Maxima and Minima

Definition

Let $c$ be in the domain $D$ of a function $f$. Then $f(c)$ is the

- absolute maximum value of $f$ on $D$ if $f(c) \geq f(x)$
  for all $x$ in $D$.
- absolute minimum value of $f$ on $D$ if $f(c) \leq f(x)$
  for all $x$ in $D$.

For example:

\[ y = f(x) \]

\[ f(c) \]

\[ x \]

\[ \text{absolute minimum} \]

\[ \text{absolute minimum} \]

\[ \text{Not absolute maximum} \]

\[ \text{Not absolute minimum} \]

Remarks

- An absolute minimum/maximum is always a "y-value"; that is to say, an absolute maximum/minimum is always in the range, not the domain.
- The points in blue in the graphs above are absolute minima. But the points in green/red in the second graph are not absolute minima/maxima; simply because they are not "absolute." But they are local minima/maxima, by the following definition.

Definition

$f(c)$ is a

- local maximum value of $f$ if $f(c) \geq f(x)$ when $x$ is near $c$
- local minimum value of $f$ if $f(c) \leq f(x)$ when $x$ is near $c$.

Remarks

- "for $x$ near $c"$ means "for $x$ is some open interval containing $c". In our second example above, we can draw such open intervals:

\[ \text{local maximum} \]

\[ \text{local minimum} \]

\[ \text{(there can be many such open intervals that work.)} \]
• Every absolute maximum/minimum is also a local maximum/minimum.
• A function can have many different local maxima and minima. In terms of the graph, this usually just means it’s fluctuating a lot.
• A function can have many different points where it takes absolute maximum and minimum values. For example:

\[
y = \cos(x)
\]

\(f(x) = \cos(x)\) takes its absolute maximum value of 1 at \(x = 2\pi n\), \(n\) any integer.
It takes its absolute minimum value of -1 at \(x = (2n+1)\pi\), \(n\) any integer.

**Theorem (Extreme Value Theorem)** If \(f\) is continuous on a closed interval \([a, b]\), then \(f\) attains an absolute maximum value \(f(c)\) and an absolute minimum value \(f(d)\) for some numbers \(c\) and \(d\) in \([a, b]\).

**Examples**

![Graph of a function](image)

(An absolute max/min can be obtained at an endpoint.)

![Graph of a function](image)

(A function could have more than one point where it attains an absolute max/min.)

This is all well and good, but how do we find maxima and minima?

The following theorem is good for finding local maxima and minima:

**Theorem (Fermat)** If \(f\) has a local maximum or minimum at \(x = c\), and \(f'(c)\) exists, then \(f'(c) = 0\).

The intuitive reason for this theorem is that a local maximum or minimum have horizontal tangent lines where they are attained.

**Example** \(y = x^2 + 1\)

The function \(f(x) = x^2 + 1\) has derivative \(f'(x) = 2x\). The fraction \(2x\) is zero exactly when \(x = 0\). So Fermat's Theorem tells us that \(x = 0\) is the only place where \(f\) can attain a maximum or minimum. Indeed, \(f\) has an absolute minimum at \(x = 0\), and the minimum value is \(f(0) = 0^2 + 1 = 1\).
The tangent line at \( x = 0 \) is still horizontal.

Fermat's Theorem tells us if \( f \) attains a local maximum/minimum at \( x = c \), and if \( f \) is differentiable at \( x = c \), then \( f'(c) = 0 \), **not** the other way around.

It's still useful because it tells you anywhere where \( f'(x) \) exists and is nonzero cannot be a local maximum/minimum. So to find local max and min, you only need to check points \( x \) where \( f'(x) \) is zero or doesn't exist.

Another bad example \( y = |x| \):

\[ f(x) = |x| \text{ attains an absolute minimum at } x = 0, \] but \( f'(0) \) doesn't exist, so Fermat's theorem can't tell you anything, and you would have to show directly that \( f \) attains a minimum at \( x = 0 \).

