Indefinite integrals.

Recall If $f$ is continuous on an interval, and if $F' = f$ and $G' = f$, then

$$G = F + C,$$

where $C$ is a constant. Therefore, all the antiderivatives of $f$ can be found by taking one antiderivative $F$ of $f$ and adding any constant $C$ to it.

**Notation (Indefinite integral).** We write

$$\int f(x) \, dx = F(x)$$

to mean

$$F'(x) = f(x) \quad \text{(on a given interval)}.$$ 

The “function” $F(x)$ in the first expression is really a **family** of functions; It is all functions with derivative $f(x)$.

**FTC 2 (from last lecture.)** If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x) \, dx = \left[ F(x) \right]_a^b.$$

**Example:**

$$\int x^2 \, dx = \frac{x^3}{3} + C.$$

You must always include the “$+C$”.

If $f(x)$ is continuous on a union of different intervals, then $F(x)$ will have more antiderivatives.

**Example:**

$$\int \frac{1}{x^2} \, dx = \begin{cases} -\frac{1}{x} + C_1 & \text{if } x<0 \\ \frac{1}{x} + C_2 & \text{if } x>0 \end{cases}$$

because, no matter what $C_1$ and $C_2$ are, these are all antiderivatives of $\frac{1}{x^2}$. The reason this works is that $\frac{1}{x^2}$ is discontinuous at $0$, and $0$ is not in the domain.
The Substitution Rule

This is like the chain rule, but in reverse. It says:

If \( u = g(x) \) is differentiable with range on interval \( I \), and if \( f \) is continuous on \( I \), then

\[
\int f(g(x)) g'(x) \, dx = \int f(u) \, du.
\]

Example. Find

\[
\int 2xe^{x^2} \, dx.
\]

Solution. Here we want \( g'(x) = 2x \). Therefore, if we take \( u = g(x) = x^2 \) and \( f(x) = e^x \), then we can use the substitution rule. Write \( u = x^2 \). Then \( du = \frac{du}{dx} \cdot dx = \left(\frac{d}{dx} x^2\right) dx = 2x dx \). So:

\[
\int 2xe^{x^2} \, dx = \int e^u \, du
\]

\[
= e^u + C
\]

\[
= e^{x^2} + C.
\]

Example. Let's find \( \int x^2 \cos(x^4+2) \, dx \).

Solution. \( u = x^4 + 2 \), so \( du = 4x^3 \, dx \), and so \( \frac{1}{4} \, du = x^3 \, dx \).

\[
\int x^2 \cos(x^4+2) \, dx = \int \frac{1}{4} \cos(u) \, du
\]

\[
= \frac{1}{4} \sin u + C
\]

\[
= \frac{1}{4} \sin(x^4+2) + C.
\]

Example. Let's find \( \int \sqrt{3x+1} \, dx \).

Solution. Let \( u = 3x + 1 \), so \( du = 3 \, dx \), or \( dx = \frac{1}{3} \, du \). So

\[
\int \sqrt{3x+1} \, dx = \int \sqrt{u} \frac{1}{3} \, du
\]

\[
= \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C
\]

\[
= \frac{1}{3} \sqrt{u}^3 + C
\]

\[
= \frac{1}{3} (3x+1)^{3/2} + C.
\]
Example
Find \( \int e^{5x} \, dx \)

Solution
You can do this quickly by recognizing \( \frac{1}{5} e^{5x} = 5e^{5x} \). So

\[
\int e^{5x} \, dx = \frac{1}{5} \int 5e^{5x} \, dx = \frac{1}{5} e^{5x} + C.
\]

Example
Let's find \( \int \frac{x^5}{1 + x^2} \, dx \).

