Implicit differentiation

Last time we differentiated the relation defining a circle of radius 5:

\[ x^2 + y^2 = 25 \]

We got

\[ \frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (25) \]

\[ \frac{d}{dx} x^2 + \frac{d}{dx} y^2 = 0 \]

\[ 2x + 2y \cdot y' = 0 \]

\[ 2x \cdot y' = -2y \]

\[ y' = -\frac{y}{x} . \]

The point of doing this is: \( \frac{dy}{dx} \) gives the slope of the tangent line to our circle at a point \( x, y \).

So, for example, if \((x, y) = (3, 4)\), we get that at \((3, 4)\),

\[ y' \frac{dx}{dy} = -3 \]

\[ \frac{dy}{dx} = -\frac{3}{4} . \]

On the graph:

The tangent line at \((3, 4)\) is

\[ y - 4 = -\frac{3}{4} (x - 3) \]

\[ y = -\frac{3}{4} x + \frac{9}{4} + 4 \]

\[ y = -\frac{3}{4} x + \frac{25}{4} . \]

Which looks roughly right.

Note: If \( y \neq 0 \), we can solve for \( y' \) as \( y' = -\frac{y}{x} \). This depends on \( y \), and \( y \) that's okay!
Example Consider \( x^3 + y^3 = 6xy \). This is a curve. It looks like

This graph is the just the set of \((x,y)\) satisfying this equation.

1. Let's find \( y' \). We first differentiate both sides with respect to \( x \). This is the calculus step:

\[
\frac{d}{dx} x^3 + \frac{d}{dx} y^3 = \frac{d}{dx} 6xy
\]

\[
3x^2 + 3y^2 \cdot y' = 6x \cdot y' + \left( \frac{d}{dx} 6x \right) y
\]

\[
3x^2 + 3y^2 \cdot y' = 6xy' + 6y
\]

Now we solve for \( y' \) in terms of the variable \( x \) and \( y \). This is the algebra step:

\[
3x^2 + 3y^2 \cdot y' = 6xy' + 6y
\]

\[
x^2 + y^2 \cdot y' = 2xy' + 2y
\]

\[
y^2 \cdot y' - 2xy' = 2y - x^2
\]

\[
y^2 - 2x)y' = 2y - x^2
\]

\[
y' = \frac{2y - x^2}{y^2 - 2x}
\]

1. Let's find the tangent line to this curve at \((3,3)\).

[Check: Is \((3,3)\) really on this curve? That's the same as asking if \( 3^3 + 3^3 = 6 \cdot 3 \cdot 3 \), which is true since both sides equal 54. So yes, \((3,3)\) is on this curve.]

So we just need find \( y' \) when \( x=3 \) and \( y=3 \). But we have our formula \( y' = \frac{2y - x^2}{y^2 - 2x} \) which gives:

\[
y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = \frac{6 - 9}{9 - 6} = \frac{-3}{3} = -1,
\]

at \((x,y)=(3,3)\).

This gives us the slope of our tangent line, and so the equation is
Let's check the graph:

Let's find all points where the tangent line is horizontal. This means $y' = 0$. Let's use both our equations $y' = \frac{2y - x^3}{y^2 - 2x}$ and $x^3 + y^3 = 6xy$. Substituting $y' = 0$ in the first equation gives

$$0 = \frac{2y - x^3}{y^2 - 2x}$$

Multiplying both sides by $y^2 - 2x$ (assuming this is nonzero) gives

$$0 = 2y - x^3$$
$$2y = x^3$$
$$y = \frac{1}{2} x^3.$$

Now that we have $y$ by itself, we can substitute this into $x^3 + y^3 = 6xy$:

$$x^3 + \left(\frac{1}{2} x^3\right)^3 = 6x \left(\frac{1}{2} x^3\right)$$
$$x^3 + \frac{1}{8} x^6 = 3x^3$$
$$\frac{1}{8} x^6 = 2x^3$$
$$x^6 = 16x^3$$

If $x \neq 0$, then divide by $x^3$:

$$x^3 = 16$$
$$x = (16)^{1/3}$$
$$= (2^4)^{1/3}$$
$$= 2^{4/3}.$$
So we've solved for $k$, now we can solve for $y$ using $y = \frac{1}{2}x^2$:

$$y = \frac{1}{2} \left(2^{4/3}\right)^2$$

$$= \frac{1}{2} \cdot 2^{8/3}$$

$$= \frac{1}{2} \cdot 2^{5/3}$$

So the tangent line should be horizontal at $(k,y) = \left(2^{4/3}, 2^{5/3}\right) \approx (2.52, 3.17)$

What if $k=0$? Then using $y = \frac{1}{2}x^2$ again, we get $y = \frac{1}{2} \cdot 0^2 = 0$. So we would need to check if $(0,0)$ has horizontal tangent line. There's a problem there though, namely that the graph intersects itself in a horizontal and vertical direction. It's like there are two tangents at $(0,0)$. What's going on here? Well, remember when we cancelled $y^2 - 2x$ in the equation $y' = 0 = \frac{2y - k^2}{y^2 - 2x}$? At $(0,0)$, $y^2 - 2x = 0^2 - 2 \cdot 0 = 0$, so we couldn't cancel. In fact, the numerator is $2y - k^2 = 0$, so we get a zero-over-zero situation which tells us $y'$ is undefined at $(0,0)$ according to our expression. So we just ignore $(0,0)$ here.

