MINKOWSKI FORMULAE AND ALEXANDROV THEOREMS IN SPACETIME

MU-TAO WANG, YE-KAI WANG, AND XIANGWEN ZHANG

Abstract. The classical Minkowski formula is extended to spacelike codimension-two submanifolds in spacetimes which admit "hidden symmetry" from conformal Killing-Yano two-forms. As an application, we obtain an Alexandrov type theorem for spacelike codimension-two submanifolds in a static spherically symmetric spacetime: a codimension-two submanifold with constant normalized null expansion (null mean curvature) must lie in a shear-free (umbilical) null hypersurface. These results are generalized for higher order curvature invariants. In particular, the notion of *mixed higher order mean curvature* is introduced to highlight the special null geometry of the submanifold. Finally, Alexandrov type theorems are established for spacelike submanifolds with constant mixed higher order mean curvature, which are generalizations of hypersurfaces of constant Weingarten curvature in the Euclidean space.

1. Introduction

For a smooth closed oriented hypersurface $X : \Sigma \to \mathbb{R}^n$, the $k-$th Minkowski formula reads

$$(n-k)\int_\Sigma \sigma_{k-1} d\mu = k\int_\Sigma \sigma_k \langle X, \nu\rangle d\mu \tag{1.1}$$

where σ_k is the k-th elementary symmetric function of the principal curvatures and ν is the outward unit normal vector field of Σ. (1.1) was proved by Minkowski [22] for convex hypersurfaces and generalized by Hsiung [17] to all hypersurfaces stated above. There are also generalizations for various ambient spaces and higher codimensional submanifolds [14, 20, 32].

The Minkowski formula is closely related to the conformal symmetry of the ambient space. Indeed, the position vector X in (1.1) should be regarded as the restriction of the conformal Killing vector field $r\frac{\partial}{\partial r}$ on the hypersurface Σ. In this paper, we make use of the conformal Killing-Yano two-forms (see Definition 2.1) and discover several new Minkowski formulae for spacelike codimension-two submanifolds in Lorentzian manifolds. Unlike conformal Killing vector fields, conformal Killing-Yano two-forms are the so-called "hidden symmetry" which may not correspond to any continuous symmetry of the ambient space.

M.-T. Wang is supported by NSF grant DMS-1105483 and DMS-1405152. X.W. Zhang is supported by NSF grant DMS-1308136.

In the introduction, we specialize our discussion to the Schwarzschild spacetime and spacetimes of constant curvature. Several theorems proved in this article hold in more general spacetimes. The $(n+1)$-dimensional Schwarzschild spacetime with mass $m \geq 0$ is equipped with the metric

$$\bar{g} = -(1 - \frac{2m}{r^{n-2}})dt^2 + \frac{1}{1 - \frac{2m}{r^{n-2}}}dr^2 + r^2 g_{S^{n-1}}, \; r^{n-2} > 2m. \tag{1.2}$$

It is the unique spherically symmetric spacetime that satisfies the vacuum Einstein equation. Let $Q = rdr \wedge dt$ be the conformal Killing-Yano two-form (see Definition 2.1) on the Schwarzschild spacetime. The curvature tensor of $\bar{g}$ can be expressed in terms the conformal Killing-Yano two-form Q (see Appendix C). We also denote the Levi-Civita connection of $\bar{g}$ by D.

Let Σ be a closed oriented spacelike submanifold and $\{e_a\}_{a=1,\cdots,n-1}$ be an oriented orthonormal frame of the tangent bundle. Let $\vec{H}$ denote the mean curvature vector of Σ. We assume the normal bundle of Σ is also equipped with an orientation. Let L be a null normal vector field along Σ. We define the connection one-form ζ_L with respect to L by

$$\zeta_L(V) = \frac{1}{2}\langle D_V L, \underline{L}\rangle \quad \text{for any tangent vector } V \in T\Sigma, \tag{1.3}$$

where $\underline{L}$ is another null normal such that $\langle L, \underline{L}\rangle = -2$. Σ is said to be *torsion-free* with respect to L if $\zeta_L = 0$, or equivalently, $(DL)^\perp = 0$, where $(\cdot)^\perp$ denotes the normal component.

We prove the following Minkowski formula in the Schwarzschild spacetime.

Theorem A. *(Theorem 2.2) Consider the two-form $Q = rdr \wedge dt$ on the Schwarzschild spacetime. For a closed oriented spacelike codimension-two submanifold Σ in the Schwarzschild spacetime and a null normal field $\underline{L}$ along Σ, we have*

$$-(n-1)\int_\Sigma \langle \frac{\partial}{\partial t}, \underline{L}\rangle \, d\mu + \int_\Sigma Q(\vec{H}, \underline{L})\, d\mu + \sum_{a=1}^{n-1}\int_\Sigma Q(e_a, (D_{e_a}\underline{L})^\perp)\, d\mu = 0.$$

If Σ is torsion free with respect to $\underline{L}$ for a null frame $L, \underline{L}$ that satisfies $\langle L, \underline{L}\rangle = -2$, the formula takes the form:

$$-(n-1)\int_\Sigma \langle \frac{\partial}{\partial t}, \underline{L}\rangle \, d\mu - \frac{1}{2}\int_\Sigma \langle \vec{H}, \underline{L}\rangle Q(L, \underline{L})\, d\mu = 0. \tag{1.4}$$

This formula corresponds to the $k = 1$ case in (1.1) (see (4.17)) and is proved in a more general setting, see Theorem 2.2. The quantity $-\langle \vec{H}, L\rangle$ for a null normal L corresponds to the null expansion of the surface in the direction of L. codimension-two submanifolds play a special role in general relativity and their null expansions are closely related to gravitation energy as seen in Penrose's singularity theorem [26].

The Minkowski formula has been applied to various problems in global Riemannian geometry (see, for example, the survey paper [27] and references therein). One important application is a proof of Alexandrov theorem which states that every closed embedded hypersurface of constant mean curvature (CMC) in $\mathbb{R}^n$ must be a round sphere. For the proof of Alexandrov Theorem and its generalization to various ambient manifolds using the Minkowski formula, see [4, 23, 24, 28, 29].

In general relativity, the causal future or past of a geometric object is of great importance. It is interesting to characterize when a surface lies in the null hypersurface generated by a "round sphere". These are called "shearfree" null hypersurfaces (see Definition 3.10) in general relativity literature, and are analogues of umbilical hypersurfaces in Riemannian geometry.

As an application of the Minkowski formula in the Schwarzschild spacetime, we give a characterization of spacelike codimension-two submanifolds in a null hypersurface of symmetry in terms of constant null expansion.

Theorem B. *(Theorem 3.15) Let Σ be a future incoming null smooth (see Definition 3.9) closed embedded spacelike codimension-two submanifold in the $(n+1)$-dimensional Schwarzschild spacetime. Suppose there is a future incoming null normal vector field $\underline{L}$ along Σ such that $\langle \vec{H}, \underline{L} \rangle$ is a positive constant and $(D\underline{L})^{\perp} = 0$. Then Σ lies in a null hypersurface of symmetry.*

A natural substitute of CMC condition for higher codimensional submanifolds is to require the mean curvature vector field to be parallel as a section of the normal bundle. Yau [37] and Chen [7] proved that a closed immersed spacelike 2-sphere with parallel mean curvature vector in the Minkowski spacetime must be a round sphere. We are able to generalize their results to the Schwarzschild spacetime.

Corollary C. *(Corollary 3.16) Let Σ be a closed embedded spacelike codimension-two submanifold with parallel mean curvature vector in the $(n+1)$-dimensional Schwarzschild spacetime. Suppose Σ is both future and past incoming null smooth. Then Σ is a sphere of symmetry.*

Besides the Minkowski formula mentioned above, another important ingredient of the proof for Theorem B is a spacetime version of the Heintze-Karcher type inequality of Brendle [4]. Brendle's inequality was used to prove the rigidity property of CMC hypersurfaces in warped product manifolds including the important case of a time slice in the Schwarzschild spacetime. Following his approach and generalizing a monotonicity formula of his, we establish a spacetime version of this inequality (see Theorem 3.11) in Section 3.

Formula (1.4) can be viewed as a spacetime version of the Minkowski formula (1.1) with $k = 1$. In the second part of this paper, we take care of the case for general k. We introduce the notion of mixed higher order mean curvature $P_{r,s}(\chi, \underline{\chi})$ for codimension-two submanifolds in spacetime, which generalizes the notion of Weingarten curvatures for hypersurfaces in the Euclidean space. The mixed higher order mean curvature is derived from the two null second fundamental forms χ and $\underline{\chi}$ of the submanifold with respect to the null normals L and $\underline{L}$, respectively. In the hypersurface case, there is only one second fundamental form A and the Weingarten curvatures are defined as the elementary symmetric functions of A, $\sigma_k(A)$. Here, motivated by an idea of Chern [9] (see also [15]) in his study of Alexandrov's uniqueness theorem, we define the mixed higher order mean curvature $P_{r,s}(\chi, \underline{\chi})$, with $1 \leq r+s \leq n-1$, see Definition 4.1. It turns out that those quantities share some nice properties of $\sigma_k(A)$. First, we establish the following spacetime Minkowski formulae.

Theorem D. *(Theorem 4.3) Let Σ be a closed spacelike codimension-two submanifold in a spacetime of constant curvature. Suppose Σ is torsion-free with respect to the null frame L and $\underline{L}$. Then*

$$2\int_\Sigma P_{r-1,s}(\chi,\underline{\chi})\langle L, \frac{\partial}{\partial t}\rangle d\mu + \frac{r+s}{n-(r+s)}\int_\Sigma P_{r,s}(\chi,\underline{\chi})Q(L,\underline{L})d\mu = 0 \tag{1.5}$$

and

$$2\int_\Sigma P_{r,s-1}(\chi,\underline{\chi})\langle \underline{L}, \frac{\partial}{\partial t}\rangle d\mu - \frac{r+s}{n-(r+s)}\int_\Sigma P_{r,s}(\chi,\underline{\chi})Q(L,\underline{L})d\mu = 0. \tag{1.6}$$

(1.4) is a special case of (1.6) for torsion-free submanifolds ($r = 0, s = 1$) in a spacetime of constant curvature. Moreover, the classical Minkowski formulae (1.1) for hypersurfaces in Riemannian space forms (Euclidean space, hemisphere, hyperbolic space) can be recovered by (1.5) and (1.6), see (4.17). As applications of these Minkowski formulae, we obtain Alexandrov type theorems with respect to mixed higher order mean curvature for torsion-free submanifolds in a spacetime of constant curvature, as a generalization of Theorem B.

Theorem E. *(Theorem 5.1) Let Σ be a past (future) incoming null smooth, closed, embedded, spacelike codimension-two submanifold in an $(n+1)$-dimensional spacetime of constant curvature. Suppose Σ is torsion-free with respect to L and $\underline{L}$ and the second fundamental form $\chi \in \Gamma_r$ ($-\underline{\chi} \in \Gamma_s$). If $P_{r,0}(\chi,\underline{\chi}) = C$ ($P_{0,s}(\chi,\underline{\chi}) = (-1)^s C$) for some positive constant C on Σ, then Σ lies in a null hypersurface of symmetry.*

Moreover, we show that a codimension-two submanifold in the Minkowski spacetime with $P_{r,s}(\chi,\underline{\chi}) = constant$ for $r > 0, s > 0$, satisfying other mild conditions, must be a sphere of symmetry. See Theorem 5.5 for details.

In the proof of the spacetime Minkowski formulae (1.5) (1.6), we make crucial use of a certain divergence property of $P_{r,s}(\chi,\underline{\chi})$ for torsion-free submanifolds in spacetimes of constant curvature. Unfortunately, this property no longer holds for codimension-two submanifolds in the Schwarzschild spacetime because of non-trivial ambient curvature. However, under some assumption on the restriction of the conformal Killing-Yano two-form Q to the submanifold, we establish integral inequalities (see Theorem 6.2) that imply Alexandrov type theorems in the Schwarzschild spacetime (see Corollary 6.3).

For 2-surfaces in the 4-dimensional Schwarzschild spacetime, we obtain a clean integral formula involving the total null expansion of Σ.

Theorem F. *(Theorem 6.5) Consider the two-form $Q = rdr \wedge dt$ on the 4-dimensional Schwarzschild spacetime with $m \geq 0$. For a closed oriented spacelike 2-surface Σ, we have*

$$\begin{aligned}(1.7)\qquad & 2\int_\Sigma \langle \vec{H}, L\rangle\langle \underline{L}, \frac{\partial}{\partial t}\rangle d\mu \\ &= -16\pi m + \int_\Sigma \left(R + \frac{1}{4}\bar{R}_{L\underline{L}L\underline{L}}\right) Q(L,\underline{L}) + \sum_{b,c=1}^{2}\left(\frac{1}{2}\bar{R}_{bc\underline{L}L} - 2(d\zeta_L)_{bc}\right) Q_{bc} d\mu.\end{aligned}$$

where ζ_L is the connection 1-form of the normal bundle with respect to L, $\bar{R}$ is the curvature tensor of the Schwarzschild spacetime, R is the scalar curvature of Σ, and $Q_{bc} = Q(e_b, e_c)$, $(d\zeta_L)_{bc} = (d\zeta_L)(e_b, e_c)$, etc.

The total null expansion $-\int_\Sigma \langle \vec{H}, L\rangle\langle \underline{L}, \frac{\partial}{\partial t}\rangle d\mu$ appears in the Gibbons-Penrose inequality, see for example [6].