Solution
Let \( u = 1 + x^2 \). Then \( du = 2x \, dx \). Since \( x^5 \, dx = \frac{1}{2} x^9 \, 2x \, dx \), we must express \( \frac{1}{2} x^4 \) in terms of \( u \). We have \( u - 1 = x^2 \), so \( \frac{1}{2} (u - 1)^2 = \frac{1}{2} x^4 \). Therefore,

\[
\int \frac{x^5}{1 + x^2} \, dx = \int \frac{1}{2} x^4 \cdot 2x \, dx = \int \frac{1}{2} x^9 \cdot 2x \, dx = \int \frac{1}{2} x^{10} \, dx = \frac{1}{2} \int (u - 1)^2 \, du = \frac{1}{2} \int (u - 1)(u - 1) \, du = \frac{1}{2} \int (u^2 - 2u + 1) \, du = \frac{1}{2} \left( \frac{2}{7} u^{7/2} - 2 \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right) + C = \frac{1}{7} u^{7/2} - \frac{2}{5} u^{5/2} + \frac{1}{3} u^{3/2} + C = \frac{1}{7} (1 + x^2)^{7/2} - \frac{2}{5} (1 + x^2)^{5/2} + \frac{1}{3} (1 + x^2)^{3/2} + C.
\]

Example
Let's find \( \int \tan x \, dx \).

Solution
\( \tan x = \frac{\sin x}{\cos x} \). Let \( u = \cos x \), then \( du = -\sin x \, dx \), so

\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{-1}{\cos x} (-\sin x) \, dx = \int \frac{-1}{u} \, du = -\ln |u| + C = -\ln |\cos x| + C.
\]
Substitution for definite integrals.

If you want to use substitution to evaluate a definite integral, you must change the limits of integration.

If \( g' \) is continuous on \([a,b]\) and \( f \) is continuous on the range of \( u=g(x) \), then

\[
\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
\]

**Proof** If \( F \) is an antiderivative of \( F \), then \( \frac{d}{dx} F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x) \). So FTC 2 gives

\[
\int_a^b f(g(x))g'(x) \, dx = F(g(b)) - F(g(a)).
\]

But FTC 2 also gives us

\[
\int_{g(a)}^{g(b)} f(u) \, du = F(u) \bigg|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)).
\]

So the two expressions are equal.

**Example** Evaluate \( \int_0^1 \sqrt{2x+1} \, dx \).

**Solution** Take \( u=2x+1 \). Then \( du=2dx \), or \( dx=\frac{1}{2}du \), and \( 2.0+1=3 \), and \( 2.4+1=9 \). So

\[
\int_0^1 \sqrt{2x+1} \, dx = \int_1^3 \sqrt{u} \frac{1}{2} \, du
\]

\[
= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \bigg|_1^3
\]

\[
= \frac{1}{3} \cdot 9^{3/2} - \frac{1}{3} \cdot 1^{3/2}
\]

\[
= \frac{1}{3} \cdot 27 - \frac{1}{3} \cdot 1
\]

\[
= 9 - \frac{1}{3} = \frac{26}{3}.
\]

**Example** Evaluate \( \int_1^2 \frac{dx}{(3-5x)^2} \).

**Solution** \( u=3-5x \), so \( du=-5dx \) or \( \frac{-1}{5}du=dx \). Also, \( 3-5 \cdot 1=-2 \), and \( 3-5 \cdot 2=-7 \). So

\[
\int_1^2 \frac{dx}{(3-5x)^2} = \int_{-2}^{-7} \frac{1}{u^2} \cdot \frac{-1}{5} \, du
\]

\[
= \left[ -\frac{1}{u} \right]_{-2}^{-7}
\]

\[
= -\frac{1}{5} \left( -\frac{1}{7} - \frac{1}{2} \right)
\]

\[
= -\frac{1}{5} \left( -\frac{5}{14} \right) = \frac{1}{14}.
\]
Example Evaluate \( \int_1^e \frac{\ln x}{x} \, dx \)

Solution Let \( u = \ln x \) so \( du = \frac{1}{x} \, dx \). \( \ln 1 = 0 \), \( \ln e = 1 \), so

\[
\int_1^e \frac{\ln x}{x} \, dx = \int_0^1 u \, du
\]

\[
= \frac{u^2}{2} \Big|_0^1
= \frac{1}{2} - 0
= \frac{1}{2}
\]

\( \approx 0.5 \).