A Gamma-Convergence Approach to the Cahn-Hilliard Equation

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Abstract

We study the asymptotic dynamics of the Cahn-Hilliard equation via the "Gamma convergence" of gradient flows scheme initiated by Sandier and Serfaty. This gives rise to an H^1-version of a conjecture by De Giorgi, namely, the slope of the Allen-Cahn functional with respect to the H^{-1}-structure Gamma-converges to a homogeneous Sobolev norm of the scalar mean curvature of the limiting interface. We confirm this conjecture in the case of constant multiplicity of the limiting interface. Finally, under suitable conditions for which the conjecture is true, we prove that the limiting dynamics for the Cahn-Hilliard equation is motion by Mullins-Sekerka law.

1 Introduction

We are interested in establishing convergence results arising in the study of the asymptotic limit, as $\varepsilon \searrow 0$, of the solutions to the Cahn-Hilliard equation

$$(1.1)\qquad \begin{cases} \partial_t u^\varepsilon = -\Delta v^\varepsilon & (x,t) \in \Omega \times (0,\infty) \\ v^\varepsilon = \varepsilon \Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon) & (x,t) \in \Omega \times [0,\infty) \\ \dfrac{\partial u^\varepsilon}{\partial n}(x,t) = \dfrac{\partial v^\varepsilon}{\partial n}(x,t) = 0 & (x,t) \in \partial\Omega \times [0,\infty) \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & x \in \Omega. \end{cases}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 2)$, $f(u) = 2u(u^2-1)$ is the derivative of the double-well potential $W(u) = \frac{1}{2}(u^2-1)^2$ and the initial data u_0^ε is a real-valued function in Ω. This equation is widely accepted as a good model to describe various phase separation and coarsening phenomena in a melted alloy with two stable phases. See [5,

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9, 11] and the references therein for more information on the physical background, the dynamics, and related issues.

It was formally derived by Pego [19] using the method of matched asymptotic expansions that the Cahn-Hilliard equation converges to motion by Mullins-Sekerka law (see [17]), i.e., as $\varepsilon \searrow 0$, the chemical potential v^ε tends to a limit v, which, together with a free boundary $\cup_{0\le t\le T}(\Gamma(t)\times\{t\})$, solves the following free-boundary problem in a time interval $[0,T]$ for some $T>0$:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega\backslash\Gamma(t),\ t\in[0,T], \\ v = \sigma\kappa & \text{on } \Gamma(t),\ t\in[0,T], \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial_T\Omega := \partial\Omega\times[0,T], \\ \partial_t\Gamma = \dfrac{1}{2}\left[\dfrac{\partial v}{\partial n}\right]_{\Gamma(t)} & \text{on } \Gamma(t),\ t\in[0,T], \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{1.2}$$

Here $\kappa(t)$ is the scalar mean curvature of the hypersurface $\Gamma(t)\subset\Omega$ with the sign convention that a convex hypersurface has positive mean curvature; $\sigma = \displaystyle\int_{-1}^{1}\sqrt{W(s)/2}ds = \frac{2}{3}$; $\partial_t\Gamma$ is the normal velocity of the hypersurface $\Gamma(t)$ with the sign convention that the normal velocity of an expanding hypersurface is positive; $\vec{n}$ is the unit outernormal either to Ω or $\Gamma(t)$; $\left[\frac{\partial v}{\partial n}\right]_{\Gamma(t)}$ denotes the jump in the normal derivative of v through the hypersurface $\Gamma(t)$, i.e., $\left[\frac{\partial v}{\partial n}\right]_{\Gamma(t)} = \frac{\partial v^+}{\partial n} - \frac{\partial v^-}{\partial n}$, where v^+ and v^- are respectively the restriction of v on Ω_t^+ and Ω_t^-, the exterior and interior of $\Gamma(t)$ in Ω; and finally, $\Gamma_0 \subset\subset \Omega$ is the initial hypersurface separating the phases of the function $u_0 \in \mathrm{BV}(\Omega,\{-1,1\})$ which is the $L^1(\Omega)$ limit of the sequence $\{u_0^\varepsilon\}_{0<\varepsilon<1}$ (after extraction).

Problem (1.2) is often called the Mullins-Sekerka problem, or also the two-phase Hele-Shaw problem. Existence of classical solutions of (1.2) when the initial hypersurface Γ_0 is sufficiently smooth can be found in [7, 10]. For general initial hypersurfaces Γ_0, existence of weak solutions of (1.2) can be found in Röger [21].

Under the assumption that (1.2) has a smooth solution on some time interval $[0,T]$, a rigorous justification of Pego's result was carried out by Alikakos, Bates and Chen in [2], using asymptotic expansions and spectral analysis. They showed that there exists a family of smooth functions $\{u_0^\varepsilon(x)\}_{0<\varepsilon\le1}$ which are uniformly bounded in $\varepsilon\in(0,1]$ and $(x,t)\in\overline{\Omega\times(0,T)}$, such that if $(u^\varepsilon,v^\varepsilon)$ satisfies the Cahn-Hilliard equation (1.1) then v^ε converges uniformly on $\overline{\Omega\times(0,T)}$ to v satisfying (1.2) (see, [2], Theorem 5.1). For the general case, Chen [6] obtained a global asymptotic solution in a rather weak varifold formulation of (1.2). He proved that v^ε converges weakly to a function v in $L^2_{loc}((0,\infty),H^1(\Omega))$ and the relation $v=\sigma\kappa$ holds up to a multiplicative function $m(x,t)\ge1$. It is not clear how to obtain the relation $v=\sigma\kappa$ in the limit except for the case of radially symmetric solutions of (1.1) (see also Stoth [27]). To our knowledge, no general result proving the strong convergence of (1.1) to (1.2) for general initial data is available yet.

In this paper, we propose another way of studying the asymptotic dynamics of (1.1) with initial data more general than those considered in [2] via the idea of Gamma-convergence (denoted Γ-convergence in what follows) of gradient flows. This abstract method, initiated by Sandier and Serfaty [25], is easy to understand and was used successfully for the dynamics of Ginzburg-Landau vortices. Formally speaking, this scheme states that if we have a family of energies E_ε that Γ-converges to a limiting energy E, then under suitable conditions, we can prove that the gradient flow of E_ε converges to the gradient flow of E. We take advantage here of the fact that we are in this situation, since it is known that the Cahn-Hilliard equation is an H^{-1} gradient flow for the Allen-Cahn or Modica-Mortola energy functional arising in the van der Waals-Cahn-Hilliard theory of phase transitions

$$E_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u), \tag{1.3}$$

while E_ε Γ-converges to the area functional as proved by Modica and Mortola [15] and Sternberg [26]; for which we see that (1.2) is also an appropriate gradient-flow. For an introduction to the notion of Γ-convergence, one may refer to the very nice book by Braides [3]. For precise formulation and proper scaling, see the following discussion.

1.1 The Abstract Scheme for Γ-Convergence of Gradient Flows

In this subsection, we recall some definitions and the abstract framework from Sandier and Serfaty [25] to establish "Γ-convergence of gradient flows". Let E_ε (resp. E) be C^1 functionals defined over $\mathcal{M}$ (resp. $\mathcal{N}$), an open subset of an affine space associated to a Banach space $\mathcal{B}$ (resp. $\mathcal{B}'$). Assume that $\mathcal{B}$ (resp. $\mathcal{B}'$) embeds continuously into a Hilbert space X_ε (resp. Y). By the C^1 character of E_ε, we can define the differential $dE_\varepsilon(u)$ of E_ε at u and denote by $\nabla_{X_\varepsilon}E_\varepsilon(u)$ the vector of X_ε that represents it (resp. $dE(u)$ and $\nabla_Y E(u)$ for E). Then we can make sense of what it means by solution of gradient flow for E_ε (resp. E) with respect to the structure X_ε (resp. Y) and its energy conservation.

Definition 1.1. *We say that E_ε Γ-converges along the trajectory $u^\varepsilon(t)$ ($t \in [0,T)$) in the sense S (to be specified in each problem) to E if there exists $u(t) \in \mathcal{N}$ and a subsequence (still denoted u^ε) such that $\forall t \in [0,T)$, $u^\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$ and $\liminf_{\varepsilon\to0} E_\varepsilon(u^\varepsilon(t)) \geq E(u(t))$.*

If E_ε Γ-converges to E, then the key conditions for which the gradient flow of E_ε with respect to the structure X_ε Γ- converges to the gradient flow of E with respect to the structure Y are the following inequalities for general functions u^ε

(C1) (*Lower bound on the velocity*) For a subsequence such that $u^\varepsilon(t) \overset{S}{\rightharpoonup} u(t)$, we have $u \in H^1((0,T),Y)$ and for every $s \in [0,T)$, $\liminf_{\varepsilon\to0} \int_0^s \|\partial_t u^\varepsilon(t)\|^2_{X_\varepsilon}\, dt \geq \int_0^s \|\partial_t u(t)\|^2_Y\, dt$.

(C2) (*Lower bound on the slope*) If $u^\varepsilon \overset{S}{\rightharpoonup} u$ then $\liminf_{\varepsilon\to0} \|\nabla_{X_\varepsilon}E_\varepsilon(u^\varepsilon)\|^2_{X_\varepsilon} \geq \|\nabla_Y E(u)\|^2_Y$.

The abstract result on Γ-convergence for the gradient flows in [25] states that

Theorem 1.1. *([25], Theorem 1.4) Let E_ε and E be C^1 functionals defined over $\mathcal{M}$ and $\mathcal{N}$ respectively, and let $u^\varepsilon \in H^1((0,T),X_\varepsilon)$ be a family of conservative solutions of the flow* *for* E_ε *on* $[0,T)$ *(i.e., for a.e.* $t \in (0,T)$*, we have* $\partial_t u^\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u^\varepsilon) \in X_\varepsilon$*; and for all* $t \in [0,T)$*,* $E_\varepsilon(u^\varepsilon(0)) - E_\varepsilon(u^\varepsilon(t)) = \int_0^t \|\partial_t u^\varepsilon(s)\|^2_{X_\varepsilon}\, ds$*) with* $u^\varepsilon(0) \xrightarrow{S} u^0$*, along which* E_ε $\Gamma-$*converges to* E *in the sense of definition* 1.1*. Then, under the conditions (C1) and (C2) as above and if* u^ε *is well-prepared initially, i.e,*

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u^\varepsilon(0)) = E(u^0)$$

then $\forall t \in [0,T)$*,* $u^\varepsilon(t) \xrightarrow{S} u(t)$ *where* $u(t)$ *is the solution of the gradient flow for* E *with respect to the structure* Y *on* $[0,T)$ *with initial data* u^0*.*

Although the statement and the proof of this theorem are quite simple, difficulties arise when we wish to apply it to concrete situations, especially in proving that criteria $(C1)$ and $(C2)$ are satisfied. For the case of the Cahn-Hilliard equation, due to the lack of smoothness of the limiting interface (see the discussion in the next section) the limiting functional E is in general not differentiable on Y. So, we can only apply this theorem under additional regularity assumptions. Moreover, the theorem in [25] was proved for simplicity in the case where Y was a finite-dimensional space in order to ensure the existence of solutions to the gradient flow. However, as mentioned there, the idea of the scheme works also in infinite dimensions provided that a meaning to the limiting flow is known. We will mostly follow the steps of [25] formally.