The rest of the paper is organized as follows. In section 2, we derive a simple case of spacetime Minkowski formula and give the proof of Theorem A. In section 3, we study a monotonicity formula and the spacetime Heintze-Karcher inequality. As an application, we prove the Alexandrov type theorems, Theorem B and Corollary C. In section 4, we introduce the notion of *mixed higher order mean curvatures* and establish spacetime Minkowski formulae for closed spacelike codimension-two submanifolds in constant curvature spacetimes, Theorem D. Moreover, we show the classical Minkowski formula (1.1) is recovered. As an application of Theorem D, Alexandrov type theorems for submanifolds of constant mixed mean curvature in a spacetime of constant curvature are proved in section 5. In section 6, we generalize the integral formulae to the Schwarzschild spacetime. In particular, Theorem F is proved. At last, the Appendix contains some computations used throughout the paper.

Acknowledgements. The authors would like to thank Simon Brendle, Po-Ning Chen, Shing-Tung Yau for helpful discussions. In particular, they learned a version of Theorem 3.11 in the Minkowski spacetime from Brendle. The third-named author would also like to thank Pengfei Guan and Niky Kamran for their interest and penetrating remarks. Part of the work was done while all the authors were visiting the National Center for Theoretical Sciences and Taida Institute of Mathematical Science in Taipei, Taiwan. They would like to express their gratitude to the nice working environment that was provided during their visit.

2. A Minkowski formula in spacetime

Let $F : \Sigma^{n-1} \to (V^{n+1}, \langle,\rangle)$ be a closed immersed oriented spacelike codimension-two submanifold in an oriented (n+1)-dimensional Lorentzian manifold $(V^{n+1}, \langle,\rangle)$. Denote the induced metric on Σ by σ. We assume the normal bundle is also orientable and choose a coordinate system $\{u^a | \ a = 1, 2, \cdots, n-1\}$. We identify $\frac{\partial F}{\partial u^a}$ with $\frac{\partial}{\partial u^a}$, which is abbreviated as ∂_a. Let D and ∇ denote the Levi-Civita connection of V and Σ respectively.

We recall the definition of conformal Killing-Yano two-forms.

Definition 2.1. [18, Definition 1] *Let Q be a two-form on an $(n+1)$-dimensional pseudo-Riemannian manifold $(V, \langle , \rangle)$ with Levi-Civita connection D. Q is said to be a conformal Killing-Yano two-form if*

$$\begin{aligned}&(D_X Q)(Y,Z) + (D_Y Q)(X,Z)\\ =&\frac{2}{n}\left(\langle X,Y\rangle\langle\xi,Z\rangle - \frac{1}{2}\langle X,Z\rangle\langle\xi,Y\rangle - \frac{1}{2}\langle Y,Z\rangle\langle\xi,X\rangle\right)\end{aligned} \tag{2.1}$$

for any tangent vectors X, Y and Z, where $\xi = div_V Q$.

In mathematical literature, conformal Killing-Yano two-forms were introduced by Tachibana [33], based on Yano's work on Killing forms. More generally, Kashiwada introduced the conformal Killing-Yano p-forms [19].

It is well-known that there exists a conformal Killing-Yano two-form Q on the Kerr spacetime with ξ being a multiple of the stationary Killing vector $\frac{\partial}{\partial t}$, see [18]. We also show the existence of conformal Killing-Yano forms on a class of warped product manifolds in Appendix B.

As mentioned in the introduction, we make use of the conformal Killing-Yano two-forms in Lorentzian manifolds to discover some new Minkowski formulae for the spacelike codimension-two submanifolds.

Theorem 2.2. *Let Σ be a closed immersed oriented spacelike codimension-two submanifold in an $(n+1)$-dimensional Riemannian or Lorentzian manifold V that possesses a conformal Killing-Yano two-form Q. For any normal vector field $\underline{L}$ of Σ, we have*

$$\frac{n-1}{n}\int_\Sigma \langle\xi,\underline{L}\rangle\, d\mu + \int_\Sigma Q(\vec{H},\underline{L})\, d\mu + \int_\Sigma Q(\partial_a,(D^a\underline{L})^\perp)\, d\mu = 0, \tag{2.2}$$

where $\xi = div_V Q$.

Proof. Let $h_{ab} = \langle D_a\underline{L}, \partial_b\rangle$. Consider the one-form $\mathcal{Q} = Q(\partial_a,\underline{L})du^a$ on Σ. We derive

$$\begin{aligned}div_\Sigma \mathcal{Q} &= \partial^a \mathcal{Q}_a - Q(\nabla^a\partial_a,\underline{L})\\ &= (D^a Q)(\partial_a,\underline{L}) + Q(\vec{H},\underline{L}) + Q(\partial_a, D^a\underline{L})\\ &= \frac{n-1}{n}\langle\xi,\underline{L}\rangle + Q(\vec{H},\underline{L}) + h_{ab}Q^{ab} + Q(\partial_a,(D^a\underline{L})^\perp)\\ &= \frac{n-1}{n}\langle\xi,\underline{L}\rangle + Q(\vec{H},\underline{L}) + Q(\partial_a,(D^a\underline{L})^\perp).\end{aligned} \tag{2.3}$$

The assertion follows by integrating over Σ. □

In the case of the Schwarzschild spacetime, we take $Q = rdr\wedge dt$, then $\xi = -n\frac{\partial}{\partial t}$ and Theorem A follows from the general formula (2.2).

3. An Alexandrov Theorem in spacetime

3.1. **A monotonicity formula.**

In this section, we assume that Σ is a spacelike codimension-two submanifold with spacelike mean curvature vector in a Lorentzian manifold V that possesses a conformal Killing-Yano two-form Q. We fix the sign of Q by requiring $\xi := div_V Q$ to be past-directed timelike. Let $\underline{L}$ be a future incoming null normal and L be the null normal vector with $\langle L, \underline{L}\rangle = -2$. We note that the choice of $L, \underline{L}$ is unique up to a scaling $L \to aL, \underline{L} \to \frac{1}{a}\underline{L}$. Define the null second fundamental forms with respect to $L, \underline{L}$ by

$$\chi_{ab} = \langle D_{\partial_a} L, \partial_b\rangle$$
$$\underline{\chi}_{ab} = \langle D_{\partial_a} \underline{L}, \partial_b\rangle.$$

Suppose $\langle \vec{H}, \underline{L}\rangle \neq 0$ on Σ, define the functional

$$\mathcal{F}(\Sigma, [\underline{L}]) = \frac{n-1}{n}\int_\Sigma \frac{\langle \xi, \underline{L}\rangle}{\langle \vec{H}, \underline{L}\rangle} d\mu - \frac{1}{2}\int_\Sigma Q(L, \underline{L}) d\mu. \tag{3.1}$$

Note that $\mathcal{F}$ is well-defined in that it is invariant under the change $L \to aL, \underline{L} \to \frac{1}{a}\underline{L}$.

Let $\underline{C}_0$ denote the future incoming null hypersurface of Σ and extend $\underline{L}$ arbitrarily to a future-directed null vector field along $\underline{C}_0$, still denoted by $\underline{L}$. Consider the evolution of Σ along $\underline{C}_0$ by a family of immersions $F : \Sigma \times [0, T) \to \underline{C}_0$ satisfying

$$\begin{cases} \frac{\partial F}{\partial s}(x, s) & = \varphi(x, s)\underline{L} \\ F(x, 0) & = F_0(x). \end{cases} \tag{3.2}$$

for some positive function $\varphi(x, t)$. We prove the following monotonicity formula along the flow F:

Theorem 3.1. *Let $F_0 : \Sigma \to V$ be an immersed closed oriented spacelike codimension-two submanifold in a Lorentzian manifold V with a conformal Killing-Yano two-form Q that satisfies either one of the following assumptions*

1. *V is vacuum (possibly with cosmological constant).*
2. *$\xi = div_V Q$ is a Killing field and V satisfies the null convergence condition, that is,*

$$Ric(L, L) \geq 0 \quad \textit{for any null vector}\, L. \tag{3.3}$$

Suppose that $\langle \vec{H}, \underline{L}\rangle > 0$ on Σ for some future-directed incoming null normal vector field $\underline{L}$. Then $\mathcal{F}(F(\Sigma, s), [\underline{L}])$ is monotone decreasing along the flow.

Proof. Suppose $D_{\underline{L}}\underline{L} = \underline{\omega}\underline{L}$ for a function $\underline{\omega}$. The Raychaudhuri equation [34, (9.2.32)] implies

$$\begin{aligned} \frac{\partial}{\partial s}\langle \vec{H}, \underline{L}\rangle &= \varphi\left(|\underline{\chi}|^2 + \underline{\omega}\langle \vec{H}, \underline{L}\rangle + Ric(\underline{L}, \underline{L})\right) \\ &\geq \varphi\left(|\underline{\chi}|^2 + \underline{\omega}\langle \vec{H}, \underline{L}\rangle\right), \end{aligned} \tag{3.4}$$

where $|\underline{\chi}|^2 = \underline{\chi}^{ab}\underline{\chi}_{ab}$. On the other hand,

$$\frac{\partial}{\partial s}\langle \xi, \underline{L}\rangle = \varphi\left(\langle D_{\underline{L}}\xi, \underline{L}\rangle + \underline{\omega}\langle \xi, \underline{L}\rangle\right).$$

If V satisfies assumption (1), by [18, equation (19)], we have

$$\langle D_{\underline{L}}\xi, \underline{L}\rangle = \frac{n}{n-1} Ric_{a\underline{L}} Q^a{}_{\underline{L}} = 0.$$

If V satisfies assumption (2), $\langle D_{\underline{L}}\xi, \underline{L}\rangle$ also vanishes since ξ is a Killing vector. By the Cauchy-Schwartz inequality,

$$\frac{\partial}{\partial s}\int_\Sigma \frac{\langle \xi, \underline{L}\rangle}{\langle \vec{H}, \underline{L}\rangle} d\mu \le -\int_\Sigma \varphi\left[\frac{\langle \xi, \underline{L}\rangle}{(\mathrm{tr}\underline{\chi})^2}|\underline{\chi}|^2 + \langle \xi, \underline{L}\rangle\right] d\mu \le -\frac{n}{n-1}\int_\Sigma \varphi\langle \xi, \underline{L}\rangle d\mu. \tag{3.5}$$

The evolution of $\int_\Sigma Q(L, \underline{L}) d\mu$ is given by

$$\begin{aligned} &\frac{\partial}{\partial s}\int_\Sigma Q(L, \underline{L}) d\mu \\ =&\int_\Sigma \left[\varphi\left(D_{\underline{L}}Q\right)(L, \underline{L}) + Q(D_{\partial_s}L, \underline{L}) + Q(L, D_{\partial_s}\underline{L}) - \varphi Q(L, \underline{L})\langle \vec{H}, \underline{L}\rangle\right] d\mu. \end{aligned}$$

From the conformal Killing-Yano equation (2.1), we derive

$$\left(D_{\underline{L}}Q\right)(L, \underline{L}) = \frac{1}{n}\langle \xi, \underline{L}\rangle\langle L, \underline{L}\rangle = -\frac{2}{n}\langle \xi, \underline{L}\rangle.$$

On the other hand, by standard computation

$$\begin{aligned} \langle D_{\partial_s}L, \underline{L}\rangle &= -\langle L, \varphi\underline{\omega}\underline{L}\rangle, \\ \langle D_{\partial_s}L, \partial_a\rangle &= -\langle L, D_a(\varphi\underline{L})\rangle = 2\nabla_a\varphi - \varphi\langle L, D_a\underline{L}\rangle, \end{aligned}$$

we have

$$D_{\partial_s}L = \left(2\nabla^a\varphi - \varphi\langle L, D^a\underline{L}\rangle\right)\partial_a - \varphi\underline{\omega}L.$$

The computations together yield

$$\begin{aligned} &Q(D_{\partial_s}L, \underline{L}) + Q(L, D_{\partial_s}\underline{L}) - \varphi Q(L, \underline{L})\langle \vec{H}, \underline{L}\rangle \\ =&2\nabla^a\varphi Q(\partial_a, \underline{L}) + 2\varphi Q(\partial_a, (D_a\underline{L})^\perp) + 2\varphi Q(\vec{H}, \underline{L}) \\ =&2\nabla^a\left(\varphi Q(\partial_a, \underline{L})\right) - \frac{2(n-1)}{n}\varphi\langle \xi, \underline{L}\rangle. \end{aligned}$$

In the last equality, we make use of (2.3). Consequently, we obtain

$$\frac{\partial}{\partial s}\int_\Sigma Q(L, \underline{L}) d\mu = -2\int_\Sigma \varphi\langle \xi, \underline{L}\rangle d\mu. \tag{3.6}$$

Then, the assertion follows from (3.5) and (3.6). □

3.2. **A spacetime CMC condition.**

Hypersurfaces of constant mean curvature (CMC) provide models for soap bubbles, and have been studied extensively for a long time. A common generalization of this condition for higher codimensional submanifolds is the parallel mean curvature condition. In general relativity, the most relevant physical phenomenon is the divergence of light rays emanating from a codimension-two submanifold. This is called the null expansion in physics literature. We thus impose constancy conditions on the null expansion of codimension-two submanifolds. More precisely, we are interested in the codimension-two submanifold that admits a future null normal vector field L such that

(1) $\langle \vec{H}, L\rangle$ is a constant, and
(2) $(DL)^\perp = 0$.

