive integer satisfying the inequality

$$\frac{\alpha_n}{\beta_{n+1}} \leq \epsilon < \frac{\alpha_{n-1}}{\beta_n}. \tag{32}$$

We consider the two cases.

Case 1 $n \leq n_0$ Since $n \leq n_0$, and since the sequence $\{\beta_n\}$ is increasing, we get $\epsilon\beta_n \leq \epsilon\beta_{n_0} \leq E(\epsilon)$. Since $n \geq n_1$, and since the sequence $\{\alpha_k\}_{k=1}^{\infty}$ decreases, we get $\alpha_n \leq \alpha_{n_1} \leq E(\epsilon)$. Thus $E_n(\epsilon) = \epsilon\beta_n + \alpha_n \leq 2E(\epsilon)$.

Case 2 $n > n_0$ In this case, $n-1 \geq n_0$. Thus, we have $\epsilon\beta_n < \alpha_{n-1} \leq \alpha_{n_0} \leq E(\epsilon)$, $\alpha_n \leq \alpha_{n_0} \leq E(\epsilon)$. Hence $E_n(\epsilon) \leq 2E(\epsilon)$.

Finally, we prove (24) and (25). Since

$$\frac{\alpha_m}{\beta_m} \geq \epsilon > \frac{\alpha_{m+1}}{\beta_{m+1}},$$

we have either

$$\frac{\alpha_m}{\beta_m} \geq \epsilon \geq \frac{\alpha_m}{\beta_{m+1}} \quad \text{or} \quad \frac{\alpha_m}{\beta_{m+1}} > \epsilon > \frac{\alpha_{m+1}}{\beta_{m+1}}.$$

If $(\alpha_m/\beta_m) \geq \epsilon \geq (\alpha_m/\beta_{m+1})$ then $(\alpha_m/\beta_{m+1}) \leq \epsilon \leq (\alpha_m/\beta_m) < (\alpha_{m-1}/\beta_m)$. Comparing the latter inequalities with (32) we get, in view of the uniqueness of $n = n(\epsilon, r)$, that $n(\epsilon, r) = m(\epsilon, r)$.

Similarly, if $(\alpha_m/\beta_{m+1}) > \epsilon > (\alpha_{m+1}/\beta_{m+1})$ then $(\alpha_{m+1}/\beta_{m+2}) < \epsilon < (\alpha_m/\beta_{m+1})$. It follows that $m = n-1$. This completes the proof of (24), (25) and the proof of the theorem. ■

The error estimate given by (16) is optimal for fixed z, but we do not know whether the global error estimate given by (17) is optimal. This calls for an investigation of the optimal error estimate in the next section.

4 A RESULT ON OPTIMAL ERROR ESTIMATES

In Section 3, we have used the approximation in the case of exact data to approximate H^p-functions in the case of nonexact data. This, of course, yields large errors. However, as we shall show below that in the case of near-optimal points in which $z_{n1}, \ldots, z_{nn}$ are vertices of a regular n-gon centered at the origin, the error estimate for the case of nonexact data, especially that given by (17), is optimal in an appropriate sense (see Theorem 6). The consideration of this case of near optimality is motivated by Theorem 2.

In this section, for the sake of simplicity and for the clarity in exposition, we consider the case in which $r = 1/2$ and $f \in H^1$. The general case is almost verbatim. Suppose that, if we stop at the dimension n of the approximation, which we shall call the nth step of the approximation process, we have a finite sequence $\{\mu_{nj}\}_{j=1}^n$ of measured values (of course, with noise) of f at the points $z_{nk} = re^{i(\pi/n)2k}$ $(k = 1, 2, \ldots, n)$. For simplicity, we always write $c_{n,k}(z)$ for $c_{1,n,k}(z)$, $z_k = z_{nk}$ and $\mu_k = \mu_{nk}$ $(k = 1, \ldots, n)$ if we are in the nth step of the approximation process. Suppose that the deviation of noise values $\{\mu_j\}_{j=1}^n$ from the exact values $\{f(z_j)\}_{j=1}^n$ is constrained by a positive constant $\epsilon > 0 : \max_{1\leq j\leq n} |\mu_j - f(z_j)| \leq \epsilon$. As in Theorem 1, putting

$$R_n(\mu, z) = \sum_{k=1}^{n} c_{n,k}(z)\mu_k,$$

we first have:

