The definite integral

Today, we will make sense of the expression

\[ \int_a^b f(x) \, dx. \]

This expression is called a definite integral. I will say what this means today.

The two most important things you should learn about definite integrals are:

- That they represent the area under a graph. (which I’ll talk about today)
- How to evaluate them using antiderivatives (which I’ll talk about next time).

Before we even touch these definite integrals, we need some notation.

**Sigma notation.**

Let \( x_1, x_2, x_3, \ldots, x_{n-1}, x_n \) be real numbers. We use the following shorthand notation for the sum of all these numbers:

\[ \sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n. \]

\( \Sigma \), or \( \text{Sigma} \), is the Greek analog of capital \( S \) (or rather, capital \( S \) is the Latin analog of \( \Sigma \)).

**Example.** There is a formula for this sum when \( x_1 = 1, x_2 = 2, \ldots, x_i = i, \ldots, x_{n-1} = n-1, x_n = n \). It is

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \]

In other words,

\[ 1 + 2 + 3 + \ldots + (n-1) + n = \frac{n(n+1)}{2}. \]

For example,

\[ \sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6 = 21 = \frac{6(6+1)}{2} = 21 \]

An easier example is

\[ \sum_{i=1}^{n} c = n \cdot c \quad (c \text{ constant}). \]

Here \( x_1 = x_2 = \ldots = x_{n-1} = x_n = c \).

**Areas under curves.** We want to evaluate the area under the curve \( y = f(x) \) on an interval \([a, b]\).
Step 1. Estimate the area by breaking up \([a,b]\) into smaller intervals 
\([a,x_1],[x_1,x_2], \ldots,[x_{n-1},x_n],[x_n,b]\)
and finding the area of rectangles, like in this picture.

![Diagram showing estimation of area](image)

Here we must choose \(x_1^*, x_2^*, \ldots, x_{n-1}^*, x_n^*\) some numbers in these intervals, where the top of the rectangles intersect the graph.

![Diagram showing的选择 of \(x_i^*\) values](image)

The tops of these rectangles will intersect the graph at \(f(x_1^*), f(x_2^*), \ldots, f(x_{n-1}^*), f(x_n^*)\).

Write, for convenience, \(a=x_0\) and \(b=x_n\). The area of each rectangle is base \(\times\) height = \((x_i-x_{i-1}) \cdot f(x_i^*)\).

Therefore:

\[
\text{Area under graph} \approx \text{sum of area of rectangles} = \sum_{i=1}^{n} f(x_i^*) \cdot (x_i-x_{i-1}).
\]

Step 2. Make better estimates by breaking up \([a,b]\) into more intervals with even smaller width.

![Diagram showing new \(n=5\) intervals](image)

\(n\) will get bigger, and our estimate will be finer.
Step 3 Make finer and finer estimates until we finally have the area. This means, take the limit as \( n \) gets larger and larger:

\[
\text{Area under graph} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) (x_i - x_{i-1}).
\]

We will take this as our definition of the integral. (For convenience, we take our intervals \([x_{i-1}, x_i]\) to be evenly spaced, so that \(x_i - x_{i-1} = (x_2 - x_1) = \cdots = (x_n - x_{n-1}) = (x_n - x_{n-1}) = \frac{b-a}{n}\).) So:

**Definition.**

- Let \( f \) be a function defined on \([-b,b]\).
- Break \([-b,b]\) into equally spaced intervals of length \( \frac{b-a}{n} \), \([x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n],[x_n,x]\).
- Here, \( x_i - x_{i-1} = \frac{b-a}{n} \).
- Write \( \Delta x = \frac{b-a}{n} \), the change in \( x \) from interval to interval.
- Choose \( x_i^* \) in \([x_0,x_1]\), \( x_i^* \) in \([x_1,x_2]\), \ldots, \( x_i^* \) in \([x_{n-1},x_n]\).
Then the definite integral of \( f \) from \(-b\) to \( b \) is

\[
\int_{-b}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x,
\]

Provided this limit exists and gives the same answer for all possible choices of the points \( x_1^*, x_2^*, \ldots, x_n^* \).

