Derivatives and Graphs

Last time

- If $f'(x) > 0$, then $f$ is increasing.
- If $f'(x) < 0$, then $f$ is decreasing.

So we get:

**First derivative test**

If $c$ is a critical number for $f$, then:

(a) If $f'$ changes from positive to negative at $c$ (so $f$ goes from increasing to decreasing) then $f$ has a local maximum at $c$.

(b) If $f'$ changes from negative to positive at $c$ (so $f$ goes from decreasing to increasing) then $f$ has a local minimum at $c$.

(c) Otherwise, if $f'$ is positive on both sides of $c$, or negative on both sides of $c$, then $f$ has neither a local maximum nor a local minimum at $c$.

**Example** Find the local maximum and minimum for 

$$f(x) = x + 2\sin x, \quad 0 \leq x \leq 2\pi.$$ 

**Note:** We exclude endpoints when looking for local extreme values, but not absolute extreme values.
Solution  Find critical numbers:

\[ 0 = f'(x) = 1 + 2 \cos x \]

\[-1 = 2 \cos x \]

\[-\frac{1}{2} = \cos x \]

\[ x = \frac{\pi}{3}, \frac{4\pi}{3} \] are the only solutions for \( 0 \leq x \leq 2\pi \).

To use the first derivative test, we need to check how the sign of \( f'(x) \) changes between critical numbers. Pick any number in the interval \((0, \frac{\pi}{3})\), any number in \((\frac{\pi}{3}, \frac{4\pi}{3})\), and any number in \((\frac{4\pi}{3}, 2\pi)\), and evaluate \( f' \) on these numbers. Let's choose: \( \frac{\pi}{3}, \pi, \) and \( \frac{5\pi}{3} \). We have

\[ f'(\frac{\pi}{3}) = 1 + 2 \cos(\frac{\pi}{3}) = 1 + 2 \cdot \frac{\sqrt{3}}{2} = 1 + \sqrt{3} \] positive

\[ f'(\pi) = 1 + 2 \cos(\pi) = 1 + 2(-1) = -1 \] negative

\[ f'(\frac{5\pi}{3}) = 1 + 2 \cos(\frac{5\pi}{3}) = 1 + 2 \cdot \frac{\sqrt{3}}{2} = 1 + \sqrt{3} \] positive

So the first two tell us that \( f' \) goes from positive to negative at \( \frac{\pi}{3} \), so \( f \) goes from increasing to decreasing at \( \frac{\pi}{3} \), and so \( f(\frac{\pi}{3}) \) is a local maximum.

The last two tell us that \( f' \) goes from negative to positive at \( \frac{4\pi}{3} \), so \( f \) goes from decreasing to increasing at \( \frac{4\pi}{3} \), and so \( f(\frac{4\pi}{3}) \) is a local minimum.

Local max: \( f(\frac{\pi}{3}) = \frac{\pi}{3} + 2 \sin(\frac{2\pi}{3}) = \frac{\pi}{3} + 2 \cdot \frac{\sqrt{3}}{2} = \frac{\pi}{3} + \sqrt{3} \)

Local min: \( f(\frac{4\pi}{3}) = \frac{4\pi}{3} + 2 \sin(\frac{4\pi}{3}) = 2 + 2(-\frac{\sqrt{3}}{2}) = \frac{4\pi}{3} - \sqrt{3} \).

Second derivatives

Definition

If the graph of \( f \) lies above all of its tangents on an interval \( I \), then \( f \) is called **concave upward** on \( I \).

If the graph of \( f \) lies below all of its tangents on an interval \( I \), then \( f \) is called **concave downward** on \( I \).
Example: \[ y = x^3 \]

Concave up: \[ \text{concave down:} \]

In general:
- \( f \) is concave up when the derivative \( f' \) is increasing.
- \( f \) is concave down when the derivative \( f' \) is decreasing.

Therefore we get:

Concavity test
- \( f \) is concave up when \( f'' \) is positive.
- \( f \) is concave down when \( f'' \) is negative.

Example: \( f(x) = x^3 \). Then \( f'(x) = 3x^2 \), \( f''(x) = 6x \). Therefore

\[ 6x = f''(x) > 0 \text{ exactly when } x > 0. \]
\[ 6x = f''(x) < 0 \text{ exactly when } x < 0. \]

So, \( f(x) = x^3 \) is concave up on \((0,\infty)\), and concave down on \((-\infty,0)\), as we saw from the picture.

Definition

A point \( P \) on a graph \( y = f(x) \) is called an inflection point if \( f \) is continuous there and the graph either shifts from concave up to concave down, or concave down to concave up, at \( P \).
Example On \( y = x^3 \), the point \((0,0)\) is an inflection point.

Remark Similarly to the first derivative test, inflection points happen when the second derivative changes sign. So if \( f'' \) is continuous, then all inflection points happen when \( f''(c) = 0 \). (Though \( f''(c) \) being zero doesn't always mean there necessarily is an inflection point!)

