Product Rule: If $f$ and $g$ are differentiable, so is $fg$, and
\[
(fg)' = f'g + g'f.
\]

Here is an intuitive reason: Consider a rectangle whose two side lengths are different functions, $f$ and $g$, of time $t$. So the box is expanding or contracting according to $f$ and $g$.

The area of this rectangle at time $t$ is $f(t)g(t)$. Now let time $t$ move forward to time $t+\Delta t$ for some small \(\Delta t\). Write \(\Delta f\) for the change in $f$ from time $t$ to time $t+\Delta t$ and similarly for $g$:

The change in the area is the area of shaded region: $g\Delta f + f\Delta g + \Delta f \cdot g$. The term $\Delta f \cdot g$, being a product of two small quantities, is very, very small. It's an order of magnitude smaller than the change in time \(\Delta t\). So we should expect \(\frac{\Delta f \cdot g}{\Delta t} \to 0\) even as \(\Delta t \to 0\). This gives us

\[
\frac{dfg}{dt} = g \frac{df}{dt} + f \frac{dg}{dt}.
\]

A more rigorous proof: Instead of a "change in time," we use the symbol $h$ and let $h \to 0$. We replace $\Delta f$ by $f(x+h)-f(x)$, and similarly with $g$, $fg$. The above leads us to expect:

\[
\frac{dfg}{dh} = g \frac{df}{dh} + f \frac{dg}{dh}.
\]

This term is like $dfg$ \(\Box\), like $df$ \(\Box\), like $dg$ \(\Box\), like $df \cdot dg$.

We can expand the right hand side:

\[
\begin{align*}
\frac{dfg}{dh} &= g(x)(f(x+h)-f(x)) + f(x)(g(x+h)-g(x)) + f(x+h)(g(x)-g(x+h)) + \text{higher order terms} \\
&= f(x)g(x)(f(x+h)-f(x)) + f(x)g(x)(g(x+h)-g(x)) + f(x)g(x)(g(x+h)-g(x)) + \text{higher order terms} \\
&= f(x)g(x)\Delta f + f(x)\Delta g - f(x)g(x)(\Delta x) \\
&= f(x)g(x)(f(x+h)-f(x)) - f(x)g(x)(g(x+h)-g(x)) + f(x)g(x)(g(x+h)-g(x)) + \text{higher order terms}
\end{align*}
\]

So the formula with \(\Box\) above is correct! Now divide by $h$ and take a limit:
\[
(fg)'(x) = \lim_{h \to 0} \frac{(fg)(x+h) - (fg)(x)}{h}
\]

\[
= \lim_{h \to 0} g(x) \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} f(x) \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \frac{g(x+h) - g(x)}{h}
\]

If \( f \) is differentiable, then it's continuous, so
\[
\lim_{h \to 0} f(x+h) - f(x) = f(x) - f(x) = 0.
\]

This term is therefore 0.

This proves the product rule!

Example Find \( \frac{d}{dx} xe^x \).

Solution We have
\[
\frac{d}{dx} xe^x = (\frac{d}{dx} x)e^x + x(\frac{d}{dx} e^x)
\]

\[
= 1 \cdot e^x + x \cdot e^x
\]

\[
= (x+1)e^x.
\]

Example Last time we saw \( \frac{d}{dx} x^3 \neq \frac{d}{dx} x \cdot \frac{d}{dx} x^2 \). Let's check \( \frac{d}{dx} x^3 = 3x^2 \) using the product rule.

\[
\frac{d}{dx} x^3 = \frac{d}{dx} (x \cdot x^2)
\]

\[
= (\frac{d}{dx} x) x^2 + x \cdot (\frac{d}{dx} x^2)
\]

\[
= 1 \cdot x^2 + x \cdot 2x
\]

\[
= 3x^2.
\]

Quotient rule: If \( f \) and \( g \) are differentiable, then \( f/g \) is differentiable where it is defined and
\[
(f/g)' = \frac{f'g - fg'}{g^2}
\]

(My calculus teacher had the following silly mnemonic for remembering this:
"Low-d-high minus high-d-low, square the bottom and away we go!")