1.2 Application to the Cahn-Hilliard Equation

Consider the Cahn-Hilliard equation (1.1) with the associated Modica-Mortola energy functional defined by (1.3). We asssume the following conditions on the initial data u_0^ε

$$E_\varepsilon(u_0^\varepsilon) \le M < \infty, \quad \frac{1}{|\Omega|}\int_\Omega u_0^\varepsilon = m_\varepsilon \in (-m, m) \ (0 < m < 1). \tag{1.4}$$

Observe that the Cahn-Hilliard equation (1.1) is the H_n^{-1} gradient flow of the Modica-Mortola energy functional, where the space $H_n^{-1}(\Omega)$, which is similar to $H^{-1}(\Omega)$, is defined as follows. Let $<,>$ denote the pairing between $(H^1(\Omega))^*$ and $H^1(\Omega)$. Then, define

$$H_n^{-1}(\Omega) = \{f \in (H^1(\Omega))^* \mid \exists\, g \in H^1(\Omega) \text{ such that } < f, \varphi >= \int_\Omega \nabla g \cdot \nabla \varphi \ \ \forall\, \varphi \in H^1(\Omega)\}.$$

Note that the function g in the above definition is unique up to a constant. We denote by $-\Delta_n^{-1} f$ the one with mean 0 over Ω. Then, $H_n^{-1}(\Omega)$ is a Hilbert space with inner product

$$< u, v >_{H_n^{-1}(\Omega)} = \int_\Omega \nabla(\Delta_n^{-1} u) \cdot \nabla(\Delta_n^{-1} v) \quad \forall\, u, v \in H_n^{-1}(\Omega). \tag{1.5}$$

By some simple calculations, we find that the gradients of E_ε with respect to the structures $L^2(\Omega)$ and $H_n^{-1}(\Omega)$ are respectively

$$\nabla_{L^2(\Omega)} E_\varepsilon(u) = -\varepsilon \Delta u + \varepsilon^{-1} f(u), \qquad \nabla_{H_n^{-1}(\Omega)} E_\varepsilon(u) = -\Delta(-\varepsilon \Delta u + \varepsilon^{-1} f(u)). \tag{1.6}$$

It is well-known (see Modica-Mortola [15] and Sternberg [26]) that E_ε Γ-converges to the area functional

$$E(u) = 2\sigma \mathcal{H}^{N-1}(\Gamma) := E(\Gamma).$$

Here u is a function of bounded variation taking values ± 1 which will be denoted $u \in BV(\Omega, \{-1, 1\})$ in what follows, Γ is the interface separating the phases, i.e, $\Gamma = \partial\{x \in \Omega : u(x) = 1\}$, and $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. The sense S in the statement $u_\varepsilon(t) \xrightharpoonup{S} u(t)$ used in Definition 1.1 is understood as: $u^\varepsilon(t)$ converges in $L^1(\Omega)$ to $u(t)$. It can be seen that, under some smoothness assumptions, the Mullins-Sekerka flow is the H_n^{-1} gradient flow of the area functional (see heuristic arguments in Section 2). So the convergence of the Cahn-Hilliard equation to the Mullins-Sekerka flow formally fits into the framework of Γ-convergence of gradient flows of [25].

With the choice of $X_\varepsilon = H_n^{-1}(\Omega)$, the first criterion $(C1)$ in the scheme now becomes

Proposition 1.1. *Let u^ε be defined over $\Omega \times [0, T]$ such that $\int_\Omega |u^\varepsilon(t)|\, dx \le M < \infty$ for all $t \in [0, T]$ and all $\varepsilon > 0$. Assume that, after extraction, $u^\varepsilon(t) \to u(t)$ in $L^1(\Omega)$ for all $t \in [0, T]$ where $u(t) \in BV(\Omega, \{-1, 1\})$ with interface $\Gamma(t) = \partial\{x \in \Omega : u(x, t) = 1\}$. Then, for all $t \in (0, T)$, we have*

$$\liminf_{\varepsilon \to 0} \int_0^t \|\partial_t u^\varepsilon(s)\|^2_{H_n^{-1}(\Omega)}\, ds \ge \int_0^t \|\partial_t u(s)\|^2_{H_n^{-1}(\Omega)} = 4 \int_0^t \|\partial_t \Gamma(s)\|^2_{H_n^{-1}(\Omega)}\, ds. \tag{1.7}$$

We will prove this Proposition in Section 4. Because the initial interface $\Gamma_0 \subset\subset \Omega$ and we are only interested in the motion of the interfaces, we assume from now on that, there is some time $T > 0$ so that no interfaces involved can touch the boundary in the time interval $[0, T]$. For the solutions of (1.1), this assumption will be justified in Proposition 4.1 where we prove some weak time-continuity of the limiting interfaces.

So far, we have not yet identified the limiting structure Y. For this purpose, observe that the Γ-limit functional E of E_ε depends only on the structure of the limiting interface Γ and furthermore, in the inequality (1.7), $\partial_t \Gamma$ is a distribution supported on Γ. Thus, Y should be some space defined on Γ. In fact, we can choose Y with

$$\|\cdot\|_Y^2 = 4\, \|\cdot\|^2_{H_n^{-1/2}(\Gamma)}, \tag{1.8}$$

where $H_n^{-1/2}(\Gamma)$ is sort of $H_n^{-1}(\Omega)$ restricted to Γ. It is the dual of $H_n^{1/2}(\Gamma)$, a trace space on Γ equiped with a two-sided homogeneous Sobolev norm. These spaces will be discussed in details in Section 2. On the first reading, one can view $H_n^{1/2}(\Gamma)$ as a restricted version of $H^1(\Omega)$ on Γ. With the choice of Y in (1.8), we now can interpret the second criterion $(C2)$ of the Γ-convergence scheme. Set $v^\varepsilon = \varepsilon\Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon)$. Then, from (1.6) and (1.5),

$$\|\nabla_{X_\varepsilon} E_\varepsilon(u^\varepsilon)\|^2_{X_\varepsilon} = \left\|\nabla_{H_n^{-1}(\Omega)} E_\varepsilon(u^\varepsilon)\right\|^2_{H_n^{-1}(\Omega)} = \|\Delta v^\varepsilon\|^2_{H_n^{-1}(\Omega)} = \|\nabla v^\varepsilon\|^2_{L^2(\Omega)}. \tag{1.9}$$

Let κ and $\vec{n}$ denote the mean curvature and the unit outernormal vector to Γ. Then from the calculations in Section 2, we find that $\nabla_Y E(\Gamma) = \frac{1}{2}\Delta_\Gamma(\sigma\kappa)\, \vec{n}$ (see (2.6)); and again, Δ_Γ is a restricted version of the usual Laplacian operator on Γ. Therefore, from Lemma 2.1, one has

$$\|\nabla_Y E(\Gamma)\|_Y^2 = \frac{1}{4}\|\Delta_\Gamma(\sigma\kappa)\|_Y^2 = \|\Delta_\Gamma(\sigma\kappa)\|^2_{H_n^{-1/2}(\Gamma)} = \sigma^2\|\kappa\|^2_{H_n^{1/2}(\Gamma)}.$$

So, the second criterion - the lower bound on the slope - is expected to be: for $u^\varepsilon \to u$ in $L^1(\Omega)$ where $u \in BV(\Omega, \{-1,1\})$ with interface $\Gamma = \partial\{u=1\} \subset \Omega$, we have

$$\liminf_{\varepsilon\to 0}\int_\Omega |\nabla v^\varepsilon|^2 \geq \sigma^2\|\kappa\|^2_{H_n^{1/2}(\Gamma)}. \tag{1.10}$$

This inequality is actually a sharp lower bound for the dissipation rate of the ε-problem (1.1) in terms of the corresponding quantity of the limiting problem (1.2), assuming Γ is smooth. More succinctly, for the ε-problem (1.1), the dissipation rate is

$$-\frac{d}{dt}E_\varepsilon(u^\varepsilon(t)) = -\int_\Omega \nabla_{L^2(\Omega)}E_\varepsilon(u^\varepsilon)\cdot\partial_t u^\varepsilon = -\int_\Omega(-v^\varepsilon)(-\Delta v^\varepsilon) = \int_\Omega |\nabla v^\varepsilon|^2.$$

Similarly, by using (2.5) and (2.1) in Section 2, we find that the dissipation rate for the limiting problem (1.2) is

$$\begin{aligned}-\frac{d}{dt}E(\Gamma(t)) &= -\int_{\Gamma(t)}\nabla_{L^2(\Gamma)}E(\Gamma)\cdot(\partial_t\Gamma)\,\vec{n} = \int_{\Gamma(t)} 2v.\partial_t\Gamma\\ &= \int_{\Gamma(t)} v\left[\frac{\partial v}{\partial n}\right]_{\Gamma(t)} = \int_\Omega |\nabla v|^2 = \sigma^2\|\kappa\|^2_{H_n^{1/2}(\Gamma(t))}.\end{aligned}$$

Thus, a dynamics formulation for (1.10) is

$$\liminf_{\varepsilon\to 0} -\frac{d}{dt}E_\varepsilon(u^\varepsilon(t)) \geq -\frac{d}{dt}E(\Gamma(t)).$$

Although we are not able to prove the inequality (1.10) in full generality (see the discussion after the proof of Theorem 1.2, Item A), it motivates us to study the asymptotic behavior of the functional $\int_\Omega |\nabla v^\varepsilon|^2 := \int_\Omega |\nabla(\varepsilon\Delta u^\varepsilon - \varepsilon^{-1}f(u^\varepsilon))|^2$ of the static problem via Γ-convergence. This leads us to the following.

Conjecture (CH). *Let $\{u^\varepsilon\}_{0<\varepsilon\leq 1}$ be a sequence of bounded C^3 functions satisfying (1.4) and let $u \in BV(\Omega,\{-1,1\})$ be its $L^1(\Omega)$-limit (after extraction). Asssume that $\Gamma = \partial\{u = 1\} \subset \Omega$ is C^3. Then $\int_\Omega \left|\nabla(\varepsilon\Delta u^\varepsilon - \varepsilon^{-1}f(u^\varepsilon))\right|^2$ Γ-converges to $\sigma^2\|\kappa\|^2_{H_n^{1/2}(\Gamma)}$.*

Note that this is in the spirit of De Giorgi's conjectures [8] but in an H^1-version. An L^2-version of this result was proved by Röger and Schätzle [22] for space dimensions $N = 2, 3$; see also [16, 18] for partial results. Under suitable assumptions, they proved that $T_\varepsilon(u^\varepsilon) \equiv \int_\Omega \varepsilon^{-1}(\varepsilon\Delta u^\varepsilon - \varepsilon^{-1}f(u^\varepsilon))^2$ Γ-converges to $\sigma^2\|\kappa\|^2_{L^2(\Gamma)}$. This result is typical of the Allen-Cahn equation $\partial_t u^\varepsilon = \Delta u^\varepsilon - \varepsilon^{-2}f(u^\varepsilon)$ because the quantity $T_\varepsilon(u^\varepsilon)$ is exactly its dissipation rate and generically uniformly bounded in ε. Meanwhile, conjecture (CH) is typical of the Cahn-Hilliard equation; the quantity $T_\varepsilon(u^\varepsilon)$ there is unbounded in ε.

We will partially prove conjecture (CH) in the following theorem.

Theorem 1.2. **A.** *Let $\{u^\varepsilon\}_{0<\varepsilon\leq 1}$ be a sequence of C^3 functions which are uniformly bounded in ε and satisfy (1.4). Let $u \in BV(\Omega, \{-1,1\})$ be its $L^1(\Omega)$-limit and let $v^\varepsilon = \varepsilon\Delta u^\varepsilon - \varepsilon^{-1}f(u^\varepsilon)$. Assume that the limiting interface $\Gamma = \partial\{u = 1\} \subset \Omega$ is C^3 and that the following conditions are satisfied:*
(i) Γ has constant multiplicity, i.e, there exists a constant m such that, in the sense of Radon measures,

$$\left(\frac{\varepsilon\left|\nabla u^\varepsilon\right|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon}\right) dx \rightharpoonup 2m\sigma d\mathcal{H}^{N-1}\lfloor\Gamma, \tag{1.11}$$

(ii) either $m = 1$ or the space dimension $N \leq 3$ or the limiting equipartition of energy holds, i.e, in the sense of Radon measures

$$\left|\frac{\varepsilon\left|\nabla u^\varepsilon\right|^2}{2} - \frac{W(u^\varepsilon)}{\varepsilon}\right| dx \rightharpoonup 0, \tag{1.12}$$

then we have the following inequality

$$\liminf_{\varepsilon\to 0} \int_\Omega \left|\nabla v^\varepsilon\right|^2 \geq \sigma^2 \left\|\kappa\right\|^2_{H_n^{1/2}(\Gamma)}. \tag{1.13}$$

B. *Assume that Γ is any smooth hypersurface enclosing $\Omega^- \subset\subset \Omega$. Then we can find a family of smooth functions $\{u_0^\varepsilon(x)\}_{0<\varepsilon<1}$ which are uniformly bounded for $0 < \varepsilon < 1$ such that (1.4) holds, $u_0^\varepsilon(x) \to 1 - 2\chi_{\Omega^-}(x)$ in $L^1(\Omega)$,*

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_0^\varepsilon) = 2\sigma\mathcal{H}^{N-1}(\Gamma), \tag{1.14}$$

and for the corresponding chemical potentials $v_0^\varepsilon = \varepsilon\Delta u_0^\varepsilon - \varepsilon^{-1}f(u_0^\varepsilon)$, we have

$$\lim_{\varepsilon\to 0} \int_\Omega \left|\nabla v_0^\varepsilon\right|^2 dx = \sigma^2 \left\|\kappa\right\|^2_{H_n^{1/2}(\Gamma)}. \tag{1.15}$$

Remark 1.1. *The single-multiplicity assumption (i.e., (1.11) with $m = 1$) can be verified in many situations. The simplest example is the case when u^ε is a minimizer of E_ε.*

This theorem is related to the Cahn-Hilliard equation but it also has its own interest. Following the idea of [25], we deduce the limiting dynamics of the Cahn-Hilliard equation from Proposition 1.1 and Theorem 1.2 as announced.

