Logarithms

Last time we differentiated inverse trig functions using implicit differentiation. For example:

\[ y = \tan^{-1} x \text{ means } \tan y = x, \]

so

\[ \sec^2 y \cdot y' = 1. \text{ (because } \frac{d}{dt} \tan t = \sec^2 t) \]

Therefore:

\[ y' = \frac{1}{\sec^2 y} \]

But \( y = \tan^{-1} x \), so we could write \( \sec^2 y = \sec^2(\tan^{-1} x) \), but we can do better: remember

\[ \sec^2 y = 1 + \tan^2 y, \]

so

\[ y' = \frac{1}{1 + \tan^2 y} \]

\[ = \frac{1}{1 + (\tan^{-1} x)^2} \]

\[ = \frac{1}{1 + x^2}. \]

We can try the same technique to differentiate \( \log_b x \).

Now let \( b > 0 \). Then

\[ y = \log_b x \text{ means } b^y = x. \]

So

\[ \frac{d}{dx} b^y = \frac{d}{dx} x \]

\[ b^y \ln b \cdot y' = 1. \]

Substitute \( y = \log_b x \) back in the exponent \( y \):

\[ b^{\log_b x} (\ln b) \cdot y' = 1 \]

\[ x (\ln b) y' = 1 \]

\[ y' = \frac{1}{x \ln b}. \]

Therefore:

\[ \frac{d}{dx} \log_b x = \frac{1}{x \ln b}. \]
If \( b = e \), then \( \ln b = \ln e = 1 \), so
\[
\frac{d}{dx} \ln x = \frac{1}{x}.
\]

**Remark** It is nice to prove this directly:
\[
\begin{align*}
Y &= \ln x \\
e^y &= x \\
\frac{d}{dx} e^y &= \frac{d}{dx} x \\
e^y y' &= 1 \\
e^{\ln x} y' &= 1 \\
x y' &= 1 \\
y' &= \frac{1}{x}.
\end{align*}
\]

You can actually get \( \frac{d}{dx} \log_b(x) \) from this identity using \( \log_b(x) = \frac{\ln x}{\ln b} \):
\[
\frac{d}{dx} \log_b(x) = \frac{d}{dx} \frac{\ln x}{\ln b}
= \frac{1}{\ln b} \frac{d}{dx} \ln x \quad (\text{because } \frac{1}{\ln b} \text{ is a constant})
= \frac{1}{\ln b} \frac{1}{x}
= \frac{1}{x \ln b}.
\]

**Example** Let’s find \( \frac{d}{dx} \ln(\sin x) \).

**Solution** Chain rule. \( \sin x \) is the inner function and \( \ln x \) is the outer function.
\[
\frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \cdot \frac{d}{dx} \sin x
= \frac{1}{\sin x} \cos x
= \cot x.
\]

**Example** Let’s find \( \frac{d}{dx} \log_{10}(2+\sin x) \).

**Solution** Chain rule again. Inner function = \( 2 + \sin x \), outer function = \( \log_{10} x \).
\[
\frac{d}{dx} \log_b (2+\sin x) = \frac{1}{(2+\sin x) \ln b} \cdot \frac{d}{dx} (2+\sin x)
\]
\[
= \frac{1}{(2+\sin x) \ln b} \cdot \cos x
\]
\[
= \frac{\cos x}{(2+\sin x) \ln b}.
\]

**Remark** If you forget the formula for \( \frac{d}{dx} \log_b x \), you could again write \( \log_b x = \frac{\ln x}{\ln b} \) and get
\[
\frac{d}{dx} \log_{10} (2+\sin x) = \frac{d}{dx} \left( \frac{1}{\ln 10} \ln (2+\sin x) \right)
\]
\[
= \frac{1}{\ln 10} \frac{d}{dx} \ln (2+\sin x)
\]
\[
= \frac{1}{\ln 10} \frac{1}{2+\sin x} \frac{d}{dx} (2+\sin x)
\]
\[
= \frac{1}{\ln 10} \frac{1}{2+\sin x} \cos x.
\]
\[
= \frac{\cos x}{(2+\sin x) \ln 10}.
\]