THEOREM 5 *Let* ϵ *be in* $(0,1)$ *and let* $x(\epsilon)$ *be the largest solution of the equation* $\epsilon 2^x - x\ln 2 = 0$. *Put* $n(\epsilon) = \lfloor x(\epsilon)\rfloor$, *where* $\lfloor a\rfloor$ *denotes the greatest integer not exceeding* a. *Then* $\lim_{\epsilon\downarrow 0} x(\epsilon) = +\infty$, $\lim_{\epsilon\downarrow 0} n(\epsilon) = +\infty$, *and*

$$\sup_{|z|\leq r} |R_{n(\epsilon)}(\mu, z) - f(z)| \leq \frac{2}{2^{x(\epsilon)}}(x(\epsilon)\ln(2e^2x(\epsilon))\ln 2 + 4\|f\|_1) \equiv E(f,\epsilon).$$

Remark When there is no noise, using the de la Vallée Poussin means, Totik gave a sharp error estimate in H^p (or in the disc algebra $A(U)$) depending on the smoothness of the analytic function on the boundary ∂U (see Corollary 2 in [14]). By a similar method, we can get a global estimate as in [14].

The proof of Theorem 5 makes use of the following lemmas.

LEMMA 1 *Given a regular n-gon* $A_1 \ldots A_n$ *whose vertices lie on a circle* (C) *of radius r. Denote by* $d(A,B)$ *the Euclidean distance between two points* A, B *of the complex plane. For any point P on* (C), *put*

$$s(P) = \sum_{j=1}^{n} \prod_{k=1,\ k\neq j}^{n} d(P, A_k).$$

Then $s(P)$ *attains its maximum at the midpoint of the small arc* A_iA_{i+1} *of* (C) *for some* $i \in \{1,\ldots,n\}$. *Here, we denote* $A_{n+1} \equiv A_1$.

LEMMA 2 *Let* $z_k = re^{i(\pi/n)2k}$ $(k = 1,\ldots,n)$. *Then for any complex number z satisfying* $|z| = r$ *we have*

$$\sum_{j=1}^{n} \prod_{k=1,\ k\neq j}^{n} |z - z_k| \leq r^{n-1} \sum_{j=1}^{n} \frac{1}{\sin((2j-1)\pi/(2n))}. \tag{33}$$

The equality holds at $z = re^{i(\pi/n)(2k-1)}$.

Lemma 2 is a direct consequence of Lemma 1 and the proof is thus omitted. We only give the proof of Lemma 1 for the case n is an even integer; the case n odd is argued similarly.

Proof of Lemma 1 Without loss of generality, we can (and shall) assume that P lies on the small arc A_1A_2 of (C).

Since n is even, i.e., $n = 2k$ for some $k \geq 2$. We have

$$s(P) = \sum_{j=2}^{k+1} \left\{ (d(P,A_j) + d(P, A_{2k+3-j})) \prod_{l=2,\ l\neq j}^{k+1} (d(P,A_l)d(P,A_{2k+3-l})) \right\}.$$

It can be seen that, in this case, the segments $A_2A_{2k+1}, A_3A_{2k}, \ldots, A_{k+1}A_{k+2}$ are mutually parallel. It follows that each of the $2k$ quantities $d(P,A_j) + d(P,A_{2k+3-j})$, $d(P,A_j)d(P,A_{2k+3-j})$ $(j = 2,\ldots,k+1)$ attains its maximum at the midpoint of the arc A_1A_2. Hence the lemma follows. ■

Proof of Theorem 5 First, we explicitly calculate and estimate $c_{n,k}(z)$. Since z_k $(k = 1, \dots, n)$ are nth roots of r^n, we have

$$B_n(z) = \frac{z^n - r^n}{1 - r^n z^n}, \qquad \prod_{j=1}^{n}(1 - \overline{z_j} z_k) = 1 - r^{2n}, \qquad \prod_{j=1,\, j\neq k}^{n}(z_k - z_j) = n z_k^{n-1}.$$

From the formulae of $c_{p,n,k}(z)$ in Theorem 1, it follows that

$$\begin{aligned} c_{n,k}(z) &= \frac{B_n(z)}{z - z_k} \cdot \prod_{j=1}^{n}(1 - \overline{z_j} z_k) \prod_{j=1,\, j\neq k}^{n} \frac{1}{z_k - z_j} \frac{1 - \bar{z} z_k}{1 - |z|^2} \\ &= \frac{1 - \bar{z} z_k}{1 - |z|^2} \frac{1 - r^{2n}}{n z_k^{n-1}(1 - r^n z^n)} \prod_{j=1,\, j\neq k}^{n} (z - z_j). \end{aligned}$$

and that

$$|c_{n,k}(z)| \leq 2 \frac{1 - r^{2n}}{n r^{n-1}} \frac{1}{|1 - r^n z^n|} \prod_{j=1,\, j\neq k}^{n} |z - z_j|.$$