Theorem 1.3. *Let $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ be the smooth solution of (1.1) on $\Omega \times [0,\infty)$ with initial data u_0^ε such that, after extraction, $u_0^\varepsilon(x)$ converges strongly in $L^1(\Omega)$ to $u^0(x,0) \in BV(\Omega, \{-1,1\})$ with interface $\Gamma(0) = \partial\{x \in \Omega : u^0(x,0) = 1\} \subset\subset \Omega$ consisting of a finite number of closed hypersurfaces. Then there exists $T_* > 0$ such that, after extraction, we have that for all $t \in [0,T_*)$, $u^\varepsilon(x,t)$ converges strongly in $L^1(\Omega)$ to $u^0(x,t) \in BV$*

$(\Omega, \{-1, 1\})$ *with interface* $\Gamma(t) = \partial\{x \in \Omega : u^0(x,t) = 1\} \subset \Omega$. *Moreover, under the following assumptions*
(A1) The initial data u_0^ε *is well-prepared, i.e.,* $\lim_{\varepsilon\to 0} E_\varepsilon(u_0^\varepsilon) = 2\sigma\mathcal{H}^{N-1}(\Gamma(0))$,
(A2) $\cup_{t\in[0,T_*)}(\Gamma(t)\times t)$ *is a* C^3 *space-time hypersurface,*
(A3) The lower bound on the slope holds, i.e, for each time slice t *we have*

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla v^\varepsilon(t)|^2 \geq \sigma^2 \left\|\kappa(t)\right\|^2_{H_n^{1/2}(\Gamma(t))},$$

the Cahn-Hilliard equation converges to motion by Mullins-Sekerka law, i.e., $v^\varepsilon(x,t)$ *converges strongly in* $L^2((0,T_*), H^1(\Omega))$ *to* $v(x,t)$ *solving (1.2) with the initial interface* $\Gamma(0)$. *Finally,* T_* *can be chosen to be the minimum of the collision time (i.e., for all* $t \in [0, T^*)$ *the hypersurfaces contained in* $\Gamma(t)$ *do not collide) and of the exit time from* Ω *of the hypersurfaces under the Mullins-Sekerka law.*

Remark 1.2. *In the case of radially symmetric solutions to (1.1), (A3) holds. This follows from the single-multiplicity property of the limiting interfaces* $\Gamma(t)$ *proved in Theorem* 2.2 *of Chen [6] and Theorem 1.2.*

Remark 1.3. *It should be noticed that under the assumptions (A1), (A2) and that for each* $t \in (0, T_*)$ *there exists a constant* $m(t) \geq 1$ *such that*

$$v(x,t) = m(t)\sigma\kappa(t) \tag{1.16}$$

then the weak formulation of the Mullins-Sekerka law (1.2) in Chen [6] becomes the classical solution; see Section 2.4 in [6]. The latter assumption requires, among other things, that the multiplicity of the limiting interface $\Gamma(t)$ *is constant for each* $t \in (0, T_*)$. *In our theorem, the assumption (A3) is automatically satisfied if we have either (1.16) or in space dimensions* $N \leq 3$ *if we assume the constant multiplicity of the limiting interface (see Theorem 1.2, Item A). The works of Hutchinson-Tonegawa [12] and Tonegawa [28, 29] showed that the multiplicity is an integer-valued function, however it could be nonconstant. They studied the equation* $\varepsilon\Delta u^\varepsilon - \varepsilon^{-1}f(u^\varepsilon) = v^\varepsilon$ *assuming the chemical potential* v^ε *to be uniformly bounded in* $W^{1,p}(\Omega)$ *where* $p > N/2$.

Remark 1.4. *The multiplicity of the limiting interface in phase transitions is a common issue; it corresponds to how many times the zero level set of* u^ε *folds into the limiting interface* Γ. *In the case of the Allen-Cahn equation* $\partial_t u^\varepsilon = \Delta u^\varepsilon - \varepsilon^{-2}f(u^\varepsilon)$, *due to the lack of control on the gradient of the chemical potential* v^ε, *the multiplicity of the limiting interface can be any positive integer, even in the radially symmetric case (see Bronsard and Stoth [4] for the profiles). However, for our equation* $v^\varepsilon = \varepsilon\Delta u^\varepsilon - \varepsilon^{-1}f(u^\varepsilon)$, *it has been conjectured by Tonegawa [28, 29] that the multiplicity of the limiting interface must be exactly one if the chemical potential* v^ε *is uniformly bounded in* $W^{1,p}(\Omega)$ $(p > N/2)$ *and the limiting chemical potential* v *is nonzero there. Recently, there has been some progress in resolving this multiplicity issue [23] where the authors assume that the chemical potential* v^ε *is uniformly bounded in* $W^{1,p}(\Omega)$ *for* $p > N$. *This leads us to believe that (1.13) may be true without any assumption.*

The rest of the paper is divided in three more sections. Section 2 will be devoted to some notation together with some heuristics about the area functional. In Section 3, we will prove the Γ-convergence of the slope, Theorem 1.2. In Section 4, we prove the dynamical law for the Cahn-Hilliard equation, Theorem 1.3.

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2 Notations and Heuristics

In this section, we collect some notations used throughout the paper together with some formal derivations of the gradient flow of the area functional with respect to different structures. It should be emphasized that although many calculations are purely formal, they can be made rigorous if we have some further regularity on the limiting interfaces.

Let Ω be a smooth bounded open domain in $I\!R^N$. Consider a subdomain Ω^- of Ω with boundary Γ of finite perimeter. This implies that Γ is contained in the union of a countable number of disjoint closed Lipschitz surfaces. Denote by Ω^+ the set $\Omega\backslash\overline{\Omega^-}$. Because the area functional $E(\Gamma)$ depends only on the interface Γ, we will need some calculus on the interface Γ. Especially, the space $H_n^{-1}(\Omega)$ will have an interface analogue $H_n^{-1/2}(\Gamma)$ which we define in the sequel.

From now on, assume further that Γ *is the union of a finite number of disjoint closed Lipschitz surfaces.* Under this mild regularity assumption, we can define the trace space $H^{1/2}(\Gamma)$ as the traces on Γ of $H^1(\Omega^-)$ functions. Now, let $f \in H^{1/2}(\Gamma)$. Then, we can extend f into some $H^1(\Omega)$ function over Ω. Let $X(f)$ be the set of these extensions. Because the trace mapping $H^1(\Omega) \hookrightarrow L^2(\Gamma, \mathcal{H}^{N-1})$ is compact, we can prove that there exists a unique function $\tilde{f} \in X(f)$ minimizing the Dirichlet functional $\int_\Omega |\nabla u|^2$ over $X(f)$. We call this $\tilde{f}$ the natural extension of f over the whole domain Ω. It satisfies

$$\Delta\tilde{f} = 0 \text{ in } \Omega,\ \ \tilde{f} = f \ \text{ on } \ \Gamma, \text{ and } \ \frac{\partial\tilde{f}}{\partial n} = 0 \ \text{ on } \ \partial\Omega.$$

With this $\tilde{f}$, we are able to define the following two-sided homogeneous Sobolev seminorm and the semi-inner product on $H^{1/2}(\Gamma)$

$$(2.1) \qquad \|f\|_{H_n^{1/2}(\Gamma)} = \left\|\nabla\tilde{f}\right\|_{L^2(\Omega)},\ \ < u, v >_{H_n^{1/2}(\Gamma)} = \int_\Omega \nabla\tilde{u}\cdot\nabla\tilde{v} \qquad \forall u, v \in H^{1/2}(\Gamma).$$

Observe that $\|f\|_{H_n^{1/2}(\Gamma)} = 0$ iff f is a constant on Γ. So we can define the equivalence relation $\sim$ in $H^{1/2}(\Gamma)$: $f_1 \sim f_2$ iff $\|f_1 - f_2\|_{H_n^{1/2}(\Gamma)} = 0$.

Notation. Let $H_n^{1/2}(\Gamma)$ be the quotient space $H^{1/2}(\Gamma)/\sim$.

Then, $H_n^{1/2}(\Gamma)$ with inner product $< \cdot, \cdot >_{H_n^{1/2}(\Gamma)}$ is a Hilbert space. It can be seen as a restriction of $H^1(\Omega)$ on Γ.

Let $f \in H^{1/2}(\Gamma)$ and let $f^\pm$ be the restrictions of $\tilde{f}$ on $\Omega^\pm$. We define the Laplacian operator Δ_Γ on Γ as follows:

$$\Delta_\Gamma(f) = -(\frac{\partial f^+}{\partial n^+} + \frac{\partial f^-}{\partial n^-}) \tag{2.2}$$

where $\overrightarrow{n}^+, \overrightarrow{n}^-$ are respectively the unit outer normals on Γ of Ω^+, Ω^- and the right hand side is understood in the $H^{-1/2}(\Gamma)$ sense. Then $\Delta_\Gamma f = -\left[\frac{\partial \tilde{f}}{\partial n}\right]_\Gamma$, and furthermore, $\Delta \tilde{f} = \Delta_\Gamma(f)\delta_\Gamma$ in the sense of distributions. The Laplacian Δ_Γ can be seen as the restriction of the usual Laplacian operator on Γ. By Green's theorem, we now can write the inner product in $H^{1/2}(\Gamma)$ in the "instrinsic" form

$$< u, v >_{H_n^{1/2}(\Gamma)} = -\int_\Gamma (\Delta_\Gamma u) v \, d\mathcal{H}^{N-1} \qquad \forall\, u, v \in H^{1/2}(\Gamma). \tag{2.3}$$

Let $H_n^{-1/2}(\Gamma)$ be the dual of $H_n^{1/2}(\Gamma)$ with the usual dual norm $\|\cdot\|_{H_n^{-1/2}(\Gamma)}$. Using the Riesz Representation Theorem, we record a characterization of $H_n^{-1/2}(\Gamma)$ in the following lemma.