We review the definition of connection one-form of mean curvature gauge from [36] and relate it to the condition introduced above.

Definition 3.2. *Let $\vec{H}$ denote the mean curvature vector of Σ. We define the normal vector field by reflecting $\vec{H}$ along the incoming light cone to be*

$$\vec{J} = \langle \vec{H}, e_{n+1}\rangle e_n - \langle \vec{H}, e_n\rangle e_{n+1}$$

as in [36]. The connection one-form ζ_{e_n} of the normal bundle with respect to e_n and e_{n+1} is defined by

$$\zeta_{e_n}(V) = \langle D_V e_n, e_{n+1}\rangle \quad \textit{for any tangent vector } V \in T\Sigma. \tag{3.7}$$

Suppose the mean curvature vector is spacelike, we take $e_n^{\vec{H}} = -\frac{\vec{H}}{|\vec{H}|}$ and $e_{n+1}^{\vec{H}} = \frac{\vec{J}}{|\vec{H}|}$ and write $\alpha_{\vec{H}}$ for the connection one-form with respect to this mean curvature gauge:

$$\alpha_{\vec{H}}(V) = \langle D_V e_n^{\vec{H}}, e_{n+1}^{\vec{H}}\rangle.$$

This is consistent with the notation in [8]. We note that $\{e_n^{\vec{H}}, e_{n+1}^{\vec{H}}\}$ determines the same orientation as $\{e_n, e_{n+1}\}$.

Recall the connection one-form with respect to L is given by

$$\zeta_L(V) = \frac{1}{2}\langle D_V L, \underline{L}\rangle \quad \text{for any tangent vector } V \in T\Sigma. \tag{3.8}$$

The two definitions in (3.7) and (3.8) give the same connection one-form if we choose $L = e_{n+1} + e_n$ and $\underline{L} = e_{n+1} - e_n$.

Proposition 3.3. *Suppose the mean curvature vector field $\vec{H}$ of Σ is spacelike.*

(1) *If $\langle \vec{H}, L\rangle = c < 0$ and $(DL)^\perp = 0$ for some future outward null normal L and some negative constant c, then $\alpha_{\vec{H}} = -d\log|\vec{H}|$.*
(2) *If $\langle \vec{H}, \underline{L}\rangle = c > 0$ and $(D\underline{L})^\perp = 0$ for some future inward null normal $\underline{L}$ and some positive constant c, then $\alpha_{\vec{H}} = d\log|\vec{H}|$.*

Proof. Recall that the dual mean curvature vector $\vec{J}$ is future timelike. For (1), the condition $\langle \vec{H}, L\rangle = c < 0$ is equivalent to

$$L = \frac{-c}{|\vec{H}|}\left(-\frac{\vec{H}}{|\vec{H}|} + \frac{\vec{J}}{|\vec{H}|}\right).$$

Choose $\underline{L} = \frac{|\vec{H}|}{-c}\left(\frac{\vec{H}}{|\vec{H}|} + \frac{\vec{J}}{|\vec{H}|}\right)$ such that $\langle L, \underline{L}\rangle = -2$. Since $(DL)^\perp = 0$, we have

$$\begin{aligned} 0 = \frac{1}{2}\langle D_a L, \underline{L}\rangle &= \frac{1}{2}\partial_a\left(\frac{-c}{|\vec{H}|}\right)\frac{|\vec{H}|}{-c}(-2) + \left\langle D_a\left(-\frac{\vec{H}}{|\vec{H}|}\right), \frac{\vec{J}}{|\vec{H}|}\right\rangle \\ &= \partial_a \log|\vec{H}| + \left\langle D_a\left(-\frac{\vec{H}}{|\vec{H}|}\right), \frac{\vec{J}}{|\vec{H}|}\right\rangle \end{aligned}$$

Hence $\alpha_{\vec{H}} = -d\log|\vec{H}|$. (2) can be proved in a similar way. $\square$

Remark 3.4. *We remark that when Σ lies in a totally geodesic time slice of a static spacetime, the condition (1) and (2) reduces to the CMC condition.*

3.3. **A Heintze-Karcher type inequality.**

In this and the next subsections, we focus our discussion on a class of spherically symmetric static spacetimes which includes the Schwarzschild spacetime.

Assumption 3.5. *We assume V is a spacetime that satisfies the null convergence condition* (3.3) *and the metric $\bar{g}$ on $V = \mathbb{R}\times M$ is of the form*

$$\bar{g} = -f^2(r)dt^2 + \frac{1}{f^2(r)}dr^2 + r^2 g_N. \tag{3.9}$$

where $M = [0, \bar{r})\times N$ equipped with metric

$$g = \frac{1}{f^2(r)}dr^2 + r^2 g_N \tag{3.10}$$

and (N, g_N) is a compact n-dimensional Riemannian manifold. We consider two cases:

(i) *$f : [0,\infty) \to \mathbb{R}$ with $f(0) = 1$, $f'(0) = 0$, and $f(r) > 0$ for $r \geq 0$.*
(ii) *$f : [r_0,\infty) \to \mathbb{R}$ with $f(r_0) = 0$ and $f(r) > 0$ for $r > r_0$.*

In case (i), $(V, \bar{g})$ is complete. In case (ii), V contains an event horizon $\mathcal{H} = \{r = r_0\}$. We note that the warped product manifolds considered in [4] are embedded as totally geodesic slices in these spacetimes.

Remark 3.6. *For a spacetime V that satisfies Assumption 3.5, a simple calculation shows that $Q = rdr\wedge dt$ is a conformal Killing-Yano two-form and $div_V Q = \xi = -n\frac{\partial}{\partial t}$ is a Killing field.*

Lemma 3.7. *Let (M, g) be a time slice in V. The null convergence condition of $(V, \bar{g})$ is equivalent to*

$$(\Delta_g f)g - Hess_g f + f Ric(g) \geq 0 \tag{3.11}$$

on M.

Proof. O'Neill's formula in our case reduces to (see [10, Proposition 2.7])

$$\begin{aligned}
\mathrm{Ric}(\bar{g})(v, w) &= \mathrm{Ric}(g)(v, w) - \frac{\mathrm{Hess}_g f(v, w)}{f},\\
\mathrm{Ric}(\bar{g})(v, \frac{\partial}{\partial t}) &= 0,\\
\mathrm{Ric}(\bar{g})(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) &= -\frac{\Delta_g f}{f}\bar{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}),
\end{aligned}$$

for any tangent vectors v and w on M. A unit tangent vector v on M corresponds to a null vector $L = \frac{1}{f}\frac{\partial}{\partial t} + v$ in the spacetime. Null convergence condition implies that

$$0 \leq \mathrm{Ric}(\bar{g})(L, L) = \frac{\Delta_g f}{f} + \mathrm{Ric}(g)(v, v) - \frac{\mathrm{Hess}_g f(v, v)}{f}$$

as claimed.

□

As in section 3.2, we denote the conformal Killing vector field X on (M, g) by $X = rf\frac{\partial}{\partial r}$. In [4], Brendle proved a Heintze-Karcher-type inequality for mean convex hypersurfaces in (M, g). In our context, it is as the following:

Theorem 3.8. [4] *Let S be a smooth, closed, embedded, orientable hypersurface in a time slice of a spacetime V that satisfies Assumption 3.5. Suppose that S has positive mean curvature $H > 0$ in the slice. Then*

$$(n-1)\int_S \frac{f}{H} d\mu \geq \int_S \langle X, \nu \rangle d\mu, \tag{3.12}$$

where ν is the outward unit normal of S in the slice and $X = rf\frac{\partial}{\partial r}$ is the conformal Killing vector field on the slice. Moreover, equality holds if and only if S is umbilical.

Proof. We first remark that since S is embedded and orientable, S is either null-homologous or homologous to $\{r_0\} \times N$. Hence $\partial\Omega = S$ or $\partial\Omega = S - \{r_0\} \times N$ for some domain $\Omega \subset M$. Inequality (3.12) is equivalent to the one in Theorem 3.5 and the one in Theorem 3.11 of Brendle's paper in the respective cases. For the reader's convenience, we trace Brendle's argument leading to (3.12).

The assumptions on (M, g) are listed in page 248 [4]:

$$\mathrm{Ric}_N \geq (n-2)\rho g_N \tag{3.13}$$

and (H1)-(H3) (note that condition (H4) is not used in the proof of (3.12)). While Brendle writes the metric in geodesic coordinates

$$d\bar{r} \otimes d\bar{r} + h^2(\bar{r}) g_N,$$

it is equivalent to ours notation (3.10) by a change of variables $r = h$ and $f = \frac{dh}{d\bar{r}}$. Moreover, as explained in the beginning of section 2 (page 252 of [4]), (H1) and (H2) are equivalent to our assumptions (i) and (ii) on f. In Proposition 2.1, (3.13) and (H3) together imply that (3.11) holds on (M, g).

The condition (3.11) turns out to be the only curvature assumption that is necessary in proving (3.12). More precisely, (3.11) is used to prove the key monotonicity formula, Proposition 3.2 (page 256). Inequality (3.12) is a direct consequence of Proposition 3.2 up to several technical lemmata, Lemma 3.6 to Corollary 3.10, in which only assumptions (H1) and (H2) are used.

Finally, the inequalities appear in Theorem 3.5 and Theorem 3.11 in [4] are equivalent to (3.12) by the divergence theorem. □

Before stating the spacetime Heintze-Karcher inequality, we define the notions of future incoming null smoothness and shearfree null hypersurface.

Definition 3.9. *A closed, spacelike codimension-two submanifold Σ in a static spacetime V is future(past) incoming null smooth if the future(past) incoming null hypersurface of Σ intersects a totally geodesic time-slice $\mathcal{M}_T = \{t = T\} \subset V$ at a smooth, embedded, orientable hypersurface S.*

Definition 3.10. *An incoming null hypersurface $\underline{\mathcal{C}}$ is shearfree if there exists a spacelike hypersurface Σ in $\underline{\mathcal{C}}$ such that the null second fundamental form $\underline{\chi}_{ab} = \langle D_a \underline{L}, \partial_b \rangle$ of Σ with respect to some null normal $\underline{L}$ satisfies $\underline{\chi}_{ab} = \psi \sigma_{ab}$ for some function ψ. A shearfree outgoing null hypersurface is defined in the same way.*

Note that being shearfree is a property of the null hypersurface, see [30, page 47-48]. The spacetime Heintze-Karcher inequality we prove is the following:

Theorem 3.11. *Let V be a spacetime as in Assumption 3.5. Let $\Sigma \subset V$ be a future incoming null smooth closed spacelike codimension-two submanifold with $\langle \vec{H}, \underline{L} \rangle > 0$ where $\underline{L}$ is a future incoming null normal. Then*

$$-(n-1) \int_\Sigma \frac{\langle \frac{\partial}{\partial t}, \underline{L} \rangle}{\langle \vec{H}, \underline{L} \rangle} d\mu - \frac{1}{2} \int_\Sigma Q(L, \underline{L}) d\mu \geq 0, \tag{3.14}$$

for a future outgoing null normal L with $\langle L, \underline{L} \rangle = -2$. Moreover, the equality holds if and only if Σ lies in an incoming shearfree null hypersurface.

Proof. We arrange φ in (3.2) such that $\underline{\omega} \geq 0$ and that $F(\Sigma, 1) = S$, the smooth hypersurface defined in Definition 3.9. We first claim that $S \subset \mathcal{M}_T$ has positive mean curvature,

$H > 0$. Recall that Raychaudhuri equation implies

$$\begin{aligned}\frac{\partial}{\partial s}\langle \vec{H}, \underline{L}\rangle &= \varphi\left(|\underline{\chi}|^2 + \underline{\omega}\langle \vec{H}, \underline{L}\rangle + Ric(\underline{L}, \underline{L})\right)\\ &\geq \varphi\left(|\underline{\chi}|^2 + \underline{\omega}\langle \vec{H}, \underline{L}\rangle\right),\end{aligned} \tag{3.15}$$

and hence $\langle \vec{H}, \underline{L}\rangle > 0$ on S. We choose $\underline{L} = \frac{1}{f}\frac{\partial}{\partial t} - e_n$ on S, where e_n is the outward unit normal of S with respect to Ω, and compute

$$\left\langle \vec{H}, \frac{1}{f}\frac{\partial}{\partial t} - e_n \right\rangle = H.$$

The claim follows since the positivity of $\langle \vec{H}, \underline{L}\rangle$ is independent of the scaling of $\underline{L}$. Next we choose $L = \frac{1}{f}\frac{\partial}{\partial t} + e_n$ on S and compute

$$\begin{aligned} -\left\langle \frac{\partial}{\partial t}, \frac{1}{f}\frac{\partial}{\partial t} - e_n \right\rangle &= f,\\ Q\left(\frac{1}{f}\frac{\partial}{\partial t} + e_n, \frac{1}{f}\frac{\partial}{\partial t} - e_n\right) &= 2\langle X, e_n\rangle.\end{aligned}$$

Remark 3.6 implies the monotonicity formula (Theorem 3.1) holds with $\xi = -n\frac{\partial}{\partial t}$ and thus

$$\begin{aligned} &-(n-1)\int_\Sigma \frac{\langle \frac{\partial}{\partial t}, \underline{L}\rangle}{\langle \vec{H}, \underline{L}\rangle} d\mu - \frac{1}{2}\int_\Sigma Q(L, \underline{L}) d\mu\\ \geq &-(n-1)\int_{F(\Sigma,1)} \frac{\langle \frac{\partial}{\partial t}, \underline{L}\rangle}{\langle \vec{H}, \underline{L}\rangle} d\mu - \frac{1}{2}\int_{F(\Sigma,1)} Q(L, \underline{L}) d\mu.\end{aligned} \tag{3.16}$$

As $F(\Sigma, 1) = S$, the above calculation on S shows the last expression is equal to

$$(n-1)\int_S \frac{f}{H} d\mu - \int_S \langle X, e_n\rangle d\mu \geq 0$$

by (3.12). Moreover, S is umbilical if the equality holds. Hence the future incoming null hypersurface generated from Σ is shearfree. $\square$

By reversing the time orientation, we also obtain the Heintze-Karcher inequality for past incoming null smooth submanifolds.