By (16), we get

$$\begin{aligned} |R_n(\mu, z) - f(z)| &\leq \epsilon \sum_{k=1}^{n} |c_{n,k}(z)| + \frac{\|f\|_1}{1 - r^2} |B_n(z)| \\ &\leq \frac{2\epsilon(1 - r^{2n})}{n r^{n-1} |1 - r^n z^n|} \sum_{k=1}^{n} \prod_{j=1,\, j\neq k}^{n} |z - z_j| + \frac{\|f\|_1}{1 - r^2} \left| \frac{z^n - r^n}{1 - r^n z^n} \right|. \end{aligned} \tag{34}$$

Thus, applying the maximum principle to the right-hand side of (34), we have in view of Lemma 2

$$\begin{aligned} \sup_{|z|\leq r} |R_n(\mu, z) - f(z)| &\leq 2\epsilon \frac{1 - r^{2n}}{n r^{n-1}(1 - r^n r^n)} r^{n-1} \sum_{j=1}^{n} \frac{1}{\sin((2j-1)\pi/(2n))} + \frac{\|f\|_1}{1 - r} \frac{2r^n}{1 + r^{2n}} \\ &\leq 2\epsilon \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\sin((2j-1)\pi/(2n))} + \frac{\|f\|_1}{2^{n-2}}. \end{aligned}$$

Put

$$S_n = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\sin((2j-1)\pi/(2n))}.$$

In view of the inequalities

$$1 < \frac{x}{\sin x} < \frac{\pi}{2} \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right),$$

and

$$\frac{1}{x+1} < \ln\frac{x+1}{x} < \frac{1}{x} \quad \text{for all } x > 0,$$

we have after some arrangements

$$\frac{2}{\pi}\ln\frac{2n+2}{e} < S_n < \ln(2e^2 n). \tag{35}$$

Combining the latter inequalities with the relation $\epsilon = x(\epsilon)\ln 2/2^{x(\epsilon)}$, we have

$$\begin{aligned}\sup_{|z|\le r}|R_{n(\epsilon)}(\mu,z) - f(z)| &\le 2\epsilon S_{n(\epsilon)} + \frac{\|f\|_1}{2^{n(\epsilon)-2}} < 2\epsilon\ln(2e^2 n(\epsilon)) + \frac{4\|f\|_1}{2^{n(\epsilon)}}\\ &< 2\epsilon\ln(2e^2 x(\epsilon)) + \frac{4\|f\|_1}{2^{x(\epsilon)-1}}\\ &= \frac{2}{2^{x(\epsilon)}}(x(\epsilon)\ln(2e^2 x(\epsilon))\ln 2 + 4\|f\|_1) \equiv E(f,\epsilon),\end{aligned}$$

which completes the proof of Theorem 5. ■

As shown in the latter proof, the global error estimate $E(f,\epsilon)$ is quite relaxing, i.e., it can be large. However, it is optimal to within a factor of $20\pi/3$, since any approximation process of f using the scheme $R_n(\cdot,z)$ would give a global error estimate $e(f,\epsilon)$ in such a way that

$$\liminf_{\epsilon\downarrow 0}\frac{e(f,\epsilon)}{E(f,\epsilon)} \ge \frac{3}{20\pi}.$$

This fact will be illustrated in Theorem 6. Here, $m(\epsilon)$ is the dimension of the approximation process at which one stops and $e(f,\epsilon) \equiv (1 + \|f\|_1)\alpha(\epsilon)$ is the error one obtains at the dimension $m(\epsilon)$.

THEOREM 6 *Suppose that there exist functions* $m : (0,1) \to \mathbb{N}$, $\alpha : (0,1) \to (0,\infty)$ *such that*

$$\lim_{\epsilon\downarrow 0} m(\epsilon) = +\infty, \qquad \lim_{\epsilon\downarrow 0}\alpha(\epsilon) = 0,$$

and that, for any $f \in H^1$ *and any finite sequence* $\mu = \{\mu_j\}_{j=1}^{m(\epsilon)} \subset \boldsymbol{C}$ *satisfying the condition* $\max_{1\le j\le m(\epsilon)}|\mu_j - f(z_j)| \le \epsilon$, *we have*

$$\sup_{|z|\le r}|R_{m(\epsilon)}(\mu,z) - f(z)| \le (1 + \|f\|_1)\alpha(\epsilon) \equiv e(f,\epsilon).$$

Then we have

$$\alpha(\epsilon) > \frac{3}{5.2^{M(\epsilon)+1}}\left((M(\epsilon)+1)\ln\frac{2(M(\epsilon)+1)}{e}\ln 2 + 1\right),$$

where $M(\epsilon)$ *is the largest solution of the equation* $\epsilon 2^{x+1} - (x+1)\pi \ln 2 = 0$. *Furthermore, for every* $f \in H^1$, *we have*

$$\liminf_{\epsilon \downarrow 0} \frac{e(f,\epsilon)}{E(f,\epsilon)} \geq \frac{3}{20\pi},$$

where $E(f,\epsilon)$ *is defined in Theorem* 5.