Lemma 2.1. *(i) For each $u \in H_n^{-1/2}(\Gamma)$, there exists a unique $u^* \in H_n^{1/2}(\Gamma)$ such that $\|u\|_{H_n^{-1/2}(\Gamma)} = \|u^*\|_{H_n^{1/2}(\Gamma)}$ and moreover, for all $v \in H_n^{1/2}(\Gamma)$ we have*

$$< u, v >_{H_n^{-1/2}(\Gamma) \times H_n^{1/2}(\Gamma)} = - < u^*, v >_{H_n^{1/2}(\Gamma)} .$$

In view of this equation and (2.3), we can formally write $u = \Delta_\Gamma u^*$. Denote u^* by $\Delta_\Gamma^{-1} u$.
(ii) $H_n^{-1/2}(\Gamma)$ is a Hilbert space with inner product

$$< u, v >_{H_n^{-1/2}(\Gamma)} = < \Delta_\Gamma^{-1} u, \Delta_\Gamma^{-1} v >_{H_n^{1/2}(\Gamma)} \qquad \forall u, v \in H_n^{-1/2}(\Gamma).$$

Comparing to $H_n^{-1}(\Omega)$, we see that $H_n^{-1/2}(\Gamma)$ is to some extent a restricted version of $H_n^{-1}(\Omega)$ on Γ. Now, let $E(\Gamma)$ be the area functional arising as the Gamma-limit of the Modica-Mortola functional: $E(\Gamma) := 2\sigma\mathcal{H}^{N-1}(\Gamma)$, where $\Gamma = \partial\{x : u^0(x) = 1\}$ is the interface separating the phases of a function $u^0 \in \mathrm{BV}(\Omega, \{-1, 1\})$. The functional E depends only on the structure of Γ so instead of finding the gradient of $E(u^0) \equiv E(\Gamma)$ with respect to the structure $H_n^{-1}(\Omega)$, we can find its gradient with respect to the restricted structure $H_n^{-1/2}(\Gamma)$. With the choice of $\|\cdot\|_Y^2 = 4\,\|\cdot\|_{H_n^{-1/2}(\Gamma)}^2$, we have

Proposition 2.1. *Assume that Γ is C^3. Then the gradient of E with respect to the structure Y at Γ is $\nabla_Y E(\Gamma) = \frac{1}{2}\Delta_\Gamma(\sigma\kappa)\,\overrightarrow{n}$, where κ is the scalar mean curvature and $\overrightarrow{n}$ the unit outernormal vector to Γ. So if $\Gamma(t)$ is C^3 in space-time then the gradient flow of E at $\Gamma(t)$ is the Mullins-Sekerka law (1.2).*

Proof. Because Γ is C^3, κ is C^1 on Γ and thus $\kappa \in H^{1/2}(\Gamma)$. Consider a smooth deformation $\Gamma(t)$ of Γ and let $V = (\partial_t\Gamma)\,\vec{n}$ be its normal velocity vector at $t = 0$. Then we have (see [1], Theorem 7.31)

$$\left.\frac{d}{dt}\right|_{t=0} E(\Gamma) = -2\sigma < \mathbf{H}, V >_{L^2(\Gamma)} \tag{2.4}$$

where $\mathbf{H} = \kappa\,\vec{n}$ is the mean curvature vector of Γ. Therefore, the gradient E with respect to the structure $L^2(\Gamma)$ at Γ is

$$\nabla_{L^2(\Gamma)} E(\Gamma) = -2\sigma\mathbf{H} = -2\sigma\kappa\,\vec{n}\ . \tag{2.5}$$

Now, we calculate the $H_n^{-1/2}$- gradient $\nabla_{H_n^{-1/2}(\Gamma)} E(\Gamma) = D\,\vec{n}$ of $E(\Gamma)$ with respect to $H_n^{-1/2}(\Gamma)$. To do this, it suffices to express the quantity $\left.\frac{d}{dt}\right|_{t=0} E(\Gamma)$ as an inner product in $H_n^{-1/2}(\Gamma)$: $\left.\frac{d}{dt}\right|_{t=0} E(\Gamma) =< D, \partial_t\Gamma >_{H_n^{-1/2}(\Gamma)}$. By Lemma 2.1, and the intrinsic form (2.3) of the inner product for $H_n^{1/2}(\Gamma)$, we have

$$\begin{aligned} < D, \partial_t\Gamma >_{H_n^{-1/2}(\Gamma)} =< \Delta_\Gamma^{-1} D, \Delta_\Gamma^{-1}\partial_t\Gamma >_{H_n^{1/2}(\Gamma)} &= -\int_\Gamma (\Delta_\Gamma^{-1} D)\cdot \Delta_\Gamma(\Delta_\Gamma^{-1}\partial_t\Gamma) d\mathcal{H}^{N-1} \\ &= -\int_\Gamma (\Delta_\Gamma^{-1} D)\cdot \partial_t\Gamma d\mathcal{H}^{N-1}. \end{aligned}$$

It follows from (2.4) that $\Delta_\Gamma^{-1} D = 2\sigma\kappa$. In other words, the $H_n^{-1/2}$- gradient $\nabla_{H_n^{-1/2}(\Gamma)} E(\Gamma)$ of E at Γ is given by $\nabla_{H_n^{-1/2}(\Gamma)} E(\Gamma) = D\,\vec{n} = \Delta_\Gamma(2\sigma\kappa)\,\vec{n}$. Recalling $\|\cdot\|_Y^2 = 4\|\cdot\|^2_{H_n^{-1/2}(\Gamma)}$, we find that

$$\nabla_Y E(\Gamma) = \frac{1}{4}\nabla_{H_n^{-1/2}(\Gamma)} E(\Gamma) = \frac{1}{2}\Delta_\Gamma(\sigma\kappa)\,\vec{n} \tag{2.6}$$

and thus the gradient flow of $E(\Gamma)$ with respect to the structure Y is $V = -\nabla_Y E(\Gamma) = -\frac{1}{2}\Delta_\Gamma(\sigma\kappa)\,\vec{n}$. Recall the definition of Δ_Γ in (2.2) to find that $\partial_t\Gamma = \frac{1}{2}\left[\frac{\partial\widetilde{\sigma\kappa}}{\partial n}\right]_\Gamma$ and this is equivalent to the Mullins-Sekerka law (1.2). ☐

3 Γ-Convergence of the Slope

In this section, we prove Theorem 1.2. In the sequel, C_0 is some generic positive constant independent of ε.

Proof. Item A. We can assume that $\sup_{0<\varepsilon<1}\int_\Omega |\nabla v^\varepsilon|^2 \leq C_0$, otherwise the inequality (1.13) is trivial. From the energy bound and the mass constraint (1.4) and in view of Lemma 3.4 in Chen [6], we have for all ε sufficiently small

$$\|v^\varepsilon\|_{H^1(\Omega)} \leq C(E_\varepsilon(u^\varepsilon) + \|\nabla v^\varepsilon\|_{L^2(\Omega)}) \leq C_0 < \infty. \tag{3.1}$$

Now, up to extraction, we have that v^ε weakly converges to some v in $H^1(\Omega)$.

In the case of single-multiplicity, i.e., $m = 1$, or, equivalently, $\lim_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon) = 2\sigma\mathcal{H}^{N-1}(\Gamma)$, the limiting equipartition of energy (1.12) is also satisfied (see, e.g., Luckhaus and Modica [13], Lemma 1). If the space dimension $N \leq 3$ then from the uniform Sobolev bound (3.1) and the works of Tonegawa [28, 29], we also have limiting equipartition of energy. So, for the rest of the proof, we assume (1.11) and (1.12). We observe the following relation between the limiting chemical potential v and the mean curvature κ of the interface Γ.

Lemma 3.1. *Suppose that (1.11) and (1.12) are satisfied. Then* $v = m\sigma\kappa$ *on* Γ *a.e* $\mathcal{H}^{N-1}$.

Assuming this lemma, we continue the proof of (1.13). By lower semicontinuity, one has

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla v^\varepsilon|^2 \geq \int_\Omega |\nabla v|^2 \geq \inf_{w\in H^1(\Omega),\ w=m\sigma\kappa\ on\ \Gamma} \int_\Omega |\nabla w|^2 . \tag{3.2}$$

The latter minimization problem has a unique solution $w = m\widetilde{\sigma\kappa}$, the natural extension of $m\sigma\kappa$ over Ω as defined in Section 2. Therefore, from (3.2) and (2.1), we obtain

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla v^\varepsilon|^2 \geq m^2\sigma^2 \|\kappa\|^2_{H_n^{1/2}(\Gamma)} . \tag{3.3}$$

Because $E_\varepsilon(u^\varepsilon)$ Γ-converges to $2\sigma\mathcal{H}^{N-1}(\Gamma)$, we have $m \geq 1$. Therefore, from (3.3), we get the inequality (1.13) as desired.

We now prove Lemma 3.1. Let $\varphi = (\varphi^1, \cdots, \varphi^N) \in (C_0^1(\Omega))^N$. Following the proof of the monotonicity formula in [28], Lemma 3.1, we multiply both sides of the equation $v^\varepsilon = \varepsilon\Delta u^\varepsilon - \varepsilon^{-1} f(u^\varepsilon)$ by $\nabla u^\varepsilon \cdot \varphi$ and integrate by parts twice to obtain

$$\int_\Omega \left(\left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon} \right) \mathrm{div}\varphi - \varepsilon \sum_{j,k} \partial_j u^\varepsilon \partial_k u^\varepsilon \partial_k \varphi^j + u^\varepsilon v^\varepsilon \mathrm{div}\varphi + u^\varepsilon \varphi\cdot\nabla v^\varepsilon \right) = 0.$$

Rearranging terms, we get

$$\begin{aligned} \int_\Omega \left(\mathrm{div}\varphi - \sum_{j,k} \frac{\partial_j u^\varepsilon}{|\nabla u^\varepsilon|} \frac{\partial_k u^\varepsilon}{|\nabla u^\varepsilon|} \partial_k \varphi^j \right) \varepsilon |\nabla u^\varepsilon|^2 & \\ = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 - \frac{W(u^\varepsilon)}{\varepsilon} \right) & \mathrm{div}\varphi - \left(u^\varepsilon v^\varepsilon \mathrm{div}\varphi + u^\varepsilon \varphi\cdot\nabla v^\varepsilon \right). \end{aligned} \tag{3.4}$$

We are going to pass to the limit in this relation. First, observe that, by the uniform bound in $L^\infty(\Omega)$ of u^ε, u is also the $L^2(\Omega)$-limit of u^ε. Second, let $\vec{n} = (\vec{n}_1, \cdots, \vec{n}_N)$ denote the outward unit normal to the region Ω^- enclosed by Γ. Then, by the constant multiplicity (1.11) and limiting equipartition of energy (1.12) assumptions, it can be proved that

$$\varepsilon \nabla u^\varepsilon \otimes \nabla u^\varepsilon dx \rightharpoonup 2m\sigma\, \vec{n} \otimes \vec{n}\, \mathcal{H}^{N-1} \lfloor \Gamma. \tag{3.5}$$

For the case of single-multiplicity (i.e., $m = 1$) with limiting equipartition of energy, this follows from the work of Reshetnyak [20]; and a simple proof in this case can be found in Kohn et al. [14] (see also Luckhaus and Modica [13]). Inspecting the proof in [14], we see that it also carries over our case. Therefore, letting $\varepsilon \searrow 0$ in (3.4), using (1.11), (1.12) and (3.5), we find that

$$(3.6)\quad 2m\sigma \int_\Gamma (\mathrm{div}\varphi - \partial_k \varphi^j \,\vec{n}_j \otimes \vec{n}_k) d\mathcal{H}^{N-1} = -\int_\Omega (uv\mathrm{div}\varphi + u\varphi \cdot \nabla v) = -\int_\Omega u\mathrm{div}(v\varphi).$$

Upon applying the divergence theorem on manifolds (see, e.g., Theorem 7.34 in [1]) to the left-hand side of (3.6), using the divergence theorem for the right-hand side of (3.6), and recalling that $u = -1$ in Ω^- and $u = 1$ in $\Omega \setminus \overline{\Omega^-}$, we get

$$2m\sigma \int_\Gamma \varphi \cdot (\kappa \,\vec{n}) d\mathcal{H}^{N-1} = 2 \int_\Gamma (v\varphi) \cdot \vec{n}\, d\mathcal{H}^{N-1},$$

and thus complete the proof of Lemma 3.1. □

Remark 3.1. *A crucial fact in the proof is the relation $v = m\sigma\kappa$ where m is a constant. To our knowledge, this relation has not been established in full generality in any dimension. However, it was shown in Tonegawa [28, 29] that for $N \leq 3$, from (1.4) and the inequality $\sup_{0<\varepsilon<1} \int_\Omega |\nabla v^\varepsilon|^2 \leq C_0$, we can conclude that $v(x) = \theta(x)\sigma\kappa$ on Γ where $\theta(x)$ is the multiplicity of the limiting interface Γ, i.e., the densities $\left(\frac{\varepsilon}{2} |\nabla u^\varepsilon(x)|^2 + \frac{1}{\varepsilon} W(u^\varepsilon(x))\right) dx$ converge to $2\theta(x)\sigma d\mathcal{H}^{N-1} \lfloor \Gamma$ in the sense of Radon measures. Remarkably, $\theta(x)$ is an odd natural number $\mathcal{H}^{N-1}$ a.e. $x \in \Gamma$. If $\theta(x)$ is a constant then (1.13) holds.*