**Example** Let’s find \( \frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} \).

**Hard way** Use the chain rule once again. Inner function = \( \frac{x+1}{\sqrt{x-2}} \), outer function = \( \ln x \).
\[
\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{1}{\frac{x+1}{\sqrt{x-2}}} \frac{d}{dx} \frac{x+1}{\sqrt{x-2}}
\]
\[
= \frac{\sqrt{x-2}}{x+1} \frac{d}{dx} \frac{x+1}{\sqrt{x-2}}
\]
\[
= \frac{\sqrt{x-2}}{x+1} \frac{d}{dx} \left( \frac{x+1}{\sqrt{x-2}} \right)
\]
\[
= \frac{\sqrt{x-2}}{x+1} \cdot \left[ \frac{1}{2} \frac{1}{\sqrt{x-2}} \left( \frac{d}{dx} (x+1) \right) - \frac{1}{2} \frac{1}{\sqrt{x-2}} \left( \frac{d}{dx} (x-2) \right) \right]
\]
\[
= \frac{\sqrt{x-2}}{x+1} \cdot \left[ \frac{1}{2} \frac{1}{\sqrt{x-2}} \cdot 1 - \frac{1}{2} \frac{1}{\sqrt{x-2}} \cdot \frac{1}{2} \right]
\]
\[
= \frac{\sqrt{x-2} \cdot \sqrt{x-2} - (x+1) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x-2}} \cdot 1}{x-2}
\]
\[
= \frac{\sqrt{x-2} \cdot \sqrt{x-2} - (x+1) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x-2}} \cdot 1}{x-2}
\]
\[
= \frac{\sqrt{x-2} \cdot \sqrt{x-2} - (x+1) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x-2}} \cdot 1}{(x+1)(x-2)}
\]
Okay, easy way We'll be a little more clever and use log identities right at the beginning, like:

\[\ln(ab) = \ln a + \ln b,\]
\[\ln a^c = c \ln a,\]
\[\ln \frac{1}{a} = -\ln a.\]

We therefore have

\[\ln \frac{x+1}{\sqrt{x-1}} = (\ln(x+1) - \ln(x-1)^{1/2})\]  
Write \(\sqrt{x-1} = (x-1)^{1/2}\) in order to use the second log identity

\[= \ln(x+1) - \frac{1}{2} \ln(x-1)\]

Now we differentiate:

\[
\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-1}} = \left[ \frac{d}{dx} \ln(x+1) \right] - \frac{1}{2} \left[ \frac{d}{dx} \ln(x-1) \right]
\]
\[
= \left[ \frac{1}{x+1} \right] - \frac{1}{2} \left[ \frac{1}{x-1} \right]
\]
\[
= \frac{1}{x+1} - \frac{1}{2} (x-1)
\]
\[
= \frac{2(x-1) - (x+1)}{2(x+1)(x-1)}
\]
\[
= \frac{2x-2-x-1}{2(x+1)(x-1)}
\]
\[
= \frac{x-3}{2(x+1)(x-1)}
\]

No quotient rule necessary! And, the simplifying is easier.

Logarithmic differentiation

The technique above, in the easier solution, can be used even if there are no ln's in sight. Just take ln's before starting.
Example: Find $\frac{d}{dx} \frac{x^{3/4} \sqrt{x^2+1}}{(3x+2)^5}$.

Differentiating this with the quotient rule looks quite awful, but we can set

$$y = \frac{x^{3/4} \sqrt{x^2+1}}{(3x+2)^5}$$

and then take ln's and implicitly differentiate.

$$\ln y = \ln x^{3/4} + \ln \sqrt{x^2+1} - \ln (3x+2)^5$$

$$\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln (x^2+1) - 5 \ln (3x+2)$$

Now here's where we differentiate:

$$\frac{1}{y} y' = \frac{3}{4} \frac{1}{x} + \frac{1}{2} \frac{1}{x^2+1} \cdot 2x - 5 \cdot \frac{1}{3x+2} \cdot 3$$

$$\frac{1}{y} y' = \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}$$

Next, move the $y$ to the other side and substitute our original expression for $y$ back in:

$$y' = y \left( \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right)$$

$$= \frac{x^{3/4} \sqrt{x^2+1}}{(3x+2)^5} \left( \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right)$$

Voilà!