If this integral exists, we say \( f \) is **integrable** on \([-b,b]\).

**Remarks.**

1. \( a,b \) are called the limits of integration and \( f(x) \) is called the integrand.
2. \( \int_{a}^{b} f(x) \, dx \) is a number, and doesn't depend on \( x \). We could have used any letter in place of \( x \):

\[
\int_{a}^{b} f(x) \, dx = \int_{b}^{a} f(t) \, dt = \ldots.
\]
3. \( \int_{a}^{b} f(x) \, dx \) measures the area under the graph of \( y = f(x) \). If this graph goes below the \( x \)-axis, we consider the area under the \( x \)-axis to be negative.

This is consistent with the definition, since \( f(x_i^*) \Delta x \) will be negative if \( f(x_i^*) \) is.

4. \( \int \) is a long “S” because it is a limit of sums.
Theorem. If $f$ is continuous on $[a, b]$, or if $f$ has only finitely many jump discontinuities on $[a, b]$, then $f$ is integrable on $[a, b]$.

Example. Evaluate $\int_0^2 3x \, dx$.

Picture:

\[ \int_0^2 3x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} 3x_i^* \, \Delta x. \]

We know $\Delta x = \frac{b-a}{n}$, but $b-a = 2-0 = 2$, so $\Delta x = \frac{2}{n}$.

Also, $x_0 = 0$, $x_1 = \frac{2}{n}$, $x_2 = \frac{4}{n}$, $x_3 = \frac{6}{n}$, ..., $x_i = \frac{2i}{n}$, ..., $x_{n-1} = \frac{2(n-1)}{n}$, $x_{n} = \frac{2n}{n} = 2$.

Choose $x_i^* = x_i$, which we can do because it doesn't matter which $x_i^*$'s we choose as long as $x_i^*$ is in $[x_{i-1}, x_i]$. So $x_i^* = x_i = \frac{2i}{n}$. Then

\[
\int_0^2 3x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} 3x_i^* \, \Delta x
= \lim_{n \to \infty} \sum_{i=1}^{n} 3 \cdot \frac{2i}{n} \cdot \frac{2}{n}
= \lim_{n \to \infty} \frac{12}{n^2} \sum_{i=1}^{n} i
= \lim_{n \to \infty} \frac{12}{n^2} \cdot \frac{n(n+1)}{2}
= \lim_{n \to \infty} \frac{12(n^2+n)}{2n^2}
= \lim_{n \to \infty} \frac{6n^2+6n}{n^2}
= \frac{6}{a}
= 6,
\]

as expected!

Should be $\frac{1}{2}$ base $\times$ height $= \frac{1}{2} \cdot 2 \cdot 6 = 6$. 

Solution. We have

\[
\int_0^2 3x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} 3x_i^* \, \Delta x.
\]
Some Properties

(1) \[ \int_{a}^{b} c \, dx = c(b-a) \]

(2) \[ \int_{a}^{b}[f(x) + g(x)] \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \]

(3) \[ \int_{a}^{b} c f(x) \, dx = c \int_{a}^{b} f(x) \, dx \]

(4) \[ \int_{a}^{b}[f(x) - g(x)] \, dx = \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx \]

(1) is just the formula for the area of a rectangle.

(2) is justified by using the sum rule for limits:

\[ \int_{a}^{b}[f(x) + g(x)] \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_{i}) + g(x_{i})] \Delta x \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i}) \Delta x + \sum_{i=1}^{n} g(x_{i}) \Delta x \]

\[ = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} f(x_{i}) \Delta x \right] + \lim_{n \to \infty} \left[ \sum_{i=1}^{n} g(x_{i}) \Delta x \right] \]

\[ = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx. \]

(3) and (4) are proved in a similar way.

Next time we will learn how to compute integrals using "anti-derivatives" instead of the limit definition.