Proof. Item B. Suppose that Γ is a smooth hypersurface enclosing Ω^- in a smooth open bounded set $\Omega \subset I\!R^N$. Then from [7, 10] we know that the Mullins-Sekerka problem (1.2) with initial data Γ has a smooth solution $(v, \cup_{0 \leq t \leq T}(\Gamma_t \times \{t\}))$ in a time interval $[0, T]$ for some $T > 0$. We now construct a family of smooth functions $\{u_0^\varepsilon(x)\}_{0<\varepsilon<1}$ which are uniformly bounded for $0 < \varepsilon < 1$ such that (1.4) holds, $u_0^\varepsilon(x) \to 1 - 2\chi_{\Omega^-}(x)$ in $L^1(\Omega)$, and for the corresponding chemical potentials $v_0^\varepsilon = \varepsilon \Delta u_0^\varepsilon - \varepsilon^{-1} f(u_0^\varepsilon)$ we have

$$(3.7)\qquad \lim_{\varepsilon \to 0} \int_\Omega |\nabla v_0^\varepsilon|^2 \, dx = \sigma^2 \|\kappa\|^2_{H_n^{1/2}(\Gamma)} = \int_\Omega |\nabla v(x, 0)|^2 \, dx.$$

For this purpose, we use the initial data of the approximate solutions to (1.1) constructed in Alikakos, Bates and Chen [2] (note that there is no time involving in our function u_0^ε though we obtain it from the solution of a time-dependent problem; furthermore, due to the sensitivity to the dimension N of the profile u_0^ε (see also Remark 3.2), we are unfortunately unable to write down an explicit formula for u_0^ε). Indeed, from the proofs of Theorems 4.12 and 5.1 in [2], we know that there exists a family of approximate solutions $\{(u_A^\varepsilon, v_A^\varepsilon)\}_{0<\varepsilon<1}$ in $C^2(\overline{\Omega_T}) \times C^2(\overline{\Omega_T})(\Omega_T = \Omega \times (0, T))$ to (1.1) that satisfy properties (3.8)-(3.15) below:

- the uniform approximation of $(1 - 2\chi_{\Omega^-})$ holds, i.e,

$$(3.8)\qquad \lim_{\varepsilon \to 0} u_A^\varepsilon(x, 0) = \begin{cases} -1 & \text{if } x \in \Omega^- \\ \ \ 1 & \text{if } x \in \Omega^+ \end{cases} \quad \text{uniformly on compact subset}$$

- the uniform zero-order approximation of $v(x,0)$ holds, i. e.,

$$\lim_{\varepsilon\to 0}\left(\varepsilon\Delta u_A^\varepsilon - \varepsilon^{-1}f(u_A^\varepsilon)\right)(x,0) = v(x,0) \text{ uniformly on } \overline{\Omega} \tag{3.9}$$

- the following differential equations are satisfied for each $\varepsilon\in(0,1]$:

$$\begin{cases} \partial_t u_A^\varepsilon = -\Delta v_A^\varepsilon & \text{in } \Omega_T \\ \quad v_A^\varepsilon = \varepsilon\Delta u_A^\varepsilon - \varepsilon^{-1}f(u_A^\varepsilon) + r_A^\varepsilon & \text{in } \Omega_T \\ \dfrac{\partial u_A^\varepsilon}{\partial n} = \dfrac{\partial v_A^\varepsilon}{\partial n} = 0 & \text{on } \partial_T\Omega := \partial\Omega\times[0,T] \end{cases} \tag{3.10}$$

- the boundedness and thin interface conditions

$$\|u_A^\varepsilon\|_{L^\infty(\Omega_T)} \le C_0 < \infty, \quad \text{measure}\{(x,t)\in\Omega_T \mid f'(u_A^\varepsilon(x,t)) < 0\} \le C_0\varepsilon \tag{3.11}$$

and the spectral estimate

$$\inf_{0<\varepsilon\le 1}\ \inf_{0\le t\le T}\ \inf_{\substack{w\in H^1(\Omega),\int_\Omega w=0, w\not\equiv 0 \\ -\Delta\Psi=w,\frac{\partial\Psi}{\partial n}=0 \ on \ \partial\Omega}} \frac{\int_\Omega \varepsilon\left|\nabla w\right|^2 + \varepsilon^{-1}f'(u_A^\varepsilon)w^2}{\int_\Omega \left|\nabla\Psi\right|^2} \ge -C_0. \tag{3.12}$$

Moreover, for some

$$k > (N+2)\frac{N^2+6N+10}{4N+10} \tag{3.13}$$

and for all $\varepsilon\in(0,1]$, the function r_A^ε which measures the accuracy in ε of the approximate solutions $\{(u_A^\varepsilon, v_A^\varepsilon)\}_{0<\varepsilon<1}$ satisfies the following inequality

$$\|r_A^\varepsilon\|_{L^q(\Omega_T)} \le \varepsilon^{\frac{(N+6)k}{N+2}-1} \tag{3.14}$$

where $q = 2\frac{N+4}{N+6}$. The integer k is roughly the order of expansion in powers of ε needed to construct $\{u_A^\varepsilon, v_A^\varepsilon\}$. Furthermore, the following smoothness condition holds for all ε small.

$$\|u_A^\varepsilon\|_{C^{9,9/4}(\Omega_T)} + \|v_A^\varepsilon\|_{C^{7,7/4}(\Omega_T)} \le \varepsilon^{-10}. \tag{3.15}$$

Our key observations on the functions $(u_A^\varepsilon, v_A^\varepsilon)$ are the following.

Lemma 3.2. A. *The functions* $\{u_A^\varepsilon(x,0)\}_{0<\varepsilon\le 1}$ *satisfy the single-multiplicity relation*

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_A^\varepsilon(\cdot,0)) = 2\sigma\mathcal{H}^{N-1}(\Gamma). \tag{3.16}$$

B. *The gradients of the functions* $\{v_A^\varepsilon(x,0)\}_{0<\varepsilon\le 1}$ *satisfy the pointwise convergence*

$$\nabla v_A^\varepsilon(x,0) \to \nabla v(x,0) \text{ for all } x\in\Omega \tag{3.17}$$

and the uniform bound

$$|\nabla v_A^\varepsilon(x,0)| \le C_0 \text{ for all } x\in\Omega. \tag{3.18}$$

Assuming this lemma, we proceed to prove (3.7). The sought-for family $\{u_0^\varepsilon(x)\}_{0<\varepsilon<1}$ is defined by $u_0^\varepsilon(x) = u_A^\varepsilon(x,0)$. Then, from (3.8) and (3.16), it follows that (1.4) holds and that $u_0^\varepsilon(x) \to 1 - 2\chi_{\Omega^-}(x)$ in $L^1(\Omega)$. Let $(u^\varepsilon(x,t), v^\varepsilon(x,t))$ be the unique solution to (1.1) with initial data $u^\varepsilon(x,0) = u_0^\varepsilon(x) = u_A^\varepsilon(x,0)$. From (3.10)-(3.15), by Theorem 2.3 in [2], we have the estimate $\|v^\varepsilon(0) - v_A^\varepsilon(0)\|_{C^1(\Omega)} \leq \varepsilon$ for ε sufficiently small. Thus, from (3.17), (3.18) and the dominated convergence theorem, one obtains

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla v^\varepsilon(x,0)|^2\,dx = \lim_{\varepsilon\to 0}\int_\Omega |\nabla v_A^\varepsilon(x,0)|^2\,dx = \int_\Omega |\nabla v(x,0)|^2\,dx.$$

Since v^ε satisfies (1.1), we see that

$$v_0^\varepsilon(x) = \varepsilon\Delta u_0^\varepsilon(x) - \varepsilon^{-1} f(u_0^\varepsilon(x)) = \varepsilon\Delta u^\varepsilon(x,0) - \varepsilon^{-1} f(u^\varepsilon(x,0)) = v^\varepsilon(x,0).$$

Thus, (3.7) holds for the family $\{u_0^\varepsilon(x)\}_{0<\varepsilon\leq 1}$. Clearly, from (3.16) and $u_0^\varepsilon(x) = u_A^\varepsilon(x,0)$, we have (1.14) and the proof of Theorem 1.2 is completed.

The remaining of this section is devoted to the proof of Lemma 3.2. For this purpose, we must go back to the actual construction of $\{u_A^\varepsilon, v_A^\varepsilon\}$ in [2]. These functions were constructed as modifications of $\left\{u_A^K, v_A^K\right\}$ where $K - 2 \geq k$ given by (3.13). The construction of $\left\{u_A^K, v_A^K\right\}$ consists of gluing together the inner approximate solution $\left\{u_I^K, v_I^K\right\}$, the outer approximate solution $\left\{u_O^K, v_O^K\right\}$ and the boundary approximate solution $\left\{u_B^K, v_B^K\right\}$. Since their constructions are quite involved, we refer the reader to the original paper. Our task here is to demonstrate that these functions actually satisfy (3.16) - (3.18). From now on to the rest of this section, the only paper we refer to is [2] and for readability, we suppress the t variable in $u_A^\varepsilon(x,t)$ and $v_A^\varepsilon(x,t)$, etc. because we only deal with $t = 0$.

Let $\delta > 0$ be a small number as in Lemma 4.9 of [2]. It was chosen so that $\mathrm{dist}(\Gamma, \partial\Omega) > 2\delta$ and d^0 is smooth in $\Gamma(2\delta)$, where for each $\alpha > 0$, we denote

$$\Gamma(\alpha) := \{x \in \Omega \mid \mathrm{dist}(x, \Gamma) < \alpha\}$$

and d^0 is the signed distance function to the interface Γ taking negative values inside Γ.

1. First, we prove (3.16). Because $\liminf_{\varepsilon\to 0} E_\varepsilon(u_A^\varepsilon(\cdot,0)) \geq 2\sigma\mathcal{H}^{N-1}(\Gamma)$, we will prove (3.16) by showing that

$$\limsup_{\varepsilon\to 0} E_\varepsilon(u_A^\varepsilon(\cdot,0)) \leq 2\sigma\mathcal{H}^{N-1}(\Gamma). \tag{3.19}$$

By construction, u_A^ε is a finite expansion in powers of ε whose coefficients are smooth functions of the form $\varphi(\frac{d_\varepsilon^K(x)}{\varepsilon}, x)$ where $d_\varepsilon^K(x)$ behaves like the signed distance function to the interface Γ. Therefore, to prove (3.19), it suffices to prove it for the expansion $\rho^\varepsilon(x)$ up to the first order in ε of $u_A^\varepsilon(x)$, namely, we have to prove that

$$\limsup_{\varepsilon\to 0} E_\varepsilon(\rho^\varepsilon) \leq 2\sigma\mathcal{H}^{N-1}(\Gamma). \tag{3.20}$$

It follows from the construction in [2] that in the zero order expansion of u_A^ε, the boundary layer is 1, the outer expansion is 1, the inner expansion is -1 and the interface expansion is the tangent hyperbolic profile. From pp.195-196, we know that for $x \in \Gamma(\delta/2)$, $\rho^\varepsilon(x) = \tanh(\frac{d^0(x)}{\varepsilon}) + \varepsilon u^1$. The function $\tanh(z)$ arises as the unique solution to the problem

$$-\varphi'' + f(\varphi) = 0 \text{ in } I\!R,\ \varphi(0) = 0,\ \varphi(\pm\infty) = \pm 1$$

(see p. 174). Furthermore, from p. 199, we know that $u^1(x) = \Delta d^0(x)\varphi_1(\frac{d^0(x)}{\varepsilon})$ where φ_1 satisfies

$$\varphi_1''(x) - f'(\tanh(x))\varphi_1(x) = \sigma - (\mathrm{tanh(x)})',\ \varphi_1(0) = 0,\ \varphi_1 \in L^\infty(I\!R).$$

Solving this ODE, we find that $\varphi_1(x) = -\frac{1}{6}\tanh(x)^2$. Away from the interface Γ, ρ^ε tends to ± 1 exponentially fast; so for the proof of (3.20), we only need to prove that

$$\text{(3.21)}\quad \limsup_{\varepsilon\to 0} \int_{|d^0|<\delta/2} \frac{\varepsilon}{2}\left|\nabla\left(\tanh(\frac{d^0}{\varepsilon}) - \frac{\varepsilon}{6}\Delta d^0 \tanh^2(\frac{d^0}{\varepsilon})\right)\right|^2 + \frac{1}{2\varepsilon}\left(1 - \left(\tanh(\frac{d^0}{\varepsilon}) - \frac{\varepsilon}{6}\Delta d^0 \tanh^2(\frac{d^0}{\varepsilon})\right)^2\right)^2 dx \le 2\sigma \mathcal{H}^{N-1}(\Gamma).$$