Theorem 3.12. *Let V be a spacetime as in Assumption 3.5. Let $\Sigma \subset V$ be a past incoming null smooth closed spacelike codimension-two submanifold such that $\langle \vec{H}, L\rangle < 0$ with respect to some future outgoing null normal L. Then*

$$(n-1)\int_\Sigma \frac{\langle \frac{\partial}{\partial t}, L\rangle}{\langle \vec{H}, L\rangle} d\mu - \frac{1}{2}\int_\Sigma Q(L, \underline{L}) d\mu \geq 0, \tag{3.17}$$

for a future incoming null normal $\underline{L}$ with $\langle L, \underline{L}\rangle = -2$. Moreover, the equality holds if and only if Σ lies in an outgoing shearfree null hypersurface.

Proof. When we reverse the time orientation, we replace $\frac{\partial}{\partial t}$ by $-\frac{\partial}{\partial t}$, Q by $-Q$, $\underline{L}$ by $-L$, and L by $-\underline{L}$. Plug these into (3.14) and we obtain (3.17). $\square$

3.4. **A Spacetime Alexandrov Theorem.**

Together with the spacetime Heintze-Karcher inequality proved in the previous subsection, the Minkowski formula (2.2) implies the following spacetime Alexandrov type theorem.

Theorem 3.13. *Let V be a spherically symmetric spacetime as in Assumption 3.5 and Σ be a future incoming null smooth, closed, embedded, spacelike codimension-two submanifold in V. Suppose there is a future incoming null normal vector field $\underline{L}$ along Σ such that $\langle \vec{H}, \underline{L}\rangle$ is a positive constant and $(D\underline{L})^\perp = 0$. Then Σ lies in a shearfree null hypersurface.*

Proof. Write $\vec{H} = -\frac{1}{2}\langle \vec{H}, \underline{L}\rangle L - \frac{1}{2}\langle \vec{H}, L\rangle \underline{L}$. From the assumption, $(D_a\underline{L})^\perp = 0$, the spacetime Minkowski formula (2.2) becomes

$$-(n-1)\int_\Sigma \langle \frac{\partial}{\partial t}, \underline{L}\rangle d\mu - \frac{1}{2}\int_\Sigma \langle \vec{H}, \underline{L}\rangle Q(L, \underline{L}) = 0$$

Again from the assumption, $\langle \vec{H}, \underline{L}\rangle$ is a positive constant function and we can divide both sides by $\langle \vec{H}, \underline{L}\rangle$ to get

$$-(n-1)\int_\Sigma \frac{\langle \frac{\partial}{\partial t}, \underline{L}\rangle}{\langle \vec{H}, \underline{L}\rangle} d\mu - \frac{1}{2}\int_\Sigma Q(L, \underline{L})\rangle d\mu = 0.$$

Hence the equality is achieved in the spacetime Heintze-Karcher inequality (3.14) and we conclude that Σ lies in a shearfree null hypersurface. $\square$

Definition 3.14. *A null hypersurface in an $(n+1)$-dimensional spherically symmetric spacetime is called a null hypersurface of symmetry if it is invariant under the $SO(n)$ isometry that defines the spherical symmetry. In other word, it is generated by a sphere of symmetry.*

We remark that in the Lorentzian space forms, the additional boost isometry sends a null hypersurface of symmetry into another one defined by a conjugate $SO(n)$-action.

An important example of the spacetime satisfying Assumption 3.5 is the exterior Schwarzschild spacetime for which the metric has the form (1.2). Since the spheres of symmetry are the only closed umbilical hypersurfaces in the totally geodesic time slice of the Schwarzschild spacetimes [4, Corollary 1.2], as a direct corollary of the above spacetime Alexandrov theorem, we obtain

Theorem 3.15 (Theorem B)**.** *Let Σ be a future incoming null smooth (see Definition 3.9) closed embedded spacelike codimension-two submanifold in the $(n+1)$-dimensional Schwarzschild spacetime. Suppose there is a future incoming null normal vector field $\underline{L}$ along Σ such that $\langle \vec{H}, \underline{L}\rangle$ is a non-zero constant and $(D\underline{L})^\perp = 0$. Then Σ lies in a null hypersurface of symmetry.*

As observed in Proposition 3.3, the condition in the above theorem can be characterized in terms of the norm of the mean curvature vector and the connection one-form in the mean curvature gauge.

Theorem B'. *Let Σ be a future incoming null smooth (see Definition 3.9) closed embedded spacelike codimension-two submanifold in the Schwarzschild spacetime with spacelike mean curvature $\vec{H}$. Suppose $\alpha_{\vec{H}} = d\log|\vec{H}|$ on Σ. Then Σ lies in a null hypersurface of symmetry.*

Finally, we generalize a result of Yau [37] and Chen [7] to the Schwarzschild spacetime.

Corollary 3.16 (Corollary C)**.** *Let Σ be a closed embedded spacelike codimension-two submanifold with parallel mean curvature vector in the Schwarzschild spacetime. Suppose Σ is both future and past incoming null smooth. Then Σ is a sphere of symmetry.*

Proof. The condition of parallel mean curvature vector implies $|\vec{H}|$ is constant and $\alpha_{\vec{H}}$ vanishes. The previous theorem implies Σ is the intersection of one incoming and one outgoing null hypersurface of symmetry. Therefore, Σ is a sphere of symmetry. $\square$

4. General Minkowski Formulae for mixed Higher order Mean Curvature

In this section, we introduce the notion of mixed higher order mean curvature of a codimension-two submanifold Σ in a spacetime V of dimension $(n+1)$. Let $\bar{R}$ denote the curvature tensor of V. Let L and $\underline{L}$ be two null normals of Σ such that $\langle L, \underline{L}\rangle = -2$. Recall the null second fundamental forms with respect to $L, \underline{L}$:

$$\begin{aligned}\chi_{ab} &= \langle D_{\partial_a} L, \partial_b\rangle\\ \underline{\chi}_{ab} &= \langle D_{\partial_a} \underline{L}, \partial_b\rangle\end{aligned}$$

and write $\zeta = \zeta_L$ for the connection 1-form with respect to L:

$$\zeta_a = \frac{1}{2}\langle D_{\partial_a} L, \underline{L}\rangle.$$

Definition 4.1. *For any two non-negative integers r and s with $0 \le r+s \le n-1$, the mixed higher order mean curvature $P_{r,s}(\chi,\underline{\chi})$ with respect to L and $\underline{L}$ is defined through the following expansion:*

$$\det(\sigma + y\chi + \underline{y}\underline{\chi}) = \sum_{0\le r+s\le n-1} \frac{(r+s)!}{r!s!} y^r \underline{y}^s P_{r,s}(\chi,\underline{\chi}), \tag{4.1}$$

where y and $\underline{y}$ are two real variables and σ is the induced metric on Σ. We also define symmetric 2- tensors $T^{ab}_{r,s}(\chi,\underline{\chi})$ and $\underline{T}^{ab}_{r,s}(\chi,\underline{\chi})$ on Σ by

$$T^{ab}_{r,s}(\chi,\underline{\chi}) = \frac{\delta P_{r,s}(\chi,\underline{\chi})}{\delta \chi_{ab}} \text{ and } \underline{T}^{ab}_{r,s}(\chi,\underline{\chi}) = \frac{\delta P_{r,s}(\chi,\underline{\chi})}{\delta \underline{\chi}_{ab}}.$$

In the following, we write $P_{r,s}$ for $P_{r,s}(\chi, \underline{\chi})$, $T^{ab}_{r,s}$ for $T^{ab}_{r,s}(\chi, \underline{\chi})$, etc. when there is no confusion. Note that $P_{1,0} = \mathrm{tr}\chi$ and $P_{0,1} = \mathrm{tr}\underline{\chi}$, and a simple computation yields

$$T^{ab}_{1,0} = \underline{T}^{ab}_{0,1} = \sigma^{ab}.$$

Moreover, when $n = 3$, it is easy to check that

$$\begin{aligned} 2T^{ab}_{1,1} &= \sigma^{ab} tr\underline{\chi} - \underline{\chi}^{ab}; \\ 2\underline{T}^{ab}_{1,1} &= \sigma^{ab} tr\chi - \chi^{ab}. \end{aligned} \tag{4.2}$$

It is worth to remark that the quantities $P_{r,r}$, $T^{ab}_{r,r}L$, $\underline{T}^{ab}_{r,r}\underline{L}$, $T^{ab}_{r+1,r}$ and $\underline{T}^{ab}_{r,r+1}$ are all independent of the scaling of L and $\underline{L}$ as $L \to aL$, $\underline{L} \to \frac{1}{a}\underline{L}$.

In the rest of this section, we focus on spacelike codimension-two submanifolds in a spacetime of constant curvature such as the Minkowski spacetime $\mathbb{R}^{n,1}$, the de-Sitter spacetime, or the anti de-Sitter spacetime. Before proving the Minkowski formulae for those mixed higher order mean curvatures, we observe that $P_{r,s}(\chi, \underline{\chi})$ shares the divergence free property as σ_k.

Lemma 4.2. *Let Σ be a spacelike codimension-two submanifold in a spacetime of constant curvature. Suppose Σ is torsion-free with respect to L and $\underline{L}$. Then $T^{ab}_{r,s}$ and $\underline{T}^{ab}_{r,s}$ are divergence free for any (r, s), that is,*

$$\nabla_b T^{ab}_{r,s} = \nabla_b \underline{T}^{ab}_{r,s} = 0. \tag{4.3}$$

Proof. We denote $\tilde{\sigma}_{ab} = (\sigma + y\chi + \underline{y\chi})_{ab}$ and its inverse by $(\tilde{\sigma}^{-1})^{ab}$. Formally differentiating (4.1) with respect to χ_{ab} and $\underline{\chi}_{ab}$, we obtain

$$y(\tilde{\sigma}^{-1})^{ab} \det(\tilde{\sigma}) \quad = \sum \tfrac{(r+s)!}{r!s!} y^r \underline{y}^s T^{ab}_{r,s}, \tag{4.4}$$

and

$$\underline{y}(\tilde{\sigma}^{-1})^{ab} \det(\tilde{\sigma}) \quad = \sum \tfrac{(r+s)!}{r!s!} y^r \underline{y}^s \underline{T}^{ab}_{r,s}. \tag{4.5}$$

Next, taking covariant derivative on both sides of equation (4.4) and the left-hand side becomes

$$-y(\tilde{\sigma}^{-1})^{ac}\nabla_b(y\chi + \underline{y\chi})_{cd}(\tilde{\sigma}^{-1})^{db} \det(\tilde{\sigma}) + y(\tilde{\sigma}^{-1})^{ab}\nabla_b(y\chi + \underline{y\chi})_{cd}(\tilde{\sigma}^{-1})^{cd} \det(\tilde{\sigma}).$$

Switching indices b and c in the second summand, we arrive at

$$\begin{aligned} & y\left(\tilde{\sigma}^{-1}\right)^{ac}\left[y\left(\nabla_c\chi_{bd} - \nabla_b\chi_{cd}\right) + \underline{y}(\nabla_c\underline{\chi}_{bd} - \nabla_b\underline{\chi}_{cd})\right]\left(\tilde{\sigma}^{-1}\right)^{db}\det(\tilde{\sigma}) \\ =& \sum_{r,s} \frac{(r+s)!}{r!s!} y^r \underline{y}^s \nabla_b T^{ab}_{r,s}. \end{aligned} \tag{4.6}$$

Similar computation applying to (4.5) yields

$$\begin{aligned}(4.7)\qquad &\underline{y}\left(\tilde{\sigma}^{-1}\right)^{ac}\left[y(\nabla_c\chi_{bd}-\nabla_b\chi_{cd})+\underline{y}(\nabla_c\underline{\chi}_{bd}-\nabla_b\underline{\chi}_{cd})\right]\left(\tilde{\sigma}^{-1}\right)^{db}\det(\tilde{\sigma})\\ &=\sum_{r,s}\frac{(r+s)!}{r!s!}y^r\underline{y}^s\nabla_b\underline{T}^{ab}_{r,s}.\end{aligned}$$

For submanifolds in spacetime, the Codazzi equations give

$$\begin{aligned}\nabla_c\chi_{bd}-\nabla_b\chi_{cd}&=\langle\bar{R}(\partial_c,\partial_b)L,\partial_d\rangle+\zeta_b\chi_{cd}-\zeta_c\chi_{bd}\\ \nabla_c\underline{\chi}_{bd}-\nabla_b\underline{\chi}_{cd}&=\langle\bar{R}(\partial_c,\partial_b)\underline{L},\partial_d\rangle-\zeta_b\underline{\chi}_{cd}+\zeta_c\underline{\chi}_{bd}.\end{aligned}\tag{4.8}$$

Since Σ is assumed to be torsion-free and the ambient space is of constant curvature, the left hand side of both (4.6) and (4.7) are zero. Then, the assertion follows by comparing the coefficients of the term $y^r\underline{y}^s$. $\square$

We now derive the Minkowski formulae for the mixed higher order mean curvatures of torsion-free submanifolds in constant curvature spacetimes.

