Exponential functions

Question: If $b$ is a positive real number, and $x$ any real number, what is meant by $b^x$?

Answers:

(i) If $x = n$, a positive integer, $b^x = b^n = b \cdot b \cdot \ldots \cdot b$ (n times).

(ii) If $x = \frac{1}{n}$, where $n$ is a positive integer, $b^x = b^{1/n} = \sqrt[n]{b}$, so $(\sqrt[n]{b})^n = b$.

(iii) If $x = -1$, $b^x = b^{-1} = \frac{1}{b}$.

Combining these, we know what $b^x$ is for $x$ any rational number. For example: $b^{3/5} = \frac{1}{\sqrt[5]{b^3}} = \frac{1}{\sqrt[5]{b^3}}$.

(iv) What is $b^x$ when $x$ is irrational? To define $b^x$, sandwich $x$ between a bunch of rational numbers $r < x < s$; let $r, s$ be rational and get closer and closer to $x$. Then $b^r, b^s$ will get closer to $b^x$:

$$b^r < b^x < b^s.$$ 

See book, p. 46 for more discussion. (Note: We will never compute $b^x$ this way! This is just so we have a strict definition of $b^x$.)

Properties: Let $b > 0$ as above.

0. $b^0 = 1$
1. $b^{x+y} = b^x b^y$ (These imply $b^{-x} = 1/b^x$)
2. $b^{x-y} = b^x/b^y$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$ (here $a, b > 0$ too.)

Note: Range of $f(x) = b^x$ is all positive real numbers $= (0, \infty)$.

Example: Let $f(x) = 2^x$. Graph $y = f(x)$.

Domain: $2^x$ makes sense for all real number $x$, by above, so Domain = $(-\infty, \infty)$.

Table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
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<tbody>
<tr>
<td>-3</td>
<td>$1/8$</td>
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<tr>
<td>-2</td>
<td>$1/4$</td>
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<tr>
<td>-1</td>
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<td>0</td>
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<td>2</td>
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<td>3</td>
<td>8</td>
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The number $e$: We define the number $e$ to be the positive real number so that the tangent line at $(0, 1)$ to the graph of $y = e^x$ has slope 1.

Approximately, $e \approx 2.71828$.

Remark: $\sqrt{2} = 2^{1/2} \approx 1.41$.

Can see the point $(\frac{1}{2}, 1.41)$ roughly on the graph.
A function $f$ is one-to-one if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

In other words, $f$ never takes the same value twice.

Graphically, one-to-one functions pass the horizontal line test:

![Graphical representation of one-to-one function](image)

Let $f$ be a one-to-one function with domain $D$ and range $E$. The inverse function $f^{-1}$ has domain $E$ and range $D$ and is defined by $f^{-1}(y) = x$ if and only if $f(x) = y$.

Properties of $f^{-1}$:

- Switches domain and range:
  
  Domain of $f^{-1} = \text{Range of } f$.
  
  Range of $f^{-1} = \text{Domain of } f$.

- We have
  
  $f^{-1}(f(x)) = x$ for $x$ in $D = \text{domain of } f$.
  
  $f(f^{-1}(x)) = x$ for $x$ in $E = \text{range of } f$.

- The graph of $f^{-1}$ is the graph of $f$ but flipped across the line $y = x$.

Example: Let $f(x) = x^3$. Then $f^{-1}(x) = \sqrt[3]{x}$. 

![Graph of inverse function](image)
Logarithms Let $b > 0$. Then $f(x) = b^x$ is one-to-one with range = positive real numbers $(0, \infty)$.

**Definition:** Define $g(x) = \log_b(x)$ to be the inverse function to $f(x) = b^x$.

So $\log_b(x) = y$ if and only if $x = b^y$. The function $g(x) = \log_b(x)$ has domain $(0, \infty)$ and range $(\infty, 0)$.

**Properties:**

$\log_b(1) = 0$ (0)

$\log_b(x) + \log_b(y) = \log_b(xy)$. (1)

$\log_b(x) - \log_b(y) = \log_b\left(\frac{x}{y}\right)$. (2)

$\log_b(x^n) = n \log_b(x)$. (3)

$\frac{\log_b(x)}{\log_b(c)} = \log_c(x)$. (4)

**Natural logarithm:**

Define $\ln(x) = \log_e(x)$.

Graph: $y = \ln(x)$

Inverses trig functions

Trig functions are not one-to-one, but we can restrict their domains to make them one-to-one.

**Example:** $f(x) = \sin(x)$ is one-to-one on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$\sin^{-1}(x)$ is one-to-one on $[-1, 1]$.

$\cos^{-1}(x)$ is one-to-one on $[0, \pi]$.

$\tan^{-1}(x)$ is one-to-one on $(-\infty, \infty)$.

$\csc^{-1}(x)$ is one-to-one on $(0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$.

$\sec^{-1}(x)$ is one-to-one on $[0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$.

$\cot^{-1}(x)$ is one-to-one on $(-\infty, 0) \cup (0, \infty)$.

(The book reviews this in more detail in section 1.5.)