There are many contributions on the left-hand side of (3.21). First, observe that the contribution coming from $\tanh(\frac{d^0(x)}{\varepsilon})$ - the zero order expansion in ε of u_A^ε, satisfies

$$\text{(3.22)}\qquad M_\varepsilon = \int_{|d^0|<\delta/2} \frac{1}{\varepsilon}\left(1 - \tanh^2(\frac{d^0}{\varepsilon})\right)^2 dx \le 2\sigma\mathcal{H}^{N-1}(\Gamma) + o(1).$$

(note that, by the choice of δ, $|\nabla d^0(x)| = 1$ if $x \in \Gamma_0(\delta/2)$). Indeed, we split M_ε into

$$M_\varepsilon = \int_{\sqrt{\varepsilon}\le|d^0|<\delta/2} \frac{1}{\varepsilon}\left(1 - \tanh^2(\frac{d^0}{\varepsilon})\right)^2 dx + \int_{|d^0|<\sqrt{\varepsilon}} \frac{1}{\varepsilon}\left(1 - \tanh^2(\frac{d^0}{\varepsilon})\right)^2 dx := N_\varepsilon + P_\varepsilon.$$

Then, using the coarea formula to estimate

$$\begin{aligned} P_\varepsilon &\le \left(\int_{-\frac{1}{\sqrt{\varepsilon}}}^{\frac{1}{\sqrt{\varepsilon}}} \left(1 - \tanh^2(t)\right)^2 dt\right) \max_{|s|\le\sqrt{\varepsilon}} \mathcal{H}^{N-1}(d^0(x) = s) \\ &\le 2\sigma \max_{|s|\le\sqrt{\varepsilon}} \mathcal{H}^{N-1}(d^0(x) = s) \le 2\sigma\mathcal{H}^{N-1}(\Gamma) + o(1), \end{aligned}$$

and

$$N_\varepsilon \le \left(\int_{\frac{1}{\sqrt{\varepsilon}}\le|t|\le\frac{\delta}{2\varepsilon}} \left(1 - \tanh^2(t)\right)^2 dt\right) \max_{\sqrt{\varepsilon}\le|s|\le\delta/2} \mathcal{H}^{N-1}(d^0(x) = s) = o(1),$$

and thus obtain (3.22). Now, using the coarea formula and estimating similarly, we find that other contributions in (3.21) vanish in the limit as $\varepsilon \searrow 0$. The proof of (3.21) is now complete.

2. Next, we prove (3.17) and (3.18).

From the construction of v_A^ε in pp. 195-197, we see that the outer expansion v_O^K and the boundary layer expansion v_B^K of v_A^ε have uniform gradient bounds in ε. Therefore, we shall show (3.17) and (3.18) by proving that

$$v_A^\varepsilon \text{ converges locally uniformly in the } C^1 \text{ sense to } v \textit{ in } \Omega\backslash\Gamma \tag{3.23}$$

and that

$$|\nabla v_A^\varepsilon(x)| \leq C_0 \text{ for all } x \in \Gamma(\delta/2). \tag{3.24}$$

3. First, we prove (3.24). The construction of v_A^ε can be found in pp. 195-197. We know from p.197 that $v_A^\varepsilon = v_A^K - \hat{e}^K$ but from p.196, we also know that for $x \in \Gamma(\delta/2)$, $v_A^K = v_I^K$. Therefore, for $x \in \Gamma(\delta/2)$, $v_A^\varepsilon = v_I^K - \hat{e}^K$. In the above formulas, $v_I^K(x)$ is defined by

$$v_I^K(x) = \sum_{i=0}^{K} \varepsilon^i v^i(z,x)\Bigg|_{z=\frac{d_\varepsilon^K(x)}{\varepsilon}}, \tag{3.25}$$

where $d_\varepsilon^K(x) = \sum_{i=0}^K \varepsilon^i d^i(x)$. Recall that $d^0(x)$ is the signed distance to the interface Γ and for each $i \geq 1$, $d^i(x)$ is a smooth function defined in a neighbourhood of Γ as in p. 176. We now estimate $\nabla v_I^K(x)$ and $\nabla \hat{e}^K(x)$ separately. For $v_I^K(x)$, the most dangerous term for the boundedness of its gradient comes from the term $v^0(z,x)$. So it suffices to show that

$$\left|\nabla v^0(z,x)\right| \leq C_0 \text{ for all } x \in \Gamma(\delta/2). \tag{3.26}$$

The function v^0 constructed in p. 191 is explicitly given by the formula

$$v^0(z,x) = v_0^+(x)\eta(z) + v_0^-(x)(1-\eta(z)) \tag{3.27}$$

where $\eta(z) \in C^\infty(I\!\!R)$ is a smooth function satisfying (see p. 179) $\eta^{'}(z) \geq 0$ $\forall z \in I\!\!R$, and

$$\eta(z) = 0 \textit{ if } z \leq -1;\ \eta(z) = 1 \textit{ if } z \geq 1;\ \int_{-\infty}^{\infty}[\eta(z) - 1/2](1-\tanh^2(z))dz = 0.$$

The functions $v_0^\pm$ are defined as follows. First, let $v^\pm$ be the restrictions of v on $Q^\pm$ (Q^- is the inside of Γ; while Q^+ is the ouside of Γ). Then $v_0^\pm$ are the smooth extensions of $v^\pm$ to $Q^\pm \cup \Gamma(2\delta)$. We have

$$D_x v^0(z,x) = D_x v_0^+(x)\eta(z) + D_x v_0^-(x)(1-\eta(z)) + (v_0^+(x) - v_0^-(x))\eta^{'}(z) D_x z. \tag{3.28}$$

The first two terms on the right hand side of (3.28) are clearly bounded. Consider now the third term. Note that $\eta^{'}(z) \neq 0$ if and only if $z \in [-1,1]$. Furthermore, when $t=0$ and $x \in \Gamma$ one has $v_0^+(x,0) = v_0^-(x,0) = \sigma\kappa$. We will explore this relation to prove the boundedness of the third term on the right hand side of (3.28). For simplicity, consider the case $z \in [0,1]$. Then $x \in Q^+$ and from the inequality $z = \left(\sum_{i=0}^{K} \varepsilon^i d^i(x)\right) / \varepsilon \leq 1$ we deduce that $d^0(x) \leq \varepsilon$. If ε is sufficiently small, there exists a unique $x_0 \in \Gamma$ such that $d^0(x) = \mathrm{dist}(x, x_0)$. Hence we get the following estimates

$$\begin{aligned} \left|v_0^+(x) - v_0^-(x)\right| &\leq \left|v_0^+(x) - \sigma\kappa\right| + \left|v_0^-(x) - \sigma\kappa\right| \leq \left|v_0^+(x) - v_0^+(x_0)\right| + \left|v_0^-(x) - v_0^-(x_0)\right| \\ &\leq \sup\left|\nabla v_0^+\right| \mathrm{dist}(x, x_0) + \sup\left|\nabla v_0^-\right| \mathrm{dist}(x, x_0) \leq C_0 \varepsilon. \end{aligned} \tag{3.29}$$

On the other hand, for $z = \varepsilon^{-1} \sum_{i=0}^{K} \varepsilon^i d^i(x)$, we have $|D_x z| \leq C_0/\varepsilon$. Thus, combining this with (3.28) and (3.29), we easily get (3.26).
It remains to estimate the term $\hat{e}^K$. It is defined in p. 197 as the solution to the equations

$$\Delta \hat{e}_K = e_K - \frac{1}{|\Omega|} \int_\Omega e_K(\xi) d\xi \ \text{ in } \Omega, \ \frac{\partial \hat{e}_K}{\partial n} = 0 \ \text{ on } \partial\Omega \ \text{ and } \int_\Omega \hat{e}_K(\xi) d\xi = 0.$$

Here $\|e_K\|_{C^0(\Omega_T)} = 0(\varepsilon^{K-1})$. By elliptic regularity, one has $\hat{e}_K(x) \in C^1(\Omega)$ and

$$\lim_{\varepsilon \to 0} \|\hat{e}_K\|_{C^1(\Omega)} = 0. \tag{3.30}$$

Now combining (3.26) and (3.30) yields (3.24).
4. Finally, we prove (3.23). We have by definition $v_A^\varepsilon = v_A^K - \hat{e}^K$ and thus

$$\|v_A^\varepsilon - v\|_{C^1} \leq \left\|v_A^K - v_O^K\right\|_{C^1} + \left\|v_O^K - v\right\|_{C^1} + \left\|\hat{e}^K\right\|_{C^1}, \tag{3.31}$$

where v_O^K is defined in p. 195. We have the following formulae for v (p. 191) and v_O^K :

$$v = v_0^+ \chi_{\overline{Q_0^+}} + v_0^- \chi_{Q_0^-}, \quad v_O^K(x,t) = \sum_{i=0}^{K} \varepsilon^i v_i^+ \chi_{\overline{Q_0^+}} + \sum_{i=0}^{K} \varepsilon^i v_i^- \chi_{Q_0^-} \ \forall x \in \Omega, \tag{3.32}$$

(for the construction of $v_i^\pm$'s, see pp. 177-179). Therefore, $\lim_{\varepsilon\to 0} \left\|v_O^K - v\right\|_{C^1(\Omega)} = 0$. This, together with (3.30) and (3.31), will imply (3.23) once the following is proved

$$v_A^K \text{ converges locally uniformly in the } C^1 \text{ sense to } v_O^K \ \textit{in } \Omega \backslash \Gamma. \tag{3.33}$$

This can be argued similarly as in the proof of Theorem 4.12 of [2], using the outer-boundary matching conditions, the inner-outer matching conditions and a boundary layer of order ε^4 for $u_A^\varepsilon, v_A^\varepsilon$. For the convenience of reader, we give here the details of the argument. Let $\zeta \in C_0^\infty(\mathbb{R})$ be a cut-off function such that $\zeta(z) = 1$ if $|z| < 1/2$, $\zeta(z) = 0$ if $|z| > 1$ and $z\zeta'(z) \leq 0$ in $\mathbb{R}$. Then, we have (see p. 196)

$$v_A^K = \begin{cases} v_B^K & \text{in } \overline{\partial\Omega(\delta/2)} \\ v_B^K \zeta(d_B/\delta) + (1 - \zeta(d_B/\delta)) v_O^K & \text{in } \partial\Omega(\delta) \backslash \partial\Omega(\delta/2) \\ v_O^K & \text{in } \Omega \backslash (\partial\Omega(\delta) \cup \Gamma(\delta)) \\ v_O^K \zeta(d^0/\delta) + (1 - \zeta(d^0/\delta)) v_O^K & \text{in } \Gamma(\delta) \backslash \Gamma(\delta/2) \\ v_I^K & \text{in } \Gamma(\delta/2). \end{cases}$$

As in p. 196, we have

$$\lim_{\varepsilon\to 0}\left\|v_A^K - v_O^K\right\|_{C^2(\partial\Omega(\delta)\backslash\partial\Omega(\delta/2))} + \lim_{\varepsilon\to 0}\left\|v_A^K - v_O^K\right\|_{C^2(\Gamma(\delta)\backslash\Gamma(\delta/2))} = 0. \tag{3.34}$$

By the definition of v_B^K in p. 196, we have

$$v_B^K(x) = \sum_{i=0}^{K}\varepsilon^i v_B^i(z,x)\mid_{z=d_B(x)/\varepsilon} -\varepsilon^K v_B^K(0,x)\ \forall x\in\overline{\partial\Omega(\delta)}. \tag{3.35}$$

From (3.32), (3.35) and Lemma 4.7 of [2] which asserts that $v_B^i(\cdot,x)\equiv v_i^+(x)$ on $\overline{\partial\Omega(\delta)}$ for $i=0,1,2,3,4$ we see that

$$\lim_{\varepsilon\to 0}\left\|v_A^K - v_O^K\right\|_{C^2(\overline{\partial\Omega(\delta/2)})} = 0. \tag{3.36}$$