In general:

$$\frac{d}{dx} \ln (f(x)) = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)}$$

by the chain rule. Rearranging, we get

$$f(x) \cdot \frac{d}{dx} \ln (f(x)) = f'(x)$$

The point is, sometimes it's even easier to find $\frac{1}{f(x)}$ and $\frac{d}{dx} \ln (f(x))$ than it is to find $f'(x)$ directly. Log rules are generally easier to apply than chains of power, product, and quotient rules.

Power rule: Now we can prove the power rule, that $\frac{d}{dx} x^n = nx^{n-1}$ for any real number $n$. 


proof

\[ y = x^a \]
\[ \ln y = \ln x^a \]
\[ \ln y = a \ln x \]
\[ \frac{d}{dx} \ln y = a \frac{d}{dx} \ln x \]
\[ \frac{1}{y} y' = a \frac{1}{x} \]
\[ y' = a \frac{y}{x} \]
\[ y' = a \frac{x^a}{x} \]
\[ y = a x^{a-1}. \]

Let's now do two more examples of differentiating with logs:

Example: Let's find \( \frac{d}{dx} x^x \).

Solution: We have

\[ y = x^x \]
\[ \ln y = \ln x^x \]
\[ \ln y = x \ln x \]

(Take \( \frac{d}{dx} \))
\[ \frac{1}{y} y' = \frac{d}{dx} (x \ln x) \]
\[ \frac{1}{y} y' = x \left( \frac{d}{dx} \ln x \right) + \left( \frac{d}{dx} x \right) \cdot \ln x \]
\[ \frac{1}{y} y' = x \cdot \frac{1}{x} + 1 \cdot \ln x \]
\[ \frac{1}{y} y' = 1 + \ln x \]
\[ y' = y (1 + \ln x) \]
\[ y' = x^x (1 + \ln x). \]

Example. Find \( \frac{d}{dx} \ln |x| \).

Solution: The domain of \( |x| \) is \((-\infty, 0) \cup (0, \infty)\) because \( |x| \) is positive unless \( x=0 \). We will differentiate on both these intervals.
We know \(|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}\)

So

\[|ln x| = \begin{cases} ln x & \text{if } x > 0 \\ ln(-x) & \text{if } x < 0 \end{cases} \]

We exclude the case \(x = 0\) from these cases because \(ln 1x\) is not defined at \(x = 0\).

So on \((0, \infty)\), we get

\[
\frac{d}{dx} ln |x| = \frac{d}{dx} ln x = \frac{1}{x}
\]

and on \((-\infty, 0)\), we get

\[
\frac{d}{dx} ln |x| = \frac{d}{dx} ln (-x) \\
= \frac{1}{-x} \cdot (-x) \\
= \frac{1}{x} \cdot (-1) \\
= \frac{-1}{x}
\]

Either way, we get

\[
\frac{d}{dx} ln |x| = \frac{1}{x}.
\]

Remark: We also have \(\frac{d}{dx} ln x = \frac{1}{x}\). How could \(\frac{d}{dx} ln |x| = \frac{d}{dx} ln x\)? Well, this is because these functions have different domains:

- Domain of \(ln x\) is \((0, \infty)\)
- Domain of \(ln |x|\) is \((-\infty, 0) \cup (0, \infty)\).

Where these domains overlap is \((0, \infty)\), and these functions are equal on \((0, \infty)\). So they give the same derivative on \((0, \infty)\). Taking \(ln |x|\) is just a way to extend \(ln(x)\) to the domain \((-\infty, 0) \cup (0, \infty)\).