Second derivative test

Suppose \( f'' \) is continuous near \( c \).

- If \( f'(c) = 0 \) and \( f''(c) > 0 \), then \( f \) has a local min at \( c \).
- If \( f'(c) = 0 \) and \( f''(c) < 0 \), then \( f \) has a local max at \( c \).

Example Describe the concavity, inflection points, and local maxes and mins of \( y = x^4 - 4x^3 \).

Solution

\[
\begin{align*}
 f(x) &= x^4 - 4x^3 \\
 f'(x) &= 4x^3 - 12x^2 = 4x^2(x-3) \\
 f''(x) &= 12x^2 - 24x = 12x(x-2).
\end{align*}
\]

So

\[
\begin{align*}
 f'(x) &= 0 \text{ at } x = 0, 3 \\
 f''(x) &= 0 \text{ at } x = 0, 2.
\end{align*}
\]

Concavity: choose an \( x < 0 \), a \( x \) with \( 0 < x < 2 \), and an \( x \) with \( 2 < x \), like \( x = -1, 1, 3 \), and evaluate \( f'' \) on these numbers to see their signs:

- Concave up on \((-\infty, 0)\): \( f''(1) = (12(1)^2 - 24(1)) = 12 - 24 = -12 \) (positive) \( \Rightarrow \) \( f \) has an inflection point at \( 0 \)
- Concave down on \((0, 2)\): \( f''(1) = (12(1)^2 - 24(1)) = 12 - 24 = -12 \) (negative) \( \Rightarrow \) \( f \) has an inflection point at \( 2 \).
- Concave up on \((2, \infty)\): \( f''(3) = (12(3)^2 - 24(3)) = 108 - 72 = 36 \) (positive)

Inflection points: \( f(0) = 0^4 - 4(0)^3 = 0 \), so \((0,0)\) is an inflection point.

\( f(2) = 2^4 - 4(2)^3 = 16 - 32 = -16 \), so \((2, -16)\) is an inflection point.

Local max/mins: \( f'(0) = 0 \), but \( f''(0) = 0 \). So the second derivative test doesn't apply. So we check

\[
\begin{align*}
 f'(-1) &= 4(-1)^3 - 12(-1) = -4 + 12 = 8 \text{ (negative)} \Rightarrow \text{ neither local max nor min at } x = -1. \\
 f'(1) &= 4(1)^3 - 12(1) = 4 - 12 = -8 \text{ (negative)} \Rightarrow \text{ neither local max nor min at } x = 1.
\end{align*}
\]
But at $2$, 
\[ f''(3) = 12 \cdot 3^2 - 29 \cdot 3 = 108 - 72 = 36, \quad \text{positive} \]
So by the second derivative test, $f$ has a local min at 3:
\[ f(3) = 3^4 - 9 \cdot 3^3 = 81 - 108 = -27. \]

**Sketch:**

![Graph with inflection points and local min at 3]

**Rolle's Theorem** Let $f$ be a function satisfying:

1. $f$ is continuous on $[a, b]$
2. $f$ is differentiable on $(a, b)$
3. $f(a) = f(b)$

Then there is a $c$ in $(a, b)$ so that $f'(c) = 0$.

The book proves this using the extreme value theorem and Fermat's theorem.

**Mean value theorem** Let $f$ be a function satisfying

1. $f$ is continuous on $[a, b]$
2. $f$ is differentiable on $(a, b)$

Then there is a $c$ in $(a, b)$ so that
\[ f'(c) = \frac{f(b) - f(a)}{b - a}. \]

That is, the slope of the secant line through $(a, f(a))$ and $(b, f(b))$ is the slope of the tangent line through $(c, f(c))$. 
Application: If $f'(x) = 0$ for all $x$ in $(a,b)$, then $f$ is constant on $(a,b)$.

Proof: Let $x_1$ and $x_2$ be any numbers in $(a,b)$ with $x_1 < x_2$. Then there is a $c$ in $(x_1,x_2)$ with

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

by the mean value theorem (applied with $x_2$ in place of $b$ and $x_1$ in place of $a$)

But $f'(c) = 0$ because we are assuming $f'(x) = 0$ for all $x$ in $(a,b)$.

So we get

$$0 = f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Multiply both sides by $x_2 - x_1$:

$$0 = f(x_2) - f(x_1)$$

This whole argument shows that $f(x_1) = f(x_2)$ no matter what $x_1$ and $x_2$ are. This means exactly that $f$ is constant.

Corollary: If $f'(x) = g'(x)$ for all $x$ in $(a,b)$, then $f(x) = g(x) + C$ for some constant $C$.

Proof: Apply the above to $F(x)$, where

$$F(x) = f(x) - g(x)$$

We get

$$F'(x) = f'(x) - g'(x) = 0$$

because $f'(x) = g'(x)$.

So $F$ is constant, say $C = F(x)$. Then

$$C = F(x) = f(x) - g(x)$$

Add $g(x)$ to both sides:

$$g(x) + C = f(x)$$

as desired.