On $\Gamma(\delta/2)$, we use the definition of v_I^K (see p. 195 of [2]) which is given by (3.25). Let M be a compact set in $\Omega\backslash\Gamma$. Then there exists $\beta>0$ such that for all $x\in M$ we have $|d^0|\geq 2\beta$ where we recall that d^0 is the signed distance to Γ. So, when ε is small enough, $d_\varepsilon^K = d^0+\sum_{i=1}^K\varepsilon^i d^i$ satisfies $\left|d_\varepsilon^K\right|\geq\beta$ for all $x\in M$. This implies that for $x\in M$ we have $\lim\limits_{\varepsilon\to 0}|z| = \lim\limits_{\varepsilon\to 0}\dfrac{\left|d_\varepsilon^K\right|}{\varepsilon} = +\infty$. Using the inner-outer expansions (see (4.4), p. 178 of [2] which aserts that for all $i,m,n,l\geq 0$ we have $D_x^m D_t^n D_z^l[v^i(\pm z,x,t) - v_i^\pm(x,t)] = 0(e^{-\alpha z})\ as\ z\to\infty$) we find that (3.32) and (3.25) yield

$$\lim_{\varepsilon\to 0}\left\|v_A^K - v_O^K\right\|_{C^2(M)} = 0. \tag{3.37}$$

From (3.34), (3.36) and (3.37) we get (3.33). This completes the proof of Lemma 3.2. □

Remark 3.2. *In the construction part of the L^2-version of De Giorgi's conjecture or of the fact that the Modica-Mortola functional Γ-converges to the area functional, one only needs to use the tangent hyperbolic profile $tanh(\frac{dist(x,\Gamma)}{\varepsilon})$. This is the zero-order expansion in powers of ε. On the contrary, the construction of u_0^ε in [2] requires the expansion up to the k^{th}- order of ε where $k>(N+2)\frac{N^2+6N+10}{4N+10}$ given in (3.13). So, the profile of u_0^ε is highly sensitive to the dimension.*

Remark 3.3. *It would be very interesting to construct a profile for u_0^ε that is independent of the dimension. Given a smooth interface Γ in Ω. Let $v=\widetilde{\sigma\kappa}$ and $m=\frac{1}{|\Omega|}\left(|\Omega^+|-|\Omega^-|\right)$. For each ε in $(0,1)$, let u_0^ε solve the minimization problem*

$$\min_{u\in H^1(\Omega),\ \int_\Omega u=m|\Omega|}\left(E_\varepsilon(u)+\int_\Omega uv\right). \tag{3.38}$$

Then, u_0^ε is smooth and satisfies

$$-\varepsilon\Delta u_0^\varepsilon + \varepsilon^{-1}f(u_0^\varepsilon) + v = \lambda^\varepsilon\ in\ \Omega,\ and\ \frac{\partial u_0^\varepsilon}{\partial n}(x) = 0\ on\ \partial\Omega,$$

where λ^ε *is a Lagrange multiplier. Let* $v_0^\varepsilon = \varepsilon\Delta u_0^\varepsilon - \varepsilon^{-1} f(u_0^\varepsilon) = v - \lambda^\varepsilon$*. Then clearly we have (3.7). We can show that* $\{u_0^\varepsilon\}_{0<\varepsilon<1}$ *are uniformly bounded and satisfy the uniform energy bound* $E_\varepsilon(u_0^\varepsilon) \leq M < \infty$*. The sequence* $\{u_0^\varepsilon\}$ *has an* $L^1(\Omega)$*- limit* $u^0 \in BV(\Omega, \{-1,1\})$ *with interface* $\tilde{\Gamma} = \partial\{u^0 = 1\}$ *separating the phases. The construction part is complete once we verify that* $u^0(x) = 1 - 2\chi_{\Omega^-}(x)$*. Were this true, we would find that* $\Gamma \equiv \tilde{\Gamma}$*. However, we do not know whether* $\tilde{\Gamma}$ *and* Γ *coincides. In other words, by solving the minimization problem (3.38), we may lose the interface* Γ*.*

4 The Limiting Dynamics of the Cahn-Hilliard Equation

In the first part of this section, we prove the lower bound for the velocity as stated in Proposition 1.1. Then, as a preparation for the proof of Theorem 1.3, we prove a selection result, saying that after a suitable extraction, we have that for all $t \in [0,T]$, $u^\varepsilon(x,t)$ converges strongly in $L^1(\Omega)$ to $u^0(x,t) \in$ BV $(\Omega, \{-1,1\})$. Finally, we establish the limiting dynamical law of the Cahn-Hilliard equation, Theorem 1.3.

4.1 Lower bound on the velocity and a selection result

First, we prove Proposition 1.1.

Proof. Fix $t \in (0,T)$. It suffices to prove (1.7) for the case where its left-hand side is finite. In this case, the sequence $\{\partial_t u^\varepsilon(x,s)\}$ is bounded in $L^2((0,t), H_n^{-1}(\Omega))$. Therefore, we can extract a subsequence which is still denoted by $\{\partial_t u^\varepsilon(x,s)\}$ that converges weakly to $w \in L^2((0,t), H_n^{-1}(\Omega))$. On the other hand, from the assumptions of our Proposition and the dominated convergence theorem, we find that $u^\varepsilon \to u$ in $L^1(\Omega \times [0,T])$. It follows that $\partial_t u^\varepsilon(x,s) \to \partial_t u(x,s)$ in the sense of distributions and thus $w(x,s) = \partial_t u(x,s)$. Denote by $\Omega^+(s)$ the set $\{x \in \Omega : u(x,s) = 1\}$ and recall that $\Gamma(s) = \partial\{u(s) = 1\}$ is the interface separating the phases -1 and $+1$. Then, $\partial_t u(s) = \partial_t(u(s)+1) = \partial_t(2\chi_{\Omega^+(s)}) = 2\partial_t\Gamma(s)$. Now, (1.7) follows from the lower semicontinuity property of weak convergence

$$\liminf_{\varepsilon\to 0} \int_0^t \|\partial_t u^\varepsilon(s)\|^2_{H_n^{-1}(\Omega)}\, ds \geq \int_0^t \|\partial_t u(s)\|^2_{H_n^{-1}(\Omega)} = 4\int_0^t \|\partial_t \Gamma(s)\|^2_{H_n^{-1}(\Omega)}\, ds.$$

□

Next, let us turn to the selection result. For the rest of the section, $(u^\varepsilon, v^\varepsilon)$ denotes the solution of (1.1) on $\Omega \times [0,\infty)$. Let $T > 0$ be any finite number. Then, for all $t \in [0,T]$,

$$\int_\Omega \frac{\varepsilon\,|\nabla u^\varepsilon(t)|^2}{2} + \frac{W(u^\varepsilon(t))}{\varepsilon} = E_\varepsilon(u^\varepsilon(t)) = E_\varepsilon(u^\varepsilon(0)) - \int_0^t \|\nabla v^\varepsilon(s)\|^2_{L^2(\Omega)}\, ds \leq E_\varepsilon(u^\varepsilon(0)) \leq M. \tag{4.1}$$

Using (4.1) and the compactness of BV functions in $L^1(\Omega)$, we can prove the following

Lemma 4.1. *For each* $t \in [0,T]$, $u^\varepsilon(\cdot,t)$ *converges strongly in* $L^1(\Omega)$ *(after extraction a subsequence) to* $u^0(\cdot,t)$ *where* $u^0(\cdot,t) \in BV(\Omega, \{-1,1\})$*.*

This lemma can be proved similarly as in Sternberg [26] and we thus omit the proof.

Note that the choice of the subsequence in Lemma 4.1 may differ for each t and this is not sufficient to establish our convergence result in Theorem 1.3. Therefore, we first prove that we can actually find a subsequence of ε such that the $L^1(\Omega)$-convergence is valid for all time slices. The idea of the proof is to establish some time-continuity property of the limiting function u^0. Before stating our result in this regard, we define the following norm on distributions u on Ω

$$\|u\|_1 = \sup_{\varphi\in C_0^\infty(\Omega),\ |\nabla\varphi|\le 1} \left|\int_\Omega u\varphi\right|, \tag{4.2}$$

i.e., the norm in the dual of Lipschitz functions. We are going to prove the following

Proposition 4.1. *There exists $u^0 \in L^4(\Omega\times[0,T])$ such that u^0 is $C^{0,1/2}$ in time for the $\|\cdot\|_1$-norm, and that, after extraction,*

$$u^\varepsilon \rightharpoonup u^0 \quad \text{in } L^4(\Omega\times[0,T]). \tag{4.3}$$

Moreover, for all $t\in[0,T]$, we have $u^0(t)\in BV(\Omega,\{-1,1\})$ and

$$u^\varepsilon(t) \rightharpoonup u^0(t) \quad \text{in } L^4(\Omega),\ u^\varepsilon(t) \longrightarrow u^0(t) \quad \text{in } L^1(\Omega). \tag{4.4}$$

Proof. Our proof is inspired by the proof of Theorem 3 in Sandier and Serfaty [24] in the context of Ginzburg-Landau vortices. From (4.1), we find that

$$\int_0^T \int_\Omega \frac{1}{2\varepsilon}(1-|u^\varepsilon(x,t)|^2)^2 dxdt \le \int_0^T E_\varepsilon(u^\varepsilon(t))dt \le MT < \infty.$$

Thus u^ε is uniformly bounded in $L^4(\Omega\times[0,T])$. Therefore, there exists $u^0\in L^4(\Omega\times[0,T])$ such that, after extraction, we have (4.3). Now, we prove the Hölder continuity in time for the $\|\cdot\|_1$-norm of u^0. Consider $\zeta\in C_c^1(\Omega)$ with $\|\nabla\zeta\|_{L^\infty(\Omega)}\le 1$. Let $\varphi\in C_c^\infty((0,T))$. Since ζ is independent of time and $\partial_t u^\varepsilon = -\Delta v^\varepsilon$, we get

$$\int_0^T\int_\Omega u^\varepsilon(\cdot,t)\cdot\zeta\partial_t\varphi dxdt = \int_0^T\int_\Omega u^\varepsilon(\cdot,t)\cdot\partial_t(\zeta\varphi)dxdt = \int_0^T\int_\Omega -\nabla v^\varepsilon(x,t)\cdot\nabla(\zeta\varphi)dxdt.$$

Since $\nabla(\zeta\varphi)=(\nabla\zeta)\varphi$ and using Hölder's inequality, we can estimate the last integral from above by

$$|\Omega|^{1/2}\left(\int_0^T |\varphi(t)|^2\,dt\right)^{1/2}\left\{\int_0^T\left(\int_\Omega |\nabla v^\varepsilon(x,t)\nabla\zeta(x)|\,dx\right)^2\right\}^{1/2}$$
$$\le \|\nabla\zeta\|_{L^\infty(\Omega)}|\Omega|\left(\int_0^T\int_\Omega|\nabla v^\varepsilon(x,t)|^2\,dxdt\right)^{1/2}\|\varphi\|_{L^2(0,T)} \le |\Omega|\,M^{1/2}\,\|\varphi\|_{L^2(0,T)}\,.$$

The last inequality follows from (4.1) and $\|\nabla\zeta\|_{L^\infty(\Omega)} \leq 1$. Consequently, by the weak convergence (4.3), we have

$$\int_0^T \int_\Omega u^0(\cdot,t)\cdot\zeta\partial_t\varphi(t)dxdt = \lim_{\varepsilon\to 0}\int_0^T \int_\Omega u^\varepsilon(\cdot,t)\cdot\zeta\partial_t(\varphi)dxdt \leq |\Omega|\, M^{1/2}\,\|\varphi\|_{L^2(0,T)}\,.$$

From the definition of the $\|\cdot\|_1$-norm in (4.2), we deduce that $t \longmapsto \|u^0(\cdot,t)\|_1$ is in $H^1((0,T))$ and by the continuous imbedding $H^1(0,T) \hookrightarrow C^{0,1/2}([0,T])$, the Hölder continuity in time for the $\|\cdot\|_1$-norm of u^0 follows.