Theorem 4.3 (Theorem D)**.** *Let Σ be a closed spacelike codimension-two submanifold in a spacetime of constant curvature. Suppose Σ is torsion-free with respect to the null frame L and $\underline{L}$. Then*

$$2\int_\Sigma P_{r-1,s}(\chi,\underline{\chi})\langle L,\frac{\partial}{\partial t}\rangle d\mu+\frac{r+s}{n-(r+s)}\int_\Sigma P_{r,s}(\chi,\underline{\chi})Q(L,\underline{L})d\mu=0\tag{4.9}$$

and

$$2\int_\Sigma P_{r,s-1}(\chi,\underline{\chi})\langle\underline{L},\frac{\partial}{\partial t}\rangle d\mu-\frac{r+s}{n-(r+s)}\int_\Sigma P_{r,s}(\chi,\underline{\chi})Q(L,\underline{L})d\mu=0.\tag{4.10}$$

Proof. By the divergence theorem, we have

$$\int_\Sigma\nabla_a[T^{ab}Q(L,\partial_b)]d\mu=0\tag{4.11}$$

where T^{ab} is one of $T^{ab}_{r,s}$. Since Σ is torsion-free, direct computation shows

$$\nabla_a[Q(L,\partial_b)]=(D_aQ)(L,\partial_b)+\chi_a^cQ_{cb}+\frac{1}{2}\chi_{ab}Q(L,\underline{L}).$$

By (2.1), we derive

$$(D_aQ)(L,\partial_b)+(D_bQ)(L,\partial_a)=2\langle L,\frac{\partial}{\partial t}\rangle\sigma_{ab}.$$

Therefore, we obtain

$$\nabla_a[T^{ab}Q(L,\partial_b)]=T^{ab}\sigma_{ab}\langle L,\frac{\partial}{\partial t}\rangle+\frac{1}{2}(T^{ba}\chi_a^c-T^{ca}\chi_a^b)Q_{cb}+\frac{1}{2}(T^{ab}\chi_{ab})Q(L,\underline{L}),$$

where the second term on the right hand side comes from anti-symmetrization of the indices. Recall the Ricci equation

$$\frac{1}{2}\chi_a{}^c\underline{\chi}_{cb} - \frac{1}{2}\chi_b{}^c\underline{\chi}_{ca} + (d\zeta)_{ab} = \frac{1}{2}\langle \bar{R}(\partial_a, \partial_b)L, \underline{L}\rangle. \tag{4.12}$$

Since Σ is torsion-free and the ambient space is of constant curvature, the equation implies χ and $\underline{\chi}$ commute. Note that T^{ba} is a polynomial of χ and $\underline{\chi}$, and thus also commutes with χ and $\underline{\chi}$. It follows that

$$T^{ba}\chi_a^c - T^{ca}\chi_a^b = 0.$$

Putting these together, (4.11) implies

$$\int_\Sigma \left(\sigma_{ab}T^{ab}_{r,s}\right)\langle L, \frac{\partial}{\partial t}\rangle d\mu + \frac{1}{2}\int_\Sigma \left(\chi_{ab}T^{ab}_{r,s}\right) Q(L, \underline{L}) d\mu = 0.$$

(4.9) follows from (A.5) and (A.3) in the appendix.

The second formula is derived similarly by considering

$$\int_\Sigma \nabla_a[\underline{T}^{ab}Q(\underline{L}, \partial_b)]d\mu = 0, \tag{4.13}$$

and using

$$\nabla_a\left[Q(\underline{L}, \partial_b)\right] = (D_aQ)(\underline{L}, \partial_b) + \underline{\chi}_a^c Q_{cb} - \frac{1}{2}\underline{\chi}_{ab}Q(L, \underline{L}).$$

□

From the above proof, we see that the spacetime Minkowski formulae (4.9) and (4.10) follow from

$$\int_\Sigma \nabla_a[T^{ab}Q(L, \partial_b)]d\mu = 0, \text{ and } \int_\Sigma \nabla_a[\underline{T}^{ab}Q(\underline{L}, \partial_b)]d\mu = 0.$$

In fact, we have two more possible identities:

$$\int_\Sigma \nabla_a[T^{ab}Q(\underline{L}, \partial_b)]d\mu = 0, \text{ and } \int_\Sigma \nabla_a[\underline{T}^{ab}Q(L, \partial_b)]d\mu = 0.$$

Following the same line, one can prove another two spacetime Minkowski formulae:

$$2\int_\Sigma P_{r-1,s}(\chi, \underline{\chi})\langle \underline{L}, \frac{\partial}{\partial t}\rangle d\mu - \frac{r+s}{n-(r+s)}\int_\Sigma P_{r-1,s+1}(\chi, \underline{\chi})Q(L, \underline{L})d\mu = 0, \tag{4.14}$$

and

$$2\int_\Sigma P_{r-1,s}(\chi, \underline{\chi})\langle L, \frac{\partial}{\partial t}\rangle d\mu + \frac{r+s}{n-(r+s)}\int_\Sigma P_{r+1,s-1}(\chi, \underline{\chi})Q(L, \underline{L})d\mu = 0. \tag{4.15}$$

We finish this section by showing that (4.9) and (4.10) recover the classical Minkowski formulae for hypersurfaces in Riemannain space forms - Euclidean space, hemisphere, and hyperbolic space.

It is well known that these space forms $S^n_{-\kappa}$ can be embedded as totally geodesic time-slices in the Minkowski spacetime, the de Sitter spacetime, and the anti-de Sitter spacetime respectively. We write the spacetime metric in static coordinates:

$$\bar{g} = -(1+\kappa r^2)dt^2 + \frac{1}{1+\kappa r^2}dr^2 + r^2 g_{S^{n-1}}.$$

Given a hypersurface $\Sigma \subset S^n_{-\kappa}$, we view it as a spacelike codimension-two submanifold lying in a totally geodesic time-slice. Let ν denote the outward unit normal of Σ in the totally geodesic slice. We take $L = \frac{1}{\sqrt{1+\kappa r^2}}\frac{\partial}{\partial t} + \nu$ and $\underline{L} = \frac{1}{\sqrt{1+\kappa r^2}}\frac{\partial}{\partial t} - \nu$ to get $\chi = -\underline{\chi} = h$ and $\zeta = 0$. Here h is the second fundamental form of Σ in the totally geodesic time-slice. Therefore, according to the definition (4.1),

$$\det(\sigma + (y - \underline{y})h) = \sum_{0 \le r+s \le n-1} \frac{(r+s)!}{r!s!} y^r \underline{y}^s P_{r,s}(\chi, \underline{\chi})$$

and thus $P_{r,s}(\chi,\underline{\chi}) = (-1)^s \sigma_{r+s}(h)$. Moreover, $\langle L, \frac{\partial}{\partial t}\rangle = -\sqrt{1+\kappa r^2}$ and standard computation gives

$$Q(L, \underline{L}) = 2\langle X, \nu\rangle, \quad \text{with } X = r\sqrt{1+\kappa r^2}\frac{\partial}{\partial r}. \tag{4.16}$$

Putting these expressions into the spacetime Minkowski formula (1.5), it reduces to

$$(n-r-s)\int_\Sigma \sqrt{1+\kappa r^2}\cdot \sigma_{r+s-1}(h)d\mu = (r+s)\int_\Sigma \sigma_{r+s}(h)\cdot\langle X,\nu\rangle d\mu. \tag{4.17}$$

We thus recover the classical Minkowski formula (1.1) in the Eucliean space by letting $k = r+s$ and $\kappa = 0$. In literature, the hemisphere is viewed as the hypersurface defined by

$$(x^0)^2 + \cdots + (x^n)^2 = 1, x^0 > 0$$

in the Euclidean space and the hyperbolic space is viewed as the hypersurface defined by

$$-(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = -1,\ x^0 > 0$$

in the Minkowski spacetime. The induced metric is

$$g = \frac{1}{1+\kappa r^2}dr^2 + r^2 g_{S^{n-1}},$$

where $\kappa = -1$ for the hemisphere and $\kappa = 1$ for the hyperbolic space. From these observation, it is not difficult to see that (4.17) recovers [2, Corollary 3(b), 3(c)] (see also [14, 24, 32]).

5. Alexandrov Theorems for Submanifolds of constant mixed higher order mean curvature in a spacetime of constant curvature

In Section 3, the simplest case of the spacetime Minkowski formula (Theorem A) was applied to establish the spacetime Alexandrov type theorems concerning codimension-two submanifolds with $\langle \vec{H}, L\rangle = constant$. It is interesting to replace the mean curvature by other invariants from the second fundamental form. In the hypersurface case, Ros [29] showed that any closed, embedded hypersurface in $\mathbb{R}^n$ with constant σ_k curvature is a round sphere. This result was generalized to the hyperbolic space by Montiel and Ros [24] and to the Schwarzschild manifold by Brendle and Eichmair [5].

In this section, we consider codimension-two submanifolds in a spacetime of constant curvature. More precisely, using the spacetime Minkowski formulae established in the previous section, we prove two Alexandrov-type theorems for submanifolds of constant mixed higher order mean curvatures. The first one assumes constancy of $P_{r,0}$ or $P_{0,s}$ and concludes that the submanifold lies in a null hypersurface of symmetry. The second one assumes the stronger condition of constancy of $P_{r,s}$ for $r > 0, s > 0$, which forces Σ to be a sphere of symmetry.

To state our first Alexandrov theorem, we recall the definition of Γ_k cone (also see Definition A.4 in Appendix A). For $1 \le k \le n-1$, Γ_k is a convex cone in $\mathbb{R}^{n-1}$ such that $\Gamma_k = \{\lambda \in \mathbb{R}^{n-1} \ : \ \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\}$ where

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the k-th elementary symmetric function. An $(n-1)\times(n-1)$ symmetric matrix W is said to belong to Γ_k if its spectrum $\lambda(W) \in \Gamma_k$.

Theorem 5.1. *Let Σ be a past (future) incoming null smooth, closed, embedded, spacelike codimension-two submanifold in an $(n+1)$-dimensional spacetime of constant curvature. Suppose Σ is torsion-free with respect to L and $\underline{L}$ and the second fundamental form $\chi \in \Gamma_r$ $(-\underline{\chi} \in \Gamma_s)$. If $P_{r,0} = C$ $(P_{0,s} = (-1)^s C)$ for some positive constant C on Σ, then Σ lies in a null hypersurface of symmetry.*

Proof. By the assumption that $P_{r,0} = C$, the Minkowski formula (1.5) becomes

$$\int_\Sigma \frac{P_{r-1,0}}{P_{r,0}} \langle L, \frac{\partial}{\partial t}\rangle d\mu + \frac{r}{2(n-r)} \int_\Sigma Q(L, \underline{L}) d\mu = 0.$$

Applying the Newton-Maclaurin inequality (A.8) repeatedly and noting that $\langle L, \frac{\partial}{\partial t}\rangle < 0$, we obtain

$$\int_\Sigma \frac{\langle L, \frac{\partial}{\partial t}\rangle}{\mathrm{tr}\chi} d\mu + \frac{1}{2(n-1)} \int_\Sigma Q(L, \underline{L}) d\mu \ge 0.$$

Comparing this with the spacetime Heintze-Karcher inequality (3.17), we see that the equality is achieved. Theorem 3.12 implies that Σ lies in a null hypersurface of symmetry.

For the corresponding statement for $\underline{\chi}$, note that under the assumption the Minkowski formula becomes

$$\int_\Sigma \frac{P_{0,s-1}}{P_{0,s}}\langle \underline{L}, \frac{\partial}{\partial t}\rangle d\mu - \frac{s}{2(n-s)}\int_\Sigma Q(L,\underline{L})d\mu = 0.$$

Applying the Newton-Maclaurin inequality repeatedly, we obtain

$$\frac{P_{0,s-1}(\chi,\underline{\chi})}{P_{0,s}(\chi,\underline{\chi})}\langle \underline{L}, \frac{\partial}{\partial t}\rangle = -\frac{P_{0,s-1}(\chi,-\underline{\chi})}{P_{0,s}(\chi,-\underline{\chi})}\langle \underline{L}, \frac{\partial}{\partial t}\rangle \geq -\frac{s(n-1)}{n-s}\frac{1}{P_{0,1}(\chi,-\underline{\chi})}\langle \underline{L}, \frac{\partial}{\partial t}\rangle.$$

As $P_{0,1}(\chi,-\underline{\chi}) = \langle \vec{H}, \underline{L}\rangle$, the equality of the spacetime Heintze-Karcher inequality is achieved again and Σ lies in a null hypersurface of symmetry by Theorem 3.11. $\square$

In the rest of this section, we prove a rigidity result for submanifolds with constant $P_{r,s}$ for $r>0, s>0$. We start with an algebraic lemma.

