The area of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where $f$ and $g$ are continuous and $f(x) \geq g(x)$ for all $x$ in $[a, b]$, is given by

$$A = \int_{a}^{b} [f(x) - g(x)] \, dx.$$ 

Graphically:

If we sum up a bunch of rectangles like the red one, we'd get $\sum \Delta x$, the limit of which is the area $A = \int_{a}^{b} [f(x) - g(x)] \, dx$.

Example: Find the area bounded above by $y = e^x$, below by $y = x$, and on the sides by $x = 0$, $x = 1$.

Solution

$$\int_{0}^{1} e^x - x \, dx = e^x - \frac{x^2}{2} \bigg|_{0}^{1}$$

$$= e - \frac{1}{2} - (e^0 - \frac{0^2}{2})$$

$$= e - \frac{1}{2} - 1$$

$$= e - \frac{3}{2}.$$ 

Example: Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution: The picture will look like this:

The integral will go from the $x$-value of the green point of intersection to that of the red point.
So where do these parabolas intersect? Well, this is when their equations coincide:

\[ x^2 = 2x - x^2 \]

\[ 2x^2 - 2x = 0 \]
\[ x^2 - x = 0 \]
\[ x(x-1) = 0 \]
\[ x = 0, 1. \]

So the integral will be from 0 to 1.

Since \( y = 2x - x^2 \) is above \( y = x^2 \), the area is:

\[
\int_0^1 [(2x - x^2) - x^2] \, dx = \int_0^1 (2x - 2x^2) \, dx
\]

\[
= \frac{2x^2}{2} - \frac{2x^3}{3} \bigg|_0^1
\]

\[
= \frac{2^2}{2} - \frac{2^3}{3} - \left( \frac{0^2}{2} - \frac{0^3}{3} \right)
\]

\[
= 1 - \frac{8}{3} - 0
\]

\[
= \frac{1}{3}.
\]

More generally:

The area between \( y = f(x), y = g(x) \), and \( x = a, x = b \) is given by

\[
A = \int_a^b |f(x) - g(x)| \, dx.
\]

(Here, we are not assuming \( f(x) \geq g(x) \). In fact, maybe \( g(x) \geq f(x) \) through part of \([a,b]\) and \( f(x) \geq g(x) \) through the rest.)

Example Find the area of the region bounded by the curves \( y = \sin x \) and \( y = \cos x \), and by \( x = 0 \) and \( x = \frac{\pi}{2} \).
We have
\[ A = \int_0^{\pi/2} |\sin(x) - \cos(x)| \, dx. \]

How would we compute this? Well, on part of \([0, \pi/2]\), \(\sin(x)\) will be larger than \(\cos(x)\), and on the rest, \(\cos(x)\) will be larger than \(\sin(x)\). These parts of \([0, \pi/2]\) will be separated by the point where \(\sin(x) = \cos(x)\). This happens at \(\pi/4\); in fact,
\[ \sin(\pi/4) = \frac{\sqrt{2}}{2} = \cos(\pi/4). \]

On \([0, \pi/4]\), \(\cos(x) \geq \sin(x)\), and on \([\pi/4, \pi/2]\), \(\sin(x) \geq \cos(x)\). Therefore, if we break up the integral, we get:

\[
A = \int_0^{\pi/4} (\cos(x) - \sin(x)) \, dx + \int_{\pi/4}^{\pi/2} (\sin(x) - \cos(x)) \, dx
\]

\[
= \left[ \sin(x) + \cos(x) \right]_0^{\pi/4} + \left[ -\cos(x) - \sin(x) \right]_{\pi/4}^{\pi/2}
\]

\[
= \left[ \sin(\pi/4) + \cos(\pi/4) - \sin(0) - \cos(0) \right] + \left[ -\cos(\pi/2) - \sin(\pi/2) - (-\cos(\pi/4) - \sin(\pi/4)) \right]
\]

\[
= \left[ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 \right] + \left[ 0 - 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right]
\]

\[
= (\sqrt{2} - 1) + (\sqrt{2} - 1)
\]

\[
= 2\sqrt{2} - 2.
\]

**Example** Find the area of the region enclosed by \(y = x - 1\) and the parabola \(y^2 = 2x + 6\).

**Solution** Intersection points: Substituting \(y = x - 1\) into \(y^2 = 2x + 6\) gives

\[(x-1)^2 = 2x + 6 \]

\[x^2 - 2x + 1 = 2x + 6\]
The points of intersection have \( x \) values -1 and 5. Therefore, substituting into \( y = x - 1 \), the corresponding \( y \)-values are \( y = -1 - 1 = -2 \) and \( y = 5 - 1 = 4 \). So

\[
\text{Intersection points} = (-1, -2) \text{ and } (5, 4).
\]

We decide to integrate in \( y \) instead.

So we need \( x \) as a function of \( y \) for both curves:

\[
\begin{align*}
  y &= x - 1 & y^2 &= 2x + 6 \\
  y + 1 &= x & y^2 - 6 &= 2x \\
        & & \frac{1}{2}y^2 - 3 &= x.
\end{align*}
\]

The line is higher in the \( x \)-direction than the parabola, so the area is

\[
A = \int_{-2}^{4} \left[ y + 1 - \left( \frac{1}{2}y^2 - 3 \right) \right] \, dy
\]

\[
= \int_{-2}^{4} \left[ -\frac{1}{2}y^2 + y + 4 \right] \, dy
\]

\[
= -\frac{y^3}{6} + \frac{y^2}{2} + 4y \bigg|_{-2}^{4}
\]

\[
= -\frac{4^3}{6} + \frac{4^2}{2} + 4(4) - \left( -\frac{(-2)^3}{6} + \frac{(-2)^2}{2} + 4(-2) \right)
\]

\[
= -\frac{64}{6} + \frac{16}{2} + 16 - \frac{8}{6} - \frac{4}{2} + 8
\]

\[
= \frac{-72}{6} + \frac{12}{2} + 24
\]

\[
= -12 + 6 + 24
\]

\[
= 18.
\]