We now prove (4.4). Let us choose a time $t_0 \in [0,T]$. Since $E_\varepsilon(u^\varepsilon(t_0)) \leq M$, we see that, after extraction $u^\varepsilon(t_0)$ has a weak limit u in $L^4(\Omega)$. Let us consider $\overline{u^\varepsilon}$ defined in $[-T,T]$ by $\overline{u^\varepsilon} = u^\varepsilon(t_0)$ for $t < t_0$ and $\overline{u^\varepsilon}(t) = u^\varepsilon(t)$ for $t \geq t_0$. One can easily check that $\overline{u^\varepsilon}$ is uniformly bounded in $L^4(\Omega\times[-T,T])$, thus we deduce that $\overline{u^\varepsilon}$ converges weakly in $L^4(\Omega\times[-T,T])$ (after extraction) to some limiting function $\overline{u}$ that is $C^{0,1/2}$ in time for the $\|\cdot\|_1$-norm. Using test functions, we see that $\overline{u} = u$ a.e in $(-T,t_0)$ and $\overline{u} = u^0$ a.e in (t_0,T). Because $\overline{u}$ and u^0 are both continuous in time, we must have, by continuity at the time t_0, $u = u^0(t_0)$. We deduce that the only possible limit of extracted sequences of $u^\varepsilon(t_0)$ is $u^0(t_0)$ and thus $u^\varepsilon(t_0)$ converges weakly in $L^4(\Omega)$ to $u^0(t_0)$ for all $t_0 \in [0,T]$. On the other hand, the inequality $E_\varepsilon(u^\varepsilon(t_0)) \leq M$ and Lemma 4.1 allow us to extract a further subsequence of ε such that $u^\varepsilon(t_0)$ converges strongly in $L^1(\Omega)$ to some $w \in BV(\Omega,\{-1,1\})$. Thus, we must have $u^0(t_0) = w \in BV(\Omega,\{-1,1\})$ and $u^\varepsilon(t_0)$ converges stronly in $L^1(\Omega)$ to $u^0(t_0)$ for all $t_0 \in [0,T]$. This completes the proof of (4.4) and our proposition. □

We are now in a position to prove Theorem 1.3. Here we follow the method of [25].

4.2 Proof of Theorem 1.3.

Proof. **1.** First, the existence of $T_* > 0$ in the first statement of the theorem follows from the time-continuity property of the limiting function $u^0 \in L^4(\Omega\times[0,T])$ proved in Proposition 4.1. By the selection result in Proposition 4.1, after extraction, we have that for all $t \in [0,T_*]$, $u^\varepsilon(x,t)$ converges strongly in $L^1(\Omega)$ to $u^0(x,t) \in \mathrm{BV}\ (\Omega,\{-1,1\})$ with interface $\Gamma(t) = \partial\{x \in \Omega : u^0(x,t) = 1\} \subset \Omega$. Let us prove that the interfaces $\Gamma(t)$ $(t \in [0,T_*))$ evolve by the Mullins-Sekerka law (1.2). Indeed, because the solution $(u^\varepsilon, v^\varepsilon)$ to (1.1) is smooth, the flow for $E_\varepsilon(u^\varepsilon)$ is conservative. Hence, we have for all $t \in (0,T_*)$

$$\begin{aligned}
E_\varepsilon(u^\varepsilon(0)) - E_\varepsilon(u^\varepsilon(t)) &= -\int_0^t < \nabla_{H_n^{-1}(\Omega)}E_\varepsilon(u^\varepsilon(s)), \partial_t u^\varepsilon(s) >_{H_n^{-1}(\Omega)} ds \\
&= \frac{1}{2}\int_0^t \left\|\nabla_{H_n^{-1}(\Omega)}E_\varepsilon(u^\varepsilon(s))\right\|^2_{H_n^{-1}(\Omega)} + \|\partial_t u^\varepsilon(s)\|^2_{H_n^{-1}(\Omega)}\,ds \\
&= \frac{1}{2}\int_0^t \|\nabla v^\varepsilon(s)\|^2_{L^2(\Omega)} + \|\partial_t u^\varepsilon(s)\|^2_{H_n^{-1}(\Omega)}\,ds.
\end{aligned}$$

For each $s \in (0,t)$, recall that $\kappa(s)$ is the mean curvature of $\Gamma(s)$. Define the function $w(x,s) \in H^1(\Omega)$ to be the natural extension of $\sigma\kappa(s)$ over Ω, i.e., $w(x,s)$ satisfies

$\Delta w(x,s) = 0$ in $\Omega \setminus \Gamma(s)$, $w(x,s) = \sigma\kappa(s)$ on $\Gamma(s)$ and finally $\frac{\partial w}{\partial n} = 0$ on $\partial\Omega$. By Proposition 4.1, all assumptions of Proposition 1.1 are satisfied for u^ε and u^0. Thus, using $(A3)$, the lower bound on velocity (1.7) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} (4.5)\, E_\varepsilon(u^\varepsilon(0)) - E_\varepsilon(u^\varepsilon(t)) &\geq \frac{1}{2}\int_0^t \sigma^2 \|\kappa\|^2_{H_n^{1/2}(\Gamma(s))} + 4\|\partial_t\Gamma(s)\|^2_{H_n^{-1}(\Omega)}\, ds - o(1) \\ &= \frac{1}{2}\int_0^t \int_\Omega |\nabla w(x,s)|^2 + 4\left|\nabla\Delta_n^{-1}\partial_t\Gamma(x,s)\right|^2 dxds - o(1) \\ (4.6) \qquad &\geq -2\int_0^t \int_\Omega \nabla w(x,s)\cdot\nabla(\Delta_n^{-1}\partial_t\Gamma(x,s))dxds - o(1). \end{aligned}$$

In view of the definition of Δ_n^{-1} in (1.5), the right hand side of (4.6) becomes

$$\begin{aligned} 2\int_0^t <\partial_t\Gamma(s), w> ds - o(1) &= \int_0^t \int_{\Gamma(s)} 2\sigma\kappa(s)\partial_t\Gamma(s)d\mathcal{H}^{N-1}ds - o(1) \\ (4.7) \qquad &= -\int_0^t \frac{d}{ds}E(\Gamma(s))ds - o(1) = E(\Gamma(0)) - E(\Gamma(t)) - o(1). \end{aligned}$$

Equality (4.7) follows from the smoothness assumption (A2). From (4.5)-(4.7), one gets

$$E_\varepsilon(u^\varepsilon(t)) - E(\Gamma(t)) \leq E_\varepsilon(u^\varepsilon(0)) - E(\Gamma(0)) + o(1).$$

By $(A1)$, we deduce that $\limsup_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon(t)) \leq E(\Gamma(t))$. However, since E_ε $\Gamma-$ converges to E, we have $\liminf_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon(t)) \geq E(\Gamma(t))$. Therefore, we must have

$$(4.8) \qquad \lim_{\varepsilon\to 0} E_\varepsilon(u^\varepsilon(t)) = E(\Gamma(t)).$$

This means that well-prepared initial data remains "well-prepared" in time. Furthermore, this also shows that the inequality (4.6) is actually an equality. This implies that for each $s \in (0,t)$ and for a.e $x \in \Omega$, we have $\nabla w(x,s) = -2\nabla\Delta_n^{-1}\partial_t\Gamma(x,s)$. So $w(x,s) = -2\Delta_n^{-1}\partial_t\Gamma(x,s) + c(s)$ for some function c depending only on time. Thus, in the sense of distributions $\partial_t\Gamma(x,s) = -\frac{1}{2}\Delta w$ and by the definition of the function w, this relation is exactly the limiting dynamical law we wish to establish. Our proof of this Mullins-Sekerka law is valid as long as $\Gamma(t) \subset \Omega$ and hypersurfaces contained in $\Gamma(t)$ do not collide for all $t < T_*$. Consequently, T_* can be chosen to be the minimum of the collision time and of the exit time from Ω of the hypersurfaces under the Mullins-Sekerka law.

2. Second, we show that v^ε converges weakly in $L^2((0,T_*), H^1(\Omega))$ to w. Indeed, for all $t \in (0,T_*)$ we have

$$\int_0^t \|\nabla v^\varepsilon(s)\|^2_{L^2(\Omega)}\, ds = E_\varepsilon(u^\varepsilon(0)) - E_\varepsilon(u^\varepsilon(t)) \leq M.$$

Again, using Lemma 3.4 in Chen [6], we see that for ε sufficiently small,

$$\|v^\varepsilon(s)\|_{H^1(\Omega)} \leq C(E_\varepsilon(u^\varepsilon(s)) + \|\nabla v^\varepsilon(s)\|_{L^2(\Omega)}) \leq C(M + \|\nabla v^\varepsilon(s)\|_{L^2(\Omega)}).$$

It follows that for ε sufficiently small, we have

$$\int_0^t \|v^\varepsilon(s)\|^2_{H^1(\Omega)}\,ds \le C(M^2 + \int_0^t \|\nabla v^\varepsilon(s)\|^2_{L^2(\Omega)}\,ds) \le C < \infty.$$

Therefore, up to a further extraction, we have that v^ε weakly converges to some v in $L^2((0,T_*),H^1(\Omega))$. We are going to prove that for a.e. $t \in (0,T_*)$,

$$v(x,t) = \sigma\kappa(x,t) = w(x,t) \text{ for } \mathcal{H}^{N-1} \text{ a.e. } x \in \Gamma(t). \tag{4.9}$$

Indeed, from the single-multiplicity property (4.8) of the limiting interface $\Gamma(t)$ on each time slice and the uniform bound (4.1) on the energy $E_\varepsilon(u^\varepsilon(t)) \le M$ for all $t \in [0,T_*]$ and all $\varepsilon > 0$, by the dominated convergence theorem, we have
- The single-multiplicity in space-time, i.e, in the sense of Radon measures,

$$\left(\frac{\varepsilon\,|\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon}\right) dxdt \rightharpoonup 2\sigma d\mathcal{H}^{N-1}\lfloor\Gamma(t)dt.$$

- The limiting equipartition of energy in space-time, i.e, in the sense of Radon measures

$$\left|\frac{\varepsilon\,|\nabla u^\varepsilon|^2}{2} - \frac{W(u^\varepsilon)}{\varepsilon}\right| dxdt \rightharpoonup 0.$$

Arguing as in the proof of Lemma 3.1, we get (4.9). Now, passing to the limit in the equation $\partial_t u^\varepsilon = -\Delta v^\varepsilon$ and recalling that v^ε satisfies the zero Neumann boundary condition, we find that $2\partial_t\Gamma(s) = -\Delta v$ in $\Omega \times (0,T_*)$ and $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega \times (0,T_*)$ in the sense of distributions (see the proof of Proposition 1.1). Therefore, in the sense of distributions, $\Delta(v-w) = 0$ in $\Omega \times (0,T_*)$ and $\frac{\partial(v-w)}{\partial n} = 0$ on $\partial\Omega \times (0,T_*)$. From (4.9), we conclude that $v = w$ a.e. in $\Omega\times(0,T_*)$ and this shows that v^ε converges weakly to w in $L^2((0,T_*),H^1(\Omega))$.
3. Finally, we now complete the proof of the theorem by showing that v^ε actually converges strongly in $L^2((0,T_*),H^1(\Omega))$ to w. In fact, because of the equality (4.8), the inequality (4.5) is actually an equality. Therefore

$$\lim_{\varepsilon\to 0}\int_0^{T_*} \|\nabla v^\varepsilon(s)\|^2_{L^2(\Omega)} = \int_0^{T_*}\int_\Omega |\nabla w(x,s)|^2\,dxds.$$

Since ∇v^ε converges weakly to ∇w in $L^2((0,T_*),L^2(\Omega))$, we conclude that ∇v^ε converges strongly to ∇w in $L^2((0,T_*),L^2(\Omega))$. It follows that v^ε converges strongly to w in $L^2((0,T_*),H^1(\Omega))$ and this completes the proof of Theorem 1.3. □

Remark 4.1. *It follows from the proof of this theorem and that of Theorem 1.2 that the "well-preparedness" in time of the Cahn-Hilliard equation is equivalent to the inequality (1.13) for almost every time t. Observe that this proof only relies on item A in Theorem 1.2 and Proposition 1.1 so it is quite short. The main issue seems to be really in the question of constant multiplicity and limiting equipartition of energy.*

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