**Definition**

A **critical number** for \( f \) is a number \( c \) in the domain of \( f \) such that \( f'(c) = 0 \) or \( f'(c) \) does not exist.

Fermat's Theorem, restated, is: If \( f \) has a local maximum or minimum at \( c \), then \( c \) is a critical number for \( f \).

**Closed interval method** To find the absolute maxima and minima of a continuous function \( f \) on a closed interval \([a, b]\):

1. Find all critical numbers.
2. Evaluate \( f \) at all critical numbers and at the endpoints \( a, b \).
3. The largest value from step 2 is the absolute maximum; the smallest is the absolute minimum.

**Example** Find the absolute maximum and minimum for

\[ f(x) = x^3 - 3x^2 + 1, \quad -\frac{1}{2} \leq x \leq 4. \]

**Solution**: \( f \) is continuous on \([-\frac{1}{2}, 4]\), so we use the closed interval method.

1. Critical numbers: \( f'(x) \) always exists (so no critical numbers come from \( f'(x) \) not existing), and
\[ f'(x) = 3x^2 - 6x. \]

So

\[ f'(x) = 0 \]
\[ 3x^2 - 6x = 0 \]
\[ 3x(x-2) = 0 \]

So

\[ x=0, \ x=2 \]

are our critical numbers.

2. **Plug in.** We need to evaluate \( f \) at 0 and 2, and also at the endpoints \(-\frac{1}{2}\) and 4:

\[ f(0) = 0^3 - 3 \cdot 0^2 + 1 = 1, \]
\[ f(2) = 2^3 - 3 \cdot 2^2 + 1 = 8 - 12 + 1 = -3 \]
\[ f(-\frac{1}{2}) = -\frac{1}{8} - \frac{3}{2} + 1 = -\frac{3}{8} + \frac{1}{2} = \frac{1}{8} \]
\[ f(4) = 4^3 - 3 \cdot 4^2 + 1 = 64 - 48 + 1 = 17. \]

3. **Max and min.** 17 is the largest of these, and \(-3\) is the smallest. So:

Absolute maximum = 17 (at \( x = 4 \))

Absolute minimum = -3 (at \( x = 2 \)).

**Example.** Find the absolute maximum and minimum values of \( f(x) = x - \sin x, 0 \leq x \leq 2\pi \).

**Solution.** Again, \( f \) is continuous.

1. **Critical numbers.** \( f \) is differentiable;

\[ f'(x) = 1 - 2 \cos x. \]

So

\[ f'(x) = 0 \]
\[ 1 - 2 \cos x = 0 \]
\[ 1 = 2 \cos x \]
\[ \frac{1}{2} = \cos x \]

This happens at

\[ 2n\pi + \frac{\pi}{3}, \ 2n\pi + \frac{5\pi}{3}. \]

But \( 0 \leq x \leq 2\pi \) so we just get

\[ \frac{\pi}{3}, \ \frac{5\pi}{3}. \]
2. **Critical numbers:**

- For $f(x) = \frac{\pi}{3} - 2\sin \frac{x}{3}$, $f'(x) = -2 \cos \frac{1}{3} \approx -0.68$
- For $f(x) = 5 \frac{\pi}{3} - 2 \sin \frac{5x}{3}$, $f'(x) = 2 \cos \frac{5}{3} \approx 6.97$

**End points:**

- $f(0) = -2 \sin 0 = 0$
- $f(2\pi) = 2\pi - 2 \sin 2\pi = 2\pi \approx 6.28$

3. **Max and min**

- Absolute maximum: $\frac{5\pi}{3} + \sqrt{3} \quad (at \quad \frac{5\pi}{3})$
- Absolute minimum: $\frac{\pi}{3} - \sqrt{3} \quad (at \quad \frac{\pi}{3})$

**Fact:** If $f'(x) > 0$ on an interval, then $f$ is increasing on that interval.
If $f'(x) < 0$ on an interval, then $f$ is decreasing on that interval.

