QUASI-LOCAL ENERGY IN PRESENCE OF GRAVITATIONAL RADIATION

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Abstract. We discuss our recent work [4] in which gravitational radiation was studied by evaluating the Wang-Yau quasi-local mass of surfaces of fixed size at the infinity of both axial and polar perturbations of the Schwarzschild spacetime, à la Chandrasekhar [1].

We compute the Wang-Yau quasi-local mass [7, 8] of "spheres of unit size" at null infinity to capture the information of gravitational radiation. The set-up, following Chandrasekhar [1], is a gravitational perturbation of the Schwarzschild solution, which is governed by the Regge-Wheeler equation (see below). We take a sphere of a fixed areal radius and push it all the way to null infinity. The limit of the geometric data is that of a standard configuration and thus the optimal embedding equation [7, 8, 2] can be solved.

Let us first consider the axial perturbations. The metric perturbation is of the form:

$$-(1-\frac{2m}{r})dt^2+\frac{1}{1-\frac{2m}{r}}dr^2+r^2d\theta^2+r^2\sin^2\theta(d\phi-q_2dr-q_3d\theta)^2.$$

The linearized vacuum Einstein equation is solved by a separation of variable Ansatz in which q_2 and q_3 are explicitly given by the Teukolsky function and the Legendre function.

In particular,

$$q_3=\sin(\sigma t)\frac{C_\mu(\theta)}{\sin\theta}\frac{(r^2-2mr)}{\sigma^2r^4}\frac{d}{dr}(rZ^{(-)})$$

for a solution of frequency σ and a separation of variable constant μ. Here $C_\mu(\theta)$ is related to the μ-th Legendre function P_μ by

$$C_\mu(\theta)=\sin\theta\frac{d}{d\theta}(\frac{1}{\sin\theta}\frac{dP_\mu(\cos\theta)}{d\theta}).$$

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After the change of variable

$$r_* = r + 2m \ln(\frac{r}{2m} - 1),$$

$Z^{(-)}$ satisfies the Regge-Wheeler equation:

$$(\frac{d^2}{dr_*^2} + \sigma^2) Z^{(-)} = V^{(-)} Z^{(-)},$$

where

$$V^{(-)} = \frac{r^2 - 2mr}{r^5}[(\mu^2 + 2)r - 6m],$$

and μ is a separation of variable constant.

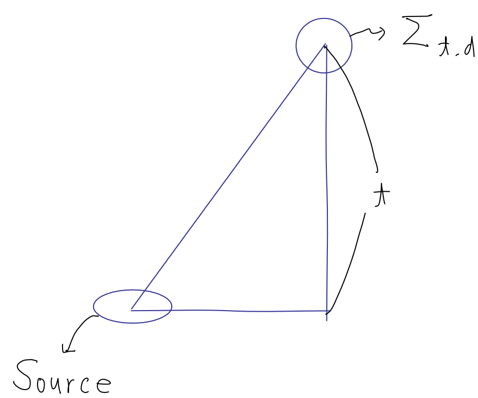
On the Schwarzschild spacetime

$$-(1 - \frac{2m}{r})dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

we consider an asymptotically flat Cartesian coordinate system (t, y_1, y_2, y_3) with $y_1 = r\sin\theta\sin\phi, y_2 = r\sin\theta\cos\phi, y_3 = r\cos\theta$. Given $(d_1, d_2, d_3) \in \mathbb{R}^3$ with $d^2 = \sum_{i=1}^3 d_i^2$, consider the 2-surface

$$\Sigma_{t,d} = \{(t, y_1, y_2, y_3) : \sum_{i=1}^3 (y_i - d_i)^2 = 1\}.$$

We compute the quasi-local mass of $\Sigma_{t,d}$ as $d \to \infty$.



Denote

$$A(r) = \frac{(r^2 - 2mr)}{\sigma^2 r^3} \frac{d}{dr}(rZ^{(-)}).$$

The linearized optimal embedding equation of $\Sigma_{t,d}$ is reduced to two linear elliptic equations on the unit 2-sphere S^2:

$$\Delta(\Delta+2)\tau = [-A''(1-Z_1^2)+6A'Z_1+12A]Z_2Z_3$$
$$(\Delta+2)N = (A''-2A'Z_1+4A)Z_2Z_3,$$

where τ and N are the respective time and radial components of the solution, and Z_1, Z_2, Z_3 are the three standard first eigenfunctions of S^2. A' and A'' are derivatives with respect to r, and r^2 is substituted by $r^2 = d^2+2Z_1+1$ in the above equations.