Lemma 5.2. *Suppose that $\chi \in \Gamma_{r+s}$ and $\underline{\chi} \in \Gamma_{r+s}$ can be diagonalized simultaneously. If*

$$\frac{\underline{\chi}_{ab}(\underline{T}_{0,s})^{bc}}{P_{0,s}}\frac{\chi_c{}^a}{P_{1,0}} \geq \frac{s}{n-1}, \tag{5.1}$$

then χ and $\underline{\chi}$ satisfy the following inequality

$$\frac{P_{r-1,s}(\chi,\underline{\chi})}{P_{r,s}(\chi,\underline{\chi})} \geq \frac{r+s}{n-(r+s)}\frac{n-1}{tr\chi}. \tag{5.2}$$

The equality holds if and only if χ is a multiple of the identity matrix.

Proof. Since $\chi \in \Gamma_{r+s}$ and $\underline{\chi} \in \Gamma_{r+s}$, $P_{r',s}(\chi,\underline{\chi}) > 0$ for any r' with $1 \leq r' \leq r$ by (A.7) and the fact that $\Gamma_{r+s} \subset \Gamma_{r'+s}$. We apply Newton-MacLaurin inequality (A.8) repeatedly to get

$$\frac{(n-(r+s))\,P_{r-1,s}(\chi,\underline{\chi})}{(r+s)P_{r,s}(\chi,\underline{\chi})} \geq \frac{(n-1-s)}{s+1}\frac{P_{0,s}(\chi,\underline{\chi})}{P_{1,s}(\chi,\underline{\chi})}.$$

It suffices to show that

$$\frac{(n-1-s)}{s+1}\frac{P_{0,s}(\chi,\underline{\chi})}{P_{1,s}(\chi,\underline{\chi})} \geq \frac{n-1}{tr\chi}. \tag{5.3}$$

Let χ_i and $\underline{\chi}_i$, $i=1,2,\cdots,n-1$ be the eigenvalues of χ and $\underline{\chi}$. By the definition of completely polarized elementary symmetric function in (A.2) and (A.6),

$$P_{0,s}(\chi,\underline{\chi}) = \sigma_{(s)}(\underline{\chi},\cdots,\underline{\chi}) =: \sigma_s(\underline{\chi}),$$

and

$$P_{1,s}(\chi,\underline{\chi}) = \sigma_{(s+1)}(\chi,\underbrace{\underline{\chi},\cdots,\underline{\chi}}_{s}) = \frac{1}{s+1}\sum_{a,b}\chi_{ab}\frac{\partial\sigma_{s+1}(\underline{\chi})}{\partial\underline{\chi}_{ab}} = \frac{1}{s+1}\sum_{i}^{n-1}\chi_i\sigma_s(\underline{\chi}|i).$$

Here $\sigma_s(\underline{\chi}|i) = \sum \underline{\chi}_{j_1}\cdots\underline{\chi}_{j_s}$ and we sum over distinct $j_k \neq i, j_k = 1,\ldots,n-1$.

On the other hand, by the simultaneous diagonalization of χ and $\underline{\chi}$, the assumption (5.1) is equivalent to

$$\frac{1}{\sigma_s(\underline{\chi})}\sum_{i=1}^{n-1}\underline{\chi}_i\sigma_{s-1}(\underline{\chi}|i)\chi_i \geq \frac{s}{n-1}\mathrm{tr}\chi.$$

Hence

$$\sum_{i=1}^{n-1}\chi_i\sigma_s(\underline{\chi}|i) = \sigma_s(\underline{\chi})\mathrm{tr}\chi - \sum_{i=1}^{n-1}\chi_i\underline{\chi}_i\sigma_{s-1}(\underline{\chi}|i) \leq \frac{n-s-1}{n-1}\sigma_s(\underline{\chi})\mathrm{tr}\chi.$$

Again, by the cone condition on χ and $\underline{\chi}$, we have $\mathrm{tr}\chi > 0$, $\sum\chi_i\sigma_s(\underline{\chi}|i) = (s+1)P_{1,s} > 0$, and thus

$$\frac{(n-1-s)}{s+1}\frac{P_{0,s}(\chi,\underline{\chi})}{P_{1,s}(\chi,\underline{\chi})} = \frac{(n-1-s)\sigma_s(\underline{\chi})}{\sum_i^{n-1}\chi_i\sigma_s(\underline{\chi}|i)} \geq \frac{n-1}{\mathrm{tr}\chi}, \tag{5.4}$$

which gives us the desired inequality. Moreover, by tracing back the proof, we notice that the equality in (5.2) holds only if this is the case in both the Newton-MacLaurin inequality and (5.1). The former one tells us that χ is a multiple of I_{n-1}. And this also implies the equality for (5.1) by the elementary identity $\sum_i\underline{\chi}_i\sigma_{s-1}\left(\underline{\chi}|i\right) = s\sigma_s\left(\underline{\chi}\right)$. On the other hand, it is easy to see that if χ is a multiple of I_{n-1}, the equality in (5.2) is achieved. $\square$

Before moving to the Alexandrov type theorem, we make a remark on the conditions in the previous algebraic lemma.

Remark 5.3. *The above algebraic lemma still holds if we replace the cone condition by $\chi\in\Gamma_{r+s}$ and $-\underline{\chi}\in\Gamma_{r+s}$, since the left hand sides of both (5.1) and (5.2) are homogeneous of degree zero in $\underline{\chi}$.*

Remark 5.4. *The technical condition (5.1) can be interpreted as follows. Consider $\vec{u} = (u_1,\cdots,u_{n-1})$ with $u_i = \frac{\underline{\chi}_i\sigma_{s-1}(\underline{\chi}|i)}{\sigma_s(\underline{\chi})}$ as a vector determined by $\underline{\chi}$ and $\vec{v} = (v_1,\cdots,v_{n-1})$ with $v_i = \frac{\chi_i}{\sigma_1(\chi)}$ as a vector determined by χ, (5.1) imposes a restriction on the angle between the vectors $\vec{u}$ and $\vec{v}$.*

Theorem 5.5. *Let Σ be a past incoming null smooth (see Definition 3.9) closed embedded spacelike codimension-two submanifold in a spacetime of constant curvature. Suppose Σ is torsion-free with respect to the null frame L and $\underline{L}$ and that the second fundamental forms $\chi\in\Gamma_{r+s}$ and $-\underline{\chi}\in\Gamma_{r+s}$ satisfy*

$$P_{r,s}(\chi,\underline{\chi}) = (-1)^sC, \text{where } C \text{ is a positive constant.} \tag{5.5}$$

and

$$\frac{\underline{\chi}_{ab}(\underline{T}_{0,s})^{bc}}{P_{0,s}}\chi_c{}^a \geq \frac{s}{n-1}P_{1,0}. \tag{5.6}$$

Then Σ is a sphere of symmetry.

Proof. For submanifolds with $P_{r,s} = constant$, the Minkowski formula (1.5) becomes

$$-\int_\Sigma \frac{P_{r-1,s}}{P_{r,s}}\langle L, \frac{\partial}{\partial t}\rangle d\mu = \frac{1}{2}\frac{r+s}{n-(r+s)}\int_\Sigma Q(L,\underline{L})d\mu.$$

It follows from Lemma 5.2 and the fact that $-\langle L, \frac{\partial}{\partial t}\rangle \geq 0$,

$$-(n-1)\int_\Sigma \frac{\langle L, \frac{\partial}{\partial t}\rangle}{tr\chi}d\mu \leq \frac{1}{2}\int_\Sigma Q(L,\underline{L})d\mu.$$

This together with the spacetime Heintze-Karcher inequality (Theorem 3.12) imply

$$(n-1)\int_\Sigma \frac{\langle L, \frac{\partial}{\partial t}\rangle}{\langle \vec{H}, L\rangle}d\mu \leq \frac{1}{2}\int_\Sigma Q(L,\underline{L})d\mu \leq (n-1)\int_\Sigma \frac{\langle L, \frac{\partial}{\partial t}\rangle}{\langle \vec{H}, L\rangle}d\mu. \tag{5.7}$$

Thus, the equality must hold. Note that the first inequality comes from the Newton-MacLaurin inequality (A.8) and the equality case implies $\chi = c_1\sigma$. And the second inequality is from the Heintze-Karcher inequality and equality holds if $\chi = c_1\sigma$.

On the other hand, equation (5.5) together with $\chi = c_1\sigma$ imply that

$$P_{0,s}(\chi,\underline{\chi}) = (-1)^s\tilde{C}, \text{ where } \tilde{C} \text{ is a positive constant.}$$

which falls in the setting of Theorem 5.1. Follow the same line of the proof there and apply Minkowski formula

$$\int_\Sigma \frac{P_{0,s-1}}{P_{0,s}}\langle \underline{L}, \frac{\partial}{\partial t}\rangle d\mu - \frac{s}{2(n-s)}\int_\Sigma Q(L,\underline{L})d\mu = 0,$$

we can arrive the equality case for the Heintze-Karcher inequality (3.14) by using the Newton-MacLaurin inequality. Then, we conclude that $\underline{\chi} = c_2\sigma$. □

As remarked above, (5.6) seems to be a technical condition. However, we believe that, another condition on χ and $\underline{\chi}$ in addition to $P_{r,s}(\chi,\underline{\chi}) = C$ is necessary in order to conclude both of them are proportional to σ.

To finish this section, we present two settings in which condition (5.6) is automatically satisfied.

The first example is: $\underline{\chi} = -\chi$. From the discussion at the end of Section 4, we see that the classical hypersurfaces cases fall in this setting. In this special situation, (5.6) is equivalent to

$$\begin{aligned} 1 &\leq \frac{(n-1)\sum_{i=1}^{n-1}\sigma_{s-1}\left((-\chi)|i\right)(-\chi)_i\chi_i}{s\sigma_1(\chi)\sigma_s(-\chi)} = \frac{(n-1)\sum_{i=1}^{n-1}\sigma_{s-1}(\chi|i)\chi_i^2}{s\sigma_1(\chi)\sigma_s(\chi)} \\ &= \frac{(n-1)\left[\sigma_s(\chi)\sigma_1(\chi) - (s+1)\sigma_{s+1}(\chi)\right]}{s\sigma_1(\chi)\sigma_s(\chi)}. \end{aligned}$$

However, this inequality follows from the standard Newton-MacLaurin inequality for symmetric functions: $(n-s-1)\sigma_s(\chi)\sigma_1(\chi) \geq (n-1)(s+1)\sigma_{s+1}(\chi)$. In view of the remark at the end of the previous section, Theorem 5.5 generalizes the classical Alexandrov theorem in Riemannian space forms [24, Theorem 7 and Theorem 10].

The second example is: one of χ and $\underline{\chi}$ is already known to be a multiple of I_{n-1}. One can easily check that (5.6) is trivial by using the elementary formula $\sum_i \underline{\chi}_i \sigma_{s-1}(\underline{\chi}|i) = s\sigma_s(\underline{\chi})$.

6. Generalization in the Schwarzschild spacetime

In this section, we discuss Minkowski type formulae and Alexandrov theorems in the Schwarzschild spacetime. The divergence equations (4.3) of $T^{ab}_{r,s}$ and $\underline{T}^{ab}_{r,s}$ play crucial roles in the proof of the Minkowski formula in a spacetime of constant curvature. Those equations no longer hold in the Schwarzschild spacetime due to the presence of a non-trivial ambient curvature contribution. However, it turns out the divergences of $T^{ab}_{r,s}$ and $\underline{T}^{ab}_{r,s}$ still posses favorable properties under natural assumptions on Σ when either $r = 0$ or $s = 0$..

Lemma 6.1. *Let Σ be a spacelike codimension-two submanifold in the Schwarzschild spacetime. Suppose Σ is torsion-free with respect to a null frame $L, \underline{L}$. Then the following statements are true:*

(1) If $Q(L, \underline{L}) \geq 0$, then $\sum_{a,b}(\nabla_b T^{ab}_{2,0})Q(L, \partial_a) \leq 0$ and $\sum_{a,b}(\nabla_b T^{ab}_{0,2})Q(\underline{L}, \partial_a) \leq 0$.

(2) Suppose χ is positive definite and $(Q^2)(L, v)Q(L, v) \leq 0$ for any vector v tangent to Σ, then $\sum_{a,b}(\nabla_b T^{ab}_{r,0})Q(L, \partial_a) \leq 0$ if $r \geq 3$.

(3) Suppose $-\underline{\chi}$ is positive definite and $(Q^2)(\underline{L}, v)Q(\underline{L}, v) \geq 0$ for any vector v tangent to Σ, then $\sum_{a,b}(\nabla_b T^{ab}_{0,s})Q(\underline{L}, \partial_a) \leq 0$ if $s \geq 3$.