The proof is somewhat similar to the proof of the product rule. We have:
\[
\frac{g(x)(f(x+h) - f(x)) - f(x)(g(x+h) - g(x))}{g(x)g(x+h)} = \frac{f(x+h)g(x) - f(x)g(x) - f(x)g(x+h) + f(x)g(x+h)}{g(x)g(x+h)}
\]

\[
= \frac{f(x+h)g(x) - f(x)g(x)}{g(x)g(x+h)} - \frac{f(x)g(x+h) - f(x)g(x)}{g(x)g(x+h)}
\]

\[
= \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} - \frac{f(x+h)}{g(x+h)} + \frac{f(x)}{g(x)} = (\frac{f}{g})(x) - (\frac{f}{g})(x).
\]
Divide by \( h \) and take the limit as \( h \to 0 \):

\[
(f/g)'(x) = \lim_{h \to 0} \frac{(f(x+h) - f(x))}{h} \quad \text{divided by} \quad \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}
\]

Again, I'm doing a few steps at once.

\[
= \lim_{h \to 0} \frac{g(x) \frac{f(x+h) - f(x)}{h} - f(x) \frac{g(x+h) - g(x)}{h}}{g(x)g(x+h)}
\]

\[
= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}
\]

This limit is \( g(x) \) by continuity of \( g \).

This proves the quotient rule.

**Example:** Find \( \frac{d}{dx} \frac{x^2 + 3x + 1}{x - 2} \).

**Solution** We have

\[
\frac{d}{dx} \frac{x^2 + 3x + 1}{x - 2} = \frac{(x-2) \left( \frac{d}{dx} (x^2 + 3x + 1) \right) - (x^2 + 3x + 1) \left( \frac{d}{dx} (x-2) \right)}{(x-2)^2}
\]

\[
= \frac{(x-2)(2x + 3) - (x^2 + 3x + 1)(1)}{(x-2)^2}
\]

Expand everything:

\[
= \frac{2x^2 + 3x - 4x - 6 - x^2 - 3x - 1}{x^2 - 4x + 4}
\]

\[
= \frac{x^2 - 4x - 7}{x^2 - 4x + 4}
\]

**Example:** Let's check the power rule for negative powers of \( x \). Let \( n \) be a positive integer.

\[
\frac{d}{dx} x^{-n} = \frac{d}{dx} \frac{1}{x^n}
\]

\[
= x^{-n} \left( \frac{d}{dx} \frac{1}{x^n} \right) - 1 \cdot \left( \frac{d}{dx} x^{-n} \right)
\]

\[
= x^{-n} \cdot 0 - 1 \cdot n x^{-n-1}
\]

\[
= -n \frac{x^{-n-1}}{x^n} = -n x^{n-1-n} = -n x^{-n-1}
\]
Trig functions

Fact

\[ \frac{d}{dx} \sin x = \cos x. \]

This is believable from the graph:

At each \( \frac{n\pi}{2} \), I have drawn in blue the tangent line to \( y=\sin x \) and plotted its slope. Without \( y=\sin x \):

Filling in this graph smoothly gives:

This blue curve looks an awful lot like \( y=\cos x! \)

The book proves this is true using a geometric argument to show that for \( 0<\theta<\frac{\pi}{4} \), we have

\[ \theta < \tan \theta = \frac{\sin \theta}{\cos \theta}, \text{ and } \sin \theta < \theta. \]

This implies

\[ \cos \theta < \frac{\sin \theta}{\theta} < 1 \quad \text{for } 0<\theta<\frac{\pi}{4}, \]

and the squeeze theorem gives \( \lim_{x \to 0} \frac{\sin \theta}{\theta} = 1 \). The even-ness of \( \frac{\sin \theta}{\theta} \) gives \( \lim_{x \to 0} \frac{\sin \theta}{\theta} = 1 \), so \( \lim_{x \to 0} \frac{\sin \theta}{\theta} = 1 \).

Now we compute:

\[ \frac{d}{dx} \sin(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \]

Angle sum formula

\[ = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \]

\[ = \lim_{h \to 0} \frac{\sin(x)\cos(h) - 1}{h} + \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h} \]
This part of the argument is just showing
\[
\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0,
\]
a formula we will use below.

\[
\begin{aligned}
\sin(x) &= \left( \lim_{h \to 0} \frac{\cos(h) - 1}{h} \right) + \cos(x) \\
&= \sin(x) \left( \lim_{h \to 0} \frac{\cos(h) - 1}{h} \right) + \cos(x) \\
&= \sin(x) \left( \lim_{h \to 0} \frac{-\sin^2(h)}{h} \right) + \cos(x) \\
&= \sin(x) \left( \lim_{h \to 0} \frac{-\sin(h)}{\cos(h) + 1} \right) + \cos(x) \\
&= \sin(x) \left( \frac{-\sin(h)}{h} \right) + \cos(x) \\
&= \sin(x) \left( \frac{-\sin(h)}{h} \right) + \cos(x) \\
&= 0 + \cos(x) \\
&= \cos(x).
\end{aligned}
\]

This proves \( \frac{d}{dx} \sin(x) = \cos(x) \).

What about \( \cos(x) \)? Well, we have:

\[
\frac{d}{dx} \cos(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}
\]

Angle sum formula:

\[
\begin{aligned}
\cos(x) \cos(h) - \sin(x) \sin(h) &= \cos(h) \\
\end{aligned}
\]

\[
\begin{aligned}
\lim_{h \to 0} \cos(x) \cos(h) - \sin(x) \sin(h) &= \lim_{h \to 0} \cos(x) \cdot 1 - \lim_{h \to 0} \sin(x) \cdot \sin(h) \\
\end{aligned}
\]

Because \( \lim_{h \to 0} \cos(h) = \cos(0) \), as above:

\[
\begin{aligned}
\lim_{h \to 0} \cos(x) \cdot 1 &= \cos(x) \\
\lim_{h \to 0} \sin(x) \cdot \sin(h) &= \sin(x) \lim_{h \to 0} \sin(h) \\
\end{aligned}
\]

Therefore \( \frac{d}{dx} \cos(x) = -\sin(x) \).

We can also compute the derivatives of other trig functions using the quotient rule:

\[
\begin{aligned}
\frac{d}{dx} \tan(x) &= \frac{d}{dx} \frac{\sin(x)}{\cos(x)} \\
&= \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot \sin(x)}{\cos^2(x)} \\
&= \frac{\cos^2(x) - \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} = \sec^2(x).
\end{aligned}
\]

\[
\begin{aligned}
\frac{d}{dx} \sec(x) &= \frac{d}{dx} \frac{1}{\cos(x)} \\
&= -\frac{\cos(x) \cdot 1 - \sin(x) \cdot \sin(x)}{\cos^2(x)} \\
&= \frac{-\cos(x) - \sin^2(x)}{\cos^2(x)} \\
&= -\tan(x) \sec(x).
\end{aligned}
\]

\[
\begin{aligned}
\frac{d}{dx} \csc(x) &= \frac{d}{dx} \frac{1}{\sin(x)} \\
&= -\frac{\sin(x) \cdot 1 - \cos(x) \cdot \cos(x)}{\sin^2(x)} \\
&= \frac{-\sin(x) - \cos^2(x)}{\sin^2(x)} \\
&= -\cot(x) \csc(x).
\end{aligned}
\]

\[
\begin{aligned}
\frac{d}{dx} \cot(x) &= \frac{d}{dx} \frac{\cos(x)}{\sin(x)} \\
&= \frac{\sin(x) \cdot 1 - \cos(x) \cdot \cos(x)}{\sin^2(x)} \\
&= \frac{\sin(x) - \cos^2(x)}{\sin^2(x)} \\
&= -\csc^2(x).
\end{aligned}
\]