Proof of Theorem 6 Consider the function

$$f(z) = \prod_{k=1}^{m(\epsilon)} \frac{z - z_k}{1 - \overline{z_k} z}.$$

Then $f \in H^1$, $\|f\|_1 = 1$ and $f(z_k) = 0$ for $k = 1, \ldots, m(\epsilon)$. We have

$$2\alpha(\epsilon) \geq \sup_{|z| \leq r} |R_{m(\epsilon)}(\mu, z) - f(z)| = \sup_{|z| \leq r} \left| \sum_{k=1}^{m(\epsilon)} \mu_k c_{m(\epsilon),k}(z) - B_{m(\epsilon)}(z) \right|.$$

Choosing $z = z_0 := re^{i(\pi/m(\epsilon))}$ and μ_k in an appropriate way with $|\mu_k| = \epsilon$ ($k = 1, \ldots, m(\epsilon)$), we find in view of the formula of $c_{m(\epsilon),k}(z)$ as in the proof of Theorem 5 that

$$\begin{aligned} 2\alpha(\epsilon) &\geq \sup_{|z| \leq r} \left| \sum_{k=1}^{m(\epsilon)} \mu_k c_{m(\epsilon),k}(z) - B_{m(\epsilon)}(z) \right| \geq \epsilon \sum_{k=1}^{m(\epsilon)} |c_{m(\epsilon),k}(z_0)| + |B_{m(\epsilon)}(z_0)| \\ &= \epsilon \sum_{k=1}^{m(\epsilon)} \frac{1}{m(\epsilon)} \frac{1}{\sin((2k-1)\pi/2m(\epsilon))} \frac{|1 - \overline{z_0} z_k|}{1 - |z_0|^2} \frac{1 - r^{2m(\epsilon)}}{1 + r^{2m(\epsilon)}} + \frac{2}{2^{m(\epsilon)} + (1/2^{m(\epsilon)})} \\ &> \frac{3}{5} \epsilon S_{m(\epsilon)} + \frac{1}{2^{m(\epsilon)}}. \end{aligned}$$

By inequality (35), we have

$$\alpha(\epsilon) > \frac{3}{5} \left(\frac{\epsilon}{\pi} \ln \frac{2m(\epsilon) + 2}{e} + \frac{1}{2^{m(\epsilon)+1}} \right).$$

Put

$$g(x) = \frac{\epsilon}{\pi} \ln \frac{2x + 2}{e} + \frac{1}{2^{x+1}}.$$

Then

$$\alpha(\epsilon) > \frac{3}{5} \min_{x > 0} g(x).$$

Since g attains its miminum at the point $M(\epsilon)$ satisfying the condition

$$g'(M(\epsilon)) = \frac{\epsilon}{\pi(M(\epsilon)+1)} - \frac{\ln 2}{2^{M(\epsilon)+1}} = 0,$$

it follows that

$$\alpha(\epsilon) > \frac{3}{5}\min_{x>0} g(x) = \frac{3}{5} g(M(\epsilon)) = \frac{3}{5.2^{M(\epsilon)+1}}\left((M(\epsilon)+1)\ln\frac{2(M(\epsilon)+1)}{e}\ln 2 + 1\right).$$

From the equations defining $x(\epsilon)$ and $M(\epsilon)$, we easily see that

$$\frac{\ln \pi}{\ln 2} - 1 \le M(\epsilon) - x(\epsilon) \le \frac{\ln \pi}{\ln 2} \quad \text{for sufficiently small } \epsilon.$$

Hence, letting $\epsilon \to 0$ in the inequality

$$\frac{e(f,\epsilon)}{E(f,\epsilon)} > \frac{3}{10.2^{M(\epsilon)-x(\epsilon)+1}} \frac{(1+\|f\|_1)((M(\epsilon)+1)\ln(2(M(\epsilon)+1)/e)\ln 2 + 1)}{(x(\epsilon)\ln(2e^2 x(\epsilon))\ln 2 + 8\|f\|_1)}$$

yields the desired result. The proof of Theorem 6 is complete. ■

Acknowledgments

The authors would like to thank the referees for their constructive comments leading to the improvement of our article. Trong would like to thank Professor Saburou Saitoh for his helpful discussion. This work was supported by the Council for Natural Sciences of Vietnam.

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