Proof. Denote the radial coordinate in the Schwarzschild metric by ρ and write the metric as

$$\overline{g} = -\left(1 - \frac{2m}{\rho^{n-2}}\right) dt^2 + \frac{1}{1 - \frac{2m}{\rho^{n-2}}} d\rho^2 + \rho^2 g_{\mathbb{S}^{n-1}}.$$

We only deal with case (2), and the other cases can be derived by the same argument. Denote $\tilde{\sigma} = \sigma + y\chi$ and write T_r for $T_{r,0}$. Setting $\underline{y} = 0$ in (4.6) and (4.8), we obtain

$$\sum_r y^r \nabla_b T^{ab}_r = y^2 \bar{R}_{Lbdc}(\tilde{\sigma}^{-1})^{ac}(\tilde{\sigma}^{-1})^{db} \det(\tilde{\sigma}).$$

From the curvature expression (C.2), we have

$$\bar{R}_{Lbdc} = -\frac{n(n-1)m}{\rho^{n+2}}\left(\frac{2}{3}Q_{Lb}Q_{dc} - \frac{1}{3}Q_{Ld}Q_{cb} - \frac{1}{3}Q_{Lc}Q_{bd}\right) - \frac{nm}{\rho^{n+2}}\left((Q^2)_{Ld}\sigma_{bc} - (Q^2)_{Lc}\sigma_{bd}\right),$$

and thus

$$\begin{aligned}&\sum_r y^r \nabla_b T^{ab}_r Q_{La}\\ = \ & -\frac{nm}{\rho^{n+2}}\left[(n-1)Q_{Lb}Q_{dc} + (Q^2)_{Ld}\sigma_{bc} - (Q^2)_{Lc}\sigma_{bd}\right] Q_{La}(\tilde{\sigma}^{-1})^{ac}(\tilde{\sigma}^{-1})^{db} \det(\tilde{\sigma}).\end{aligned}$$

Now, suppose that χ is diagonal with eigenvalues $\chi_1, \ldots, \chi_{n-1}$. Write the eigenvalues of $\tilde{\sigma}$ as $\mu_a = 1 + y\chi_a$. We obtain

$$\sum_r y^r \nabla_b T^{ab}_r Q_{La} = \tfrac{nm}{\rho^{n+2}} y^2 \left[\textstyle\sum_{a\neq b} \frac{\sigma_{n-1}(\tilde{\sigma})}{\mu_a\mu_b}(Q^2)_{La}Q_{La}\right].$$

On the other hand, by standard computation, we have

$$\sum_{a\neq b}\frac{\sigma_{n-1}(\tilde{\sigma})}{\mu_a\mu_b} = \sum_{a\neq b}\sigma_{n-3}(\mu \mid ab) = \sum_{a\neq b}\sum_{q=0}^{n-3} y^q\sigma_q(\chi \mid ab) = \sum_{q=0}^{n-3} y^q\sum_{a=1}^{n-1}(n-q-2)\sigma_q(\chi|a).$$

To get the last equality, we use the property of elementary symmetric function that $\sum_{i=1}^m \sigma_k(\lambda \mid i) = (m-k)\sigma_k(\lambda)$. Thus,

$$\begin{aligned}\sum_r y^r\nabla_b T_r^{ab}Q_{La} &= \frac{nm}{\rho^{n+2}}\sum_{a=1}^{n-1}\sum_{q=0}^{n-3} y^{q+2}(n-q-2)\sigma_q(\chi|a)(Q^2)_{La}Q_{La} \\ &= \frac{nm}{\rho^{n+2}}\sum_{a=1}^{n-1}\sum_{p=2}^{n-3} y^{p}(n-p)\sigma_{p-2}(\chi|a)(Q^2)_{La}Q_{La}\end{aligned} \tag{6.1}$$

By comparing the coefficients of y^r, we obtain

$$\sum_{a,b}(\nabla_b T_r^{ab})Q_{La} = \frac{nm(n-r)}{\rho^{n+2}}\sum_{a=1}^{n-1}\sigma_{r-2}(\chi|a)(Q^2)_{La}Q_{La}, \text{ for } r\geq 2, \tag{6.2}$$

which is negative by the assumptions that χ is positive definite and $(Q^2)(L,v)Q(L,v)\leq 0$ for any vector v tangent to Σ. This proves the second statement. The third one is proved along exactly the same line.

For $r=2$ (or $s=2$) case, by comparing the coefficients of y^2 on both sides of (6.1), we get

$$\begin{aligned}\sum_{a,b}(\nabla_b T_2^{ab})Q_{La} &= \frac{n(n-2)m}{\rho^{n+2}}\sum_{a=1}^{n-1}(Q^2)_{La}Q_{La} \\ &= \frac{n(n-2)m}{\rho^{n+2}}\left[\sum_{a,c}Q_L{}^cQ_{ca}Q_{La} - \frac{1}{2}Q_{L\underline{L}}\sum_a (Q_{La})^2\right] \\ &= -\frac{1}{2}\frac{n(n-2)m}{\rho^{n+2}}Q_{L\underline{L}}\sum_a (Q_{La})^2,\end{aligned}$$

which is non-positive by the assumption that $Q(L,\underline{L})\geq 0$. □

Notice that no condition is needed for $r=1$ or $s=1$ since $T_{1,0}^{ab}=T_{0,1}^{ab}=\sigma^{ab}$ is always divergence free. Thus, we can prove a clean Minkowski formulae for $(r,s)=(1,0)$ or $(0,1)$ in the Schwarzschild spacetime, see Theorem A or Theorem 2.2. For $r,s\geq 2$, the divergence property of $T_{r,0}$ and $T_{0,s}$ no longer hold. Fortunately, based on the above lemma, we can still establish certain inequalities for those higher order case in the Schwarzschild spacetime.

Theorem 6.2. *Let Σ be a closed spacelike codimension-two submanifold in the Schwarzschild spacetime. Suppose that Σ is torsion-free.*

If Σ satisfies assumption (1) or (2) in Lemma 6.1, then for any $1 \le r \le n-1$,

$$\int_\Sigma P_{r-1,0}(\chi,\underline{\chi})\langle L, \frac{\partial}{\partial t}\rangle + \frac{r}{2(n-r)}\int_\Sigma P_{r,0}(\chi,\underline{\chi})Q(L,\underline{L}) \ge 0. \tag{6.3}$$

If Σ satisfies assumption (1) or (3) in Lemma 6.1, then for any $1 \le s \le n-1$,

$$\int_\Sigma P_{0,s-1}(\chi,\underline{\chi})\langle \underline{L}, \frac{\partial}{\partial t}\rangle - \frac{s}{2(n-s)}\int_\Sigma P_{0,s}(\chi,\underline{\chi})Q(L,\underline{L}) \ge 0. \tag{6.4}$$

Proof. Note that $T^{ab}_{r,0}$ is a polynomial in χ only and thus $T^{ba}_{r,0}\chi^c_a - T^{ca}_{r,0}\chi^b_a = 0$. By the above lemma, we can proceed as in Theorem 4.3. □

In [5], Brendle and Eichmair considered the case that Σ is a closed embedded hypersurface contained in a totally geodesic time-slice M of the Schwarzschild spacetime. Assume Σ to be star-shaped and convex, they derived an interesting integral inequality

$$(n-k)\int_\Sigma f\sigma_{k-1}d\mu \le k\int_\Sigma \sigma_k\langle X,\nu\rangle d\mu \tag{6.5}$$

where $X : \Sigma \to M$ is the position vector, ν is the outward unit normal vector field of Σ and $f = \sqrt{1-\frac{2m}{\rho^{n-2}}}$.

In fact, (6.5) can be recovered by (6.3) or (6.4). The argument is similar as the discussion at the end of section 4 where we recover the classical Minkowski formulae (1.1) by (1.5) or (1.6) except that one needs to check the assumption in Theorem 6.2. First, it is easy to see that star-shapeness is equivalent to $Q(L,\underline{L}) \ge 0$ because of identity (4.16). On the other hand, the convexity of Σ implies χ is positive definite.

For a submanifold Σ on a totally geodesic slice M_t, the term (6.2) can be compared with the Ricci curvature term in Brendle-Eichmair's formula [5](at the end of the proof of Proposition 8). Indeed, given two vectors v, w tangent to M_t, the Ricci curvature satisfies

$$\begin{aligned} Ric_{M_t}(v,w) &= R(v,e_{n+1},w,e_{n+1}) \\ &= -\frac{2m}{r^n}\bar{g}(v,w) - \frac{n(n-1)m}{r^{n+2}}Q(v,e_{n+1})Q(w,e_{n+1}) \\ &\quad - \frac{nm}{r^{n+1}}\left(\bar{g}(v,w)Q^2(e_{n+1},e_{n+1}) - Q^2(v,w)\right), \end{aligned}$$

by the Gauss equation and (C.2). Let ν be the outward normal of Σ. We note that

$$Q^2(e_i,\nu) = Q^2(L,e_i) = Q(\nu,e_{n+1})Q(e_i,e_{n+1}),$$

where $L = e_{n+1} + \nu$. Hence, $Ric_{M_t}(e_i,\nu) = -\frac{n^2 m}{r^{n+2}}Q^2(L,e_i)$.

On the other hand, we claim that the condition in Lemma 6.1

$$(Q^2)(L,v)Q(L,v) \le 0 \quad \text{for any vector } v \text{ tangent to } \Sigma$$

is automatically satisfied under the star-shaped condition. Indeed, the main ingredient is that the tangent vector v does not have $\frac{\partial}{\partial t}$ component if Σ lies in a totally geodesic time-slice. By the definition of Q^2 (C.1) and noting that $Q(\partial_b, v) = rdr \wedge dt(\partial_b, v) = 0$ for any tangent vector ∂_b, we expand

$$(Q^2)(L, v) = \bar{g}^{\underline{L}L} Q(L, \underline{L}) Q(L, v).$$

Therefore,

$$(Q^2)(L, v) Q(L, v) = -\frac{1}{2} Q(L, \underline{L}) \left(Q(L, v)\right)^2 \leq 0,$$

provided that Σ is star-shaped.

Again, once we have the Minkwoski formulas at hand, the spacetime Alexandrov theorems follow by the spacetime Heintze-Karcher inequality as in Theorem 5.1.

Corollary 6.3. *Let Σ be a past (or future) incoming null smooth (see Definition 3.9) closed embedded spacelike codimension-two submanifold in the Schwarzschild spacetime. Suppose that Σ is torsion-free. If Σ satisfies the assumptions in either (1) or (2) ((2) or (3)) in Lemma 6.1 and*

$$P_{r,0}(\chi, \underline{\chi}) = C \qquad \left(\text{ or } P_{0,s}(\chi, \underline{\chi}) = (-1)^s C \right) \tag{6.6}$$

for some positive constant C, then Σ lies in a null hypersurface of symmetry.

Remark 6.4. *In [21], Li-Wei-Xiong show that the convexity assumption in Brendle-Eichmair's result can be removed. The same argument works here if we assume χ (or $\underline{\chi}$) is positive definite at a point.*

The above discussion on the Schwarzschild spacetime indicates that it is not easy to get a clean general form of Minkowski formulae with nontrivial curvature and torsion. In the rest of this section, we focus on a closed spacelike 2-surface in the 4-dimensional Schwarzschild spacetime which carries the two-form $Q = rdr \wedge dt$. With the same notations as in Section 4, we have

$$T^{ab}_{2,0} = (\mathrm{tr}\chi)\sigma^{ab} - \chi^{ab}, \text{ and } \underline{T}^{ab}_{0,2} = (\mathrm{tr}\underline{\chi})\sigma^{ab} - \underline{\chi}^{ab}.$$

Recall the Codazzi equations

$$\nabla_a \chi_{bc} - \nabla_b \chi_{ac} = \bar{R}_{abcL} + \chi_{ac}\zeta_b - \chi_{bc}\zeta_a; \tag{6.7}$$

$$\nabla_a \underline{\chi}_{bc} - \nabla_b \underline{\chi}_{ac} = \bar{R}_{abc\underline{L}} - \underline{\chi}_{ac}\zeta_b + \underline{\chi}_{bc}\zeta_a. \tag{6.8}$$

Taking trace of (6.7), (6.8), and using the Ricci-flatness of 4-dimensional Schwarzschild spacetime, we get

$$\nabla_a T^{ab}_{2,0} = -\sigma^{ac} \bar{R}_a{}^b{}_{cL} - T^{ab}_{2,0}\zeta_a = -\frac{1}{2} \bar{R}_L{}^b{}_{\underline{L}L} - T^{ab}_{2,0}\zeta_a; \tag{6.9}$$

$$\nabla_a \underline{T}^{ab}_{0,2} = -\sigma^{ac} \bar{R}_a{}^b{}_{c\underline{L}} + \underline{T}^{ab}_{0,2}\zeta_a = -\frac{1}{2} \bar{R}_{\underline{L}}{}^b{}_{L\underline{L}} + \underline{T}^{ab}_{0,2}\zeta_a. \tag{6.10}$$

Now, we run the same proof as Theorem 4.3 (Theorem D) by considering

$$\int_\Sigma \nabla_a [T^{ab} Q(\underline{L}, \partial_b)] = 0, \text{ and } \int_\Sigma \nabla_a [\underline{T}^{ab} Q(L, \partial_b)] d\mu = 0.$$

and get

(6.11)

$$0=\int_\Sigma -\frac{1}{2}\bar{R}_{L}{}^{b}{}_{\underline{L}L}Q(\underline{L},\partial_b)+\mathrm{tr}\chi\langle\underline{L},\frac{\partial}{\partial t}\rangle-\chi^{ab}\underline{\chi}_a^c Q_{cb}-\frac{1}{2}\left(\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\chi^{ab}\underline{\chi}_{ab}\right)Q(L,\underline{L})d\mu;$$

(6.12)

$$0=\int_\Sigma -\frac{1}{2}\bar{R}_{\underline{L}}{}^{b}{}_{L\underline{L}}Q(L,\partial_b)+\mathrm{tr}\underline{\chi}\langle L,\frac{\partial}{\partial t}\rangle-\underline{\chi}^{ab}\chi_a^c Q_{cb}+\frac{1}{2}\left(\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\underline{\chi}^{ab}\chi_{ab}\right)Q(L,\underline{L})d\mu.$$

Since $\frac{\partial}{\partial t}$ is a Killing field and $\vec{H}=\frac{1}{2}tr\underline{\chi}L+\frac{1}{2}tr\chi\underline{L}$,

$$0=\int_\Sigma\langle\vec{H},\frac{\partial}{\partial t}\rangle d\mu=\frac{1}{2}\int_\Sigma tr\underline{\chi}\langle L,\frac{\partial}{\partial t}\rangle d\mu+\frac{1}{2}\int_\Sigma tr\chi\langle\underline{L},\frac{\partial}{\partial t}\rangle d\mu.$$