The quasi-local mass of $\Sigma_{t,d}$ with respect to the optimal isometric embedding is then

$$E(\Sigma_{t,d}) = C^2\{\sin^2(\sigma t)E_1 + \sigma^2\cos^2(\sigma t)E_2\} + O(\frac{1}{d^3}),$$

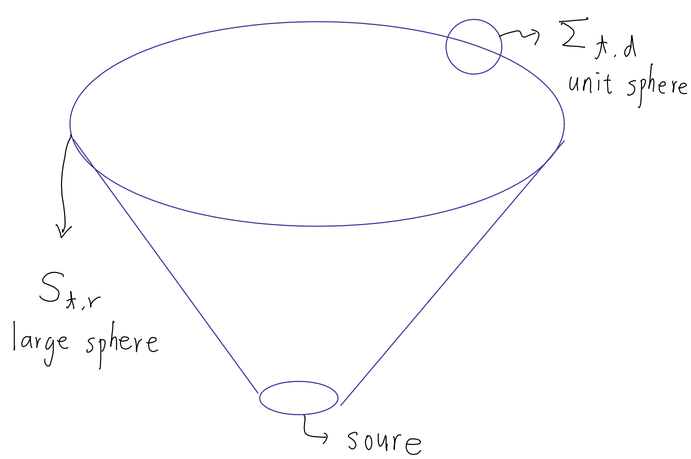
where E_1 and E_2 are two integrals on the standard unit 2-sphere, that depend on the solution τ and N of the optimal isometric embedding equation. Explicitly,

$$E_1 = \int_{S^2}(1/2)\left[A^2Z_2^2(7Z_3^2+1)+2AA'Z_1Z_3^2(3Z_2^2-1)-N(\Delta+2)N\right]$$
$$E_2 = \int_{S^2}\left[A^2Z_2^2Z_3^2-\tau\Delta(\Delta+2)\tau\right].$$

In particular,

$$\partial_t E(\Sigma_{t,d}) = \frac{\sigma\sin(2\sigma t)C^2(\theta)}{d^2}\{E_1-\sigma^2E_2\} + O(\frac{1}{d^3}).$$

Let us compare the quasi-local mass on the small spheres $\Sigma_{t,d}$ along a certain direction to the quasi-local mass of the large coordinate spheres $S_{t,r}$.



Naively, one may expect to recover $\partial_t E(S_{t,r})$ by integrating the energy radiated away at all directions $\partial_t E(\Sigma_{t,d})$. However, our calculation indicates that there are nonlinear correction terms from the quasi-local energy that should be taken into account.

We can also consider the polar perturbation of the Schwarzschild spacetime in which the metric coefficients g_{tt}, g_{rr}, $g_{\theta\theta}$, and $g_{\phi\phi}$ are perturbed in

$$-(1-\frac{2m}{r})dt^2+\frac{1}{1-\frac{2m}{r}}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2.$$

The gravitational perturbation is governed by the Zerilli equation

$$(\frac{d^2}{dr_*^2}+\sigma^2)Z^{(+)}=V^{(+)}Z^{(+)},$$

where

$$V^{(+)}=\frac{2(r^2-2mr)}{r^5(nr+3m)^2}[n^2(n+1)r^3+3mn^2r^2+9m^2nr+9m^3],$$

and n is the separation of variable constant. Again, we compute the quasi-local mass of spheres of unit-size at null infinity. The calculation is similar to the axial perturbation case but the result is different as the leading term is of the order $\frac{1}{d}$ (as opposed to $\frac{1}{d^2}$ for axial-perturbation) with nonzero coefficients. If such a linear perturbation can be realized as an actual perturbation of the Schwarzschild spacetime, the result would contradict the positivity of the quasi-local mass [6, 7, 8]. From this, we deduce the following conclusion: There does not exist any gravitational perturbation of the Schwarzschild spacetime that is of purely polar type in the sense of Chandrasekhar [1].

For an actual gravitational perturbation of the Schwarzschild solution, the vanishing of the $\frac{1}{d}$ gives a limiting integrand that integrates to zero on the limiting 2-sphere at null infinity. In fact, the quasi-local mass density ρ (see [3, equation 2.2]) of $\Sigma_{t,d}$ can be computed at the pointwise level. Up to an $O(\frac{1}{d^3})$ term

$$\begin{aligned}\rho=&(K-\frac{1}{4}|H|^2)\\&-\frac{(|H|-2)^2}{4}+\frac{1}{d^2}\{\frac{1}{2}|\nabla^2N|^2+((\Delta+2)N)^2-\frac{1}{4}(\Delta N)^2\\&-\frac{1}{4}(\Delta\tau)^2+\frac{1}{2}[\nabla^a\nabla^b(\tau_a\tau_b)-|\nabla\tau|^2-\Delta|\nabla\tau|^2]\},\end{aligned}$$

where K is the Gauss curvature of $\Sigma_{t,d}$. The first line, which integrates to zero, is of the order of $\frac{1}{d}$ and is exactly the mass aspect function of the Hawking mass [5]. The $\frac{1}{d^2}$ term of the quasi local mass $\int_{\Sigma_d}\rho\, d\mu_{\Sigma_{t,d}}$ has contributions from the second and third lines (of the order of $\frac{1}{d^2}$), the $\frac{1}{d^2}$ term of the first line, and the $\frac{1}{d}$ term of the area element $d\mu_{\Sigma_{t,d}}$. The above integral formula is obtained after performing integrations by parts and applying the optimal embedding equation several times.

To each closed loop on the limiting 2-sphere at null infinity, we can thus associate a non-vanishing arc integral that is of the order of $\frac{1}{d}$, where d is the distance from the source. We expect the freedom in varying the shape of the loop can increase the detectability of gravitational waves.

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