**Example** Find $y''$ if $x^4 + y^4 = 16$.

**Solution** Differentiate both sides (calculus step).

$$4x^3 + 4y^3 \cdot y' = 0$$

Then the algebra step gives

$$4x^3 + 4y^3 \cdot y' = 0$$

$$x^3 + y^3 \cdot y' = 0$$

$$y^3 \cdot y' = -x^3$$

$$y' = -\frac{x^3}{y^3}$$
Now differentiate again using the quotient rule. Remember to always use the chain rule on $y$. 

\[
y'' = \frac{d}{dx} \left( \frac{x^3}{y^3} \right)
\]

\[
y'' = \frac{y^3(-3x^2) - (-x^3)(3y^2y')} {y^6}
\]

\[
y'' = \frac{-3x^2y^3 + 3x^3y^2y'} {y^6}
\]

This depends on $y'$, but we already know that $y' = \frac{-x^3}{y^3}$, so

\[
y'' = \frac{-3x^2y^3 + 3x^3y^2 \left( \frac{-x^3}{y^3} \right)} {y^6}
\]

\[
y'' = \frac{-3x^2y^3 - \frac{3x^6} {y}} {y^6}
\]

\[
y'' = \frac{-3x^2y^4 - 3x^6} {y^7}
\]

\[
y'' = \frac{-3x^2 (y^4 + x^4)} {y^7}
\]

Now use $x^4 + y^4 = 16$:

\[
y'' = \frac{-3x^2 (16)} {y^7}
\]

\[
y'' = -48 \frac{x^2} {y^7}
\]

**Inverse trig functions:**

We can use implicit differentiation to find derivatives of functions like $\sin^{-1}(x)$. Set

\[
y = \sin^{-1}(x)
\]

Which means $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Then we implicitly differentiate:

\[
\frac{d}{dx} \sin y = \frac{d}{dx} x
\]

\[
\cos y \cdot y' = 1
\]
Since \(-\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}\), \(\cos \gamma \geq 0\), with \(\cos \gamma\) being zero only at the endpoints \(-\frac{\pi}{2}\) and \(\frac{\pi}{2}\). So
\[
\cos \gamma = \sqrt{1 - \sin^2 \gamma}
\]
But \(\sin \gamma = x\), so this is
\[
= \sqrt{1 - (\sin \gamma)^2} = \sqrt{1 - x^2}
\]
Therefore
\[
\cos \gamma \cdot y' = 1
\]
gives
\[
\sqrt{1 - x^2} \cdot y' = 1.
\]
The domain of \(\sin^{-1}(x)\) is \([-1, 1]\). At the endpoints, \(y = \sin^{-1}(-1) = -\frac{\pi}{2}\), and \(y = \sin^{-1}(1) = \frac{\pi}{2}\), and in either case, \(\cos(y) = \cos(\frac{\pi}{2})\) or \(\cos(\frac{\pi}{2}) = 0\). Therefore we should have \(\sqrt{1 - x^2} = 0\) at \(x = \pm 1\), and indeed this is the case. So we ignore the endpoints. So if \(x\) is in the open interval \((-1, 1)\), then \(\sqrt{1 - x^2} \cdot y' = 1\) gives
\[
y' = \frac{1}{\sqrt{1 - x^2}}
\]
because \(\sqrt{1 - x^2} > 0\) on \((-1, 1)\). Therefore
\[
\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}.
\]
Another example The range of \(\tan^{-1}(x)\) is \((-\frac{\pi}{2}, \frac{\pi}{2})\). So \(y = \tan^{-1} x\) means
\[
\tan y = x\quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.
\]
Therefore
\[
\frac{d}{dx} \tan y = \frac{d}{dx} x
\]
\[
\sec^2 y \cdot y' = 1
\]
\[
y' = \frac{1}{\sec^2 y}
\]
\[
y' = \frac{1}{1 + \tan^2 y} \quad \text{(because } \sec^2 y = 1 + \tan^2 y)\]
\[
y' = \frac{1}{1 + x^2}
\]
And so \(\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}\). (Turns out, since \(1 + x^2 > 0\), we didn't need to care about \(\frac{\pi}{2} < y < \frac{\pi}{2}\).)
Again, minus signs go with the "cotfunctions", like in the case of derivatives of usual trig functions.