**Example** $f(x) = x^2$. Then $f'(x) = 2x$. $f'(x) < 0$ on $(-\infty, 0)$, and $f'(x) > 0$ on $(0, \infty)$. This agrees with the graph:

![Graph of f(x) = x^2 showing increasing and decreasing intervals]

Notice $f'(0) = 0$, so $x=0$ is critical for $f$. Since $f(x)$ goes from decreasing to increasing at $x=0$, $f$ must have a local minimum at $x=0$.

**In general:** (First derivative test). Suppose $c$ is a critical number of a continuous function $f$.
(a) If $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$.
(b) If $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$.
(c) If $f'$ is positive to the left and right of $c$, or $f'$ is negative to the left and right of $c$, then $f$ has no local maximum or minimum at $c$.

**Example** $f(x) = x^5 - x^3$. Find all local maxima/minima.

**Solution** $f'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3)$. Then $f'(x) = 0$ at $x=0$ or when $5x^2 - 3 = 0$. The latter happens when

\[
5x^2 - 3 = 0 \\
5x^2 = 3 \\
x^2 = \frac{3}{5} \\
x = \pm \sqrt{\frac{3}{5}}.
\]
So the critical values are $x = 0, \pm \sqrt{3}$. To see how $f'$ changes, we pick points:

- To the left of $-\sqrt{3}$, like $-1$.
- Between $\sqrt{3}/3$ and $0$, like $\sqrt{3}/3$.
- Between $0$ and $\sqrt{3}/3$, like $-\sqrt{3}/3$.
- To the right of $\sqrt{3}/3$, like $1$.

We have:

\[
f'(-1) = (-1)^2(5(2)^2 - 3) = 5 \cdot 3 - 2
\]

\[
f'(-\sqrt{3}/3) = \left(-\sqrt{3}/3\right)^2 \left(5\left(-\sqrt{3}/3\right)^2 - 3\right) = \frac{1}{9}(5 \cdot \frac{3}{3} - 3) = \frac{1}{9}(1 - 3) = -\frac{2}{3}
\]

\[
f'(\sqrt{3}/3) = \left(\frac{\sqrt{3}}{3}\right)^2 \left(5\left(\frac{\sqrt{3}}{3}\right)^2 - 3\right) = \frac{1}{3}(5 \cdot \frac{3}{3} - 3) = \frac{1}{3}(1 - 3) = -\frac{2}{3}
\]

\[
f'(1) = 1^2(5 \cdot 1^2 - 3) = 5 - 3 = 2
\]

So $f(-\sqrt{3}/3)$ is a local maximum.

\[
f(-\sqrt{3}/3) = \left(\frac{\sqrt{3}}{3}\right)^5 - \left(-\frac{\sqrt{3}}{3}\right)^3 = -\frac{\sqrt{3}}{3} \left(-\left(\frac{\sqrt{3}}{3}\right)^4 - \left(\frac{\sqrt{3}}{3}\right)^2\right)
\]

\[
= -\frac{\sqrt{3}}{3} \left(\frac{9}{27} - \frac{3}{9}\right)
\]

\[
= -\frac{\sqrt{3}}{3} \left(-\frac{6}{27}\right) = \frac{6\sqrt{3}}{27}
\]

$f(0)$ is neither a local maximum nor minimum.

\[
f(0) = \left(\frac{\sqrt{3}}{3}\right)^5 - \left(\frac{\sqrt{3}}{3}\right)^3
\]

\[
= \sqrt{3}/3 \left(\left(\frac{\sqrt{3}}{3}\right)^4 - \left(\frac{\sqrt{3}}{3}\right)^2\right)
\]

\[
= \frac{\sqrt{3}}{3} \left(\frac{9}{27} - \frac{3}{9}\right)
\]

\[
= \frac{\sqrt{3}}{3} \left(-\frac{6}{27}\right)
\]

\[
= -\frac{6\sqrt{3}}{27}
\]

$f(\sqrt{3}/3)$ is a local minimum.

\[
f(\sqrt{3}/3) = \left(\frac{\sqrt{3}}{3}\right)^5 - \left(\frac{\sqrt{3}}{3}\right)^3
\]

Rough sketch: