More on Integration

Some Properties

1. \( \int_a^b c \, dx = c(b-a) \)
2. \( \int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \)
3. \( \int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx \)
4. \( \int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \)

(1) is just the formula for the area of a rectangle.
(2) is justified by using the sum rule for limits:
\[
\int_a^b [f(x) + g(x)] \, dx = \lim_{n \to \infty} \sum_{i=1}^n [f(x_i^*) + g(x_i^*)] \Delta x
\]
\[
= \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*) \Delta x + \sum_{i=1}^n g(x_i^*) \Delta x
\]
\[
= \lim_{n \to \infty} \left[ \sum_{i=1}^n f(x_i^*) \Delta x \right] + \left[ \sum_{i=1}^n g(x_i^*) \Delta x \right]
\]
\[
= \int_a^b f(x) \, dx + \int_a^b g(x) \, dx
\]

(3) and (4) are proved in a similar way.

Another important property:

5. \( \int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx \)

Picture:

Area of A and B together = (Area of A) + (Area of B).

Finally, we have:

6. \( \int_b^a f(x) \, dx = -\int_a^b f(x) \, dx \). (Switching limits of integration flips the sign.)
Fundamental Theorem of Calculus (FTC)

**FTC part 1:** If \( f \) is continuous on \([a,b]\), then the function \( g \) defined by

\[
g(x) = \int_a^x f(t) \, dt
\]

is continuous on \([a,b]\), and differentiable on \((a,b)\) with derivative \( g'(x) = f(x) \).

That is,

\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x).
\]

We won't have time to prove this, but the book gives a proof in Section 5.3.

**Example.** Last time we saw \( \int_a^b 3t \, dt = 6 \). More generally \( \int_a^b 3t \, dt = \frac{3b^2}{2} - \frac{3a^2}{2} \).

We have, therefore,

\[
\frac{d}{dx} \int_a^x 3t \, dt = \frac{d}{dx} \left( \frac{3x^2}{2} - \frac{3a^2}{2} \right) = 3x.
\]

Which verifies FTC part 1.

**Example** Find \( g'(x) \) when \( g(x) = \int_a^x \sqrt{1+t^2} \, dt \).

**Solution** Just apply FTC part 1, noting that \( \sqrt{1+t^2} \) is continuous, so FTC part 1 can really be applied!

\[
g'(x) = \sqrt{1+x^2}.
\]

**Example** Find \( \frac{d}{dx} \int_a^x \sec t \, dt \).

**Solution** We will have to use the chain rule. Let \( g(x) = \int_a^x \sec t \, dt \), \( h(x) = x^4 \). Then

\[
\frac{d}{dx} \int_a^x \sec t \, dt = \frac{d}{dx} \int_a^{h(x)} \sec t \, dt
\]

\[
= \frac{d}{dx} g(h(x)) = g'(h(x)) h'(x).
\]

But \( g'(x) = \sec x \) by FTC part 2, and \( h'(x) = 4x^3 \) by the power rule. So:

\[
g'(h(x)) h'(x) = \sec(x^4) \cdot 4x^3.
\]
If $f$ is continuous on $[a,b]$, then
$$\int_a^b f(x)\,dx = F(b) - F(a),$$
where $F$ is any antiderivative of $f$. “Antiderivative of $f$” just means a function with $F' = f$.

Again, the proof is in Section 5.3 in the book.

Fact: If $f$ is continuous, then any two antiderivatives of $f$ differ by a constant. That is:

If $F'(x) = F(x)$ and $G'(x) = F(x)$,

then $F(x) = G(x) + C$, for some constant $C$.

This fact can be proved using the mean value theorem.

Example Find $\int_0^1 x^2 \,dx$.
Solution We need to find a function $F(x)$ so that $F'(x) = x^2$. The power rule tells us
$$\frac{d}{dx} x^3 = 3x^2,$$
which is close. If we divide by 3 and try instead $\frac{x^3}{3}$, then
$$\frac{d}{dx} \frac{x^3}{3} = \frac{3x^2}{3} = x^2,$$
and we win. So now,
$$\int_0^1 x^2 \,dx = \left. \frac{x^3}{3} \right|_0^1,$$
where this bar just means “evaluate $\frac{x^3}{3}$ at the 1 on top and the 0 on bottom and subtract”.
$$= \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}.$$

Example Find $\int_0^1 e^x \,dx$.
Solution We know $\frac{d}{dx} e^x = e^x$. This means exactly that $e^x$ is its own antiderivative. So,
$$\int_0^1 e^x \,dx = \left. e^x \right|_0^1 = e - 1.$$

Example Find $\int_0^1 \frac{dx}{x}$.
Solution This expression just means $\int_0^1 \frac{1}{x} \,dx$. Since $\frac{d}{dx} \ln x = \frac{1}{x}$, we get:
$$\int_0^1 \frac{1}{x} \,dx = [\ln x]_0^1 = \ln 1 - \ln 0 = \ln \frac{6}{3} = \ln 2.$$
Example  Find \( \int \frac{1}{x^2} \, dx \).

Solution \( \frac{d}{dx} x^{-1} = -1 \cdot x^{-2} = -\frac{1}{x^2} \). So

\[
\int_1^2 \frac{1}{x^2} = -x^{-1} \bigg|_1^2 = -2^{-1} - (-1^{-1}) = -\frac{1}{2} + 1 = \frac{1}{2}.
\]

Caution: What if we tried this for \( \int_{-1}^2 \frac{1}{x^2} \, dx \)? Well, we would get

\[
\int_{-1}^2 \frac{1}{x^2} \, dx = -x^{-1} \bigg|_{-1}^2 = -2^{-1} - (-1^{-1}) = -\frac{1}{2} - 1 = -\frac{3}{2},
\]

But this is wrong!! The problem is that \( \frac{1}{x^2} \) is not continuous on \([-1, 2]\), so FTC part 2 can't be applied. Indeed, \( \frac{1}{x^2} \) has an infinite discontinuity at \( x=0 \).

In fact, \( \int_{-1}^2 \frac{1}{x^2} \, dx \) does not exist!

Table of antiderivatives

<table>
<thead>
<tr>
<th>Function</th>
<th>One particular antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^n ) if ( n \neq -1 )</td>
<td>( \frac{x^{n+1}}{n+1} )</td>
</tr>
<tr>
<td>( \frac{1}{x} )</td>
<td>( \ln</td>
</tr>
<tr>
<td>( e^x )</td>
<td>( e^x )</td>
</tr>
<tr>
<td>( b^x )</td>
<td>( \frac{b^x}{\ln b} )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( -\cos x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( \sin x )</td>
</tr>
<tr>
<td>( \sec^2 x )</td>
<td>( \tan x )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{1-x^2}} )</td>
<td>( \sin^{-1} x )</td>
</tr>
<tr>
<td>( \frac{1}{1+x^2} )</td>
<td>( \tan^{-1} x )</td>
</tr>
</tbody>
</table>

Antiderivatives respect addition, and also multiplication by constants: If \( F'=f \) and \( G'=g \), The an antiderivative of \( f(x)+g(x) \) is \( F(x)+G(x) \), and an antiderivative of \( cf(x) \) is \( cF(x) \).
Physical interpretation of integration

Since integration undoes differentiation, we have:

\[
\text{Integral of velocity (with respect to time)} = \text{Distance travelled (over time)}.
\]

Also, since \( \frac{d}{dt} \text{velocity} = \text{acceleration} \), we have

\[
\text{Integral of acceleration (with respect to time)} = \text{velocity (at a time)}.
\]