Subtracting (6.12) from (6.11), we obtain

$$\begin{aligned}
&2\int_\Sigma tr\chi\langle\underline{L},\frac{\partial}{\partial t}\rangle d\mu\\
=&\int_\Sigma\frac{1}{2}\left(\bar{R}_{L}{}^{b}{}_{\underline{L}L}Q(\underline{L},\partial_b)-\bar{R}_{\underline{L}}{}^{b}{}_{L\underline{L}}Q(L,\partial_b)\right)+\left(\chi^{ab}\underline{\chi}_a^c-\underline{\chi}^{ab}\chi_a^c\right)Q_{cb}+\left(\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\chi^{ab}\underline{\chi}_{ab}\right)Q(L,\underline{L})d\mu\\
=&\int_\Sigma\frac{1}{2}\left(\bar{R}_{L}{}^{b}{}_{\underline{L}L}Q(\underline{L},\partial_b)-\bar{R}_{\underline{L}}{}^{b}{}_{L\underline{L}}Q(L,\partial_b)\right)+\left(\bar{R}_{bc\underline{L}L}-2(d\zeta)_{bc}\right)Q^{cb}+\left(\frac{1}{2}\bar{R}_{\underline{L}L L\underline{L}}-R\right)Q(L,\underline{L})d\mu
\end{aligned}$$

where we use the following Gauss and Ricci equations to get the last equality.

$$\bar{R}+\bar{R}_{L\underline{L}}+\frac{1}{2}\bar{R}_{\underline{L}LL\underline{L}}=R+\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\chi_{ab}\underline{\chi}^{ab};\tag{6.13}$$

$$\frac{1}{2}\bar{R}_{ab\underline{L}L}=(d\zeta)_{ab}+\frac{1}{2}\left(\chi_a{}^c\underline{\chi}_{cb}-\underline{\chi}_a{}^c\chi_{cb}\right).\tag{6.14}$$

Next, from

$$\bar{R}_{\alpha\beta\underline{L}L}Q^{\alpha\beta}=\bar{R}_{bc\underline{L}L}Q^{bc}-\bar{R}_{Lb\underline{L}L}Q_{\underline{L}}{}^{b}-\bar{R}_{\underline{L}b\underline{L}L}Q_{L}{}^{b}+\frac{1}{2}\bar{R}_{\underline{L}L\underline{L}L}Q(L,\underline{L}),$$

we have

$$\frac{1}{2}\left(\bar{R}_{L}{}^{b}{}_{\underline{L}L}Q(\underline{L},\partial_b)-\bar{R}_{\underline{L}}{}^{b}{}_{L\underline{L}}Q(L,\partial_b)\right)=-\frac{1}{2}\bar{R}_{\alpha\beta\underline{L}L}Q^{\alpha\beta}+\frac{1}{2}\bar{R}_{bc\underline{L}L}Q^{bc}+\frac{1}{4}\bar{R}_{\underline{L}L\underline{L}L}Q(L,\underline{L}),$$

where $\alpha,\beta=1,\cdots,n+1$ represent the indices of the ambient space. Therefore,

$$\begin{aligned}
&2\int_\Sigma tr\chi\langle\underline{L},\frac{\partial}{\partial t}\rangle d\mu\\
=&\int_\Sigma-\frac{1}{2}\bar{R}_{\alpha\beta\underline{L}L}Q^{\alpha\beta}+\left(\frac{1}{2}\bar{R}_{bc\underline{L}L}-2(d\zeta)_{bc}\right)Q^{cb}+\left(\frac{1}{4}\bar{R}_{\underline{L}LL\underline{L}}-R\right)Q(L,\underline{L})d\mu.
\end{aligned}$$

Consider the two-form $\eta=\bar{R}_{\alpha\beta\mu\nu}Q^{\alpha\beta}dx^\mu dx^\nu$ on the spacetime. In [18, section 3.3], it is shown that $d\eta=d*\eta=0$. So $\int_\Sigma\bar{R}_{\alpha\beta\underline{L}L}Q^{\alpha\beta}$ is the same for any 2-surface bounding a

3-volume. Evaluating the integral on a sphere with $t = constant$ and $r = constant$, we have [18, (53)]

$$\int_{\Sigma} \bar{R}_{\alpha\beta \underline{L} L} Q^{\alpha\beta} = -32\pi m.$$

As a result, we reach the following Minkowski formula on the 4-dimensional Schwarzschild spacetime.

Theorem 6.5 (Theorem F). *Consider the two-form $Q = rdr \wedge dt$ on the 4-dimensional Schwarzschild spacetime with $m \geq 0$. For a closed oriented spacelike 2-surface Σ, we have*

$$\begin{aligned} &2\int_{\Sigma} \langle \vec{H}, L \rangle \langle \underline{L}, \frac{\partial}{\partial t} \rangle d\mu \\ =& -16\pi m + \int_{\Sigma} \left(R + \frac{1}{4} \bar{R}_{L\underline{L}L\underline{L}} \right) Q(L, \underline{L}) + \sum_{b,c=1}^{2} \left(\frac{1}{2} \bar{R}_{bc\underline{L}L} - 2(d\zeta_L)_{bc} \right) Q_{bc} d\mu. \end{aligned}$$

where ζ_L is the connection 1-form of the normal bundle with respect to L, $\bar{R}$ is the curvature tensor of the Schwarzschild spacetime, R is the scalar curvature of Σ, and $Q_{bc} = Q(e_b, e_c)$, $(d\zeta_L)_{bc} = (d\zeta_L)(e_b, e_c)$, etc.

Appendix A. Proof of Some Algebraic Relations

In this section, Gårding's theory for hyperbolic polynomials, in particular for elementary symmetric functions σ_k, is reviewed and applied to prove several algebraic relations for mixed higher order mean curvatures. For more detailed discussion about polarized σ_k functions, we refer to the Appendix in the lecture notes by Guan [12, 13].

Definition A.1. *Let $W^1, \cdots, W^{n-1}$ be $(n-1) \times (n-1)$ symmetric matrices, define the mixed determinant $\sigma_{(n-1)}\left(W^1, \cdots, W^{n-1}\right)$ such that $\frac{1}{(n-1)!}\sigma_{(n-1)}\left(W^1, \cdots, W^{n-1}\right)$ is the coefficient of the term $t_1 \cdots t_{n-1}$ in the polynomial*

$$\det\left(t_1 W^1 + \cdots + t_{n-1} W^{n-1}\right).$$

In general, for $1 \leq k \leq n-1$, we define the complete polarization of the symmetric function σ_k by

$$\sigma_{(k)}\left(W^1, \cdots, W^k\right) = \binom{n-1}{k} \sigma_{(n-1)}\left(W^1, \cdots, W^k, I_{n-1}, \cdots, I_{n-1}\right), \tag{A.1}$$

where the identity matrix I_{n-1} appears $(n-1-k)$ times.

Higher order mixed mean curvatures can be expressed in terms of complete polarizations of the elementary symmetric functions, σ_k:

Lemma A.2. *The following identity holds:*

$$P_{r,s}\left(\chi, \underline{\chi}\right) = \sigma_{(r+s)}(\underbrace{\chi, \cdots, \chi}_{r}, \underbrace{\underline{\chi}, \cdots, \underline{\chi}}_{s}). \tag{A.2}$$

Proof. Notice that $\frac{1}{(n-1)!}\sigma_{(n-1)}(\underbrace{\chi,\cdots,\chi}_{r},\underbrace{\underline{\chi},\cdots,\underline{\chi}}_{s},\underbrace{I_{n-1},\cdots,I_{n-1}}_{(n-1)-(r+s)})$ is the coefficient of the term $t_1\cdots t_{n-1}$ in the polynomial

$$\sigma_{n-1}\left(t_1\chi+\cdots+t_r\chi+t_{r+1}\underline{\chi}+\cdots+t_{r+s}\underline{\chi}+t_{r+s+1}+\cdots+t_{n-1}\right).$$

Denote $t=t_1+\cdots+t_r,\ \underline{t}=t_{r+1}+\cdots+t_{r+s},\ t_0=t_{r+s+1}+\cdots+t_{n-1}$ and use the equation (4.1), we get

$$\begin{aligned}
&\sigma_{n-1}\left(t_1\chi+\cdots+t_r\chi+t_{r+1}\underline{\chi}+\cdots+t_{r+s}\underline{\chi}+t_{r+s+1}+\cdots+t_{n-1}\right)\\
=&\ \det\left(t\chi+\underline{t}\underline{\chi}+t_0I_{n-1}\right)\\
=&\ t_0^{n-1}\sum_{k,l}\frac{(k+l)!}{k!l!}\left(\frac{t}{t_0}\right)^k\left(\frac{\underline{t}}{t_0}\right)^l P_{k,l}(\chi,\underline{\chi})\\
=&\ \sum_{k,l}\frac{(k+l)!}{k!l!}t_0^{n-1-(k+l)}t^k\underline{t}^lP_{k,l}(\chi,\underline{\chi})\\
=&\ \sum_{k,l}\frac{(k+l)!}{k!l!}(t_{r+s+1}+\cdots+t_{n-1})^{n-1-(k+l)}(t_1+\cdots+t_r)^k(t_{r+1}+\cdots+t_{r+s})^lP_{k,l}(\chi,\underline{\chi}).
\end{aligned}$$

For fixed r,s, the term $t_1\cdots t_{n-1}$ appears in the last expression only when $k=r$ and $l=s$, and the coefficient is

$$\frac{(r+s)!}{r!s!}\,(n-1-(r+s))!\ r!\ s!\ P_{r,s}(\chi,\underline{\chi}).$$

Thus,

$$\begin{aligned}
\sigma_{(n-1)}(\underbrace{\chi,\cdots,\chi}_{r},\underbrace{\underline{\chi},\cdots,\underline{\chi}}_{s},\underbrace{I_{n-1},\cdots,I_{n-1}}_{(n-1)-(r+s)}) &= \frac{(r+s)!\,(n-1-(r+s))!}{(n-1)!}P_{r,s}(\chi,\underline{\chi})\\
&= \frac{1}{\binom{n-1}{r+s}}P_{r,s}(\chi,\underline{\chi}).
\end{aligned}$$

(A.2) follows from this and the definition of complete polarization (A.1). $\square$

From (A.2) and the definition of $P_{r,s}$ and $T^{ab}_{r,s}$, we have the following basic identities:

$$\sum_{a,b}\chi_{ab}T^{ab}_{r,s} = \sum_{a,b}\chi_{ab}\frac{\partial P_{r,s}(\chi,\underline{\chi})}{\partial\chi_{ab}}=rP_{r,s}(\chi,\underline{\chi}), \tag{A.3}$$

$$\sum_{a,b}\underline{\chi}_{ab}\underline{T}^{ab}_{r,s} = \sum_{a,b}\underline{\chi}_{ab}\frac{\partial P_{r,s}(\chi,\underline{\chi})}{\partial\underline{\chi}_{ab}}=sP_{r,s}(\chi,\underline{\chi}). \tag{A.4}$$

Indeed, the equalities (A.3) and (A.4) follow from the fact that $P_{r,s}(\chi,\underline{\chi})$ is a polynomial in χ and $\underline{\chi}$, homogeneous of degrees r and s respectively.

In addition, the following equation $(A.5)$ can be deduced from the definition of complete polarized symmetric function and (A.2):

Lemma A.3.

$$\sum_{a,b}\sigma_{ab}T^{ab}_{r,s}=\sum_{a,b}\sigma_{ab}\frac{\partial P_{r,s}(\chi,\underline{\chi})}{\partial\chi_{ab}}=\frac{r\,(n-(r+s))}{r+s}P_{r-1,s}(\chi,\underline{\chi}).\tag{A.5}$$

Proof of (A.5). Indeed, we have the standard definition of completely polarized symmetric function as

$$\sigma_{(k)}\left(W^1,W^2,\cdots,W^k\right)=\frac{1}{k!}\sum_{i_1,\cdots,i_k=1}^{n-1}W^1_{i_1}W^2_{i_2}\cdots W^k_{i_k}\frac{\partial^k\sigma_k(W)}{\partial W_{i_1}\cdots\partial W_{i_k}},$$

where $W^j_1,\cdots,W^j_{n-1}$ are the eigenvalues of W^j. We note that $\frac{\partial^k\sigma_k(W)}{\partial W_{i_1}\cdots\partial W_{i_k}}$ is a combinatorial constant depending only on k and W can be replaced by any symmetric matrix. Thus

$$\begin{aligned}\sigma_{(k)}(\chi,\underbrace{\underline{\chi},\cdots,\underline{\chi}}_{k-1}) &= \frac{1}{k!}\sum_{i_1,\cdots,i_k=1}^{n-1}\chi_{i_1}\underline{\chi}_{i_2}\cdots\underline{\chi}_{i_k}\frac{\partial^k\sigma_k(\underline{\chi})}{\partial\underline{\chi}_{i_1}\cdots\partial\underline{\chi}_{i_k}}=\frac{1}{k}\sum_{i=1}^{n-1}\chi_i\frac{\partial\sigma_k(\underline{\chi})}{\partial\underline{\chi}_i}\\ &= \frac{1}{k}\,\frac{d}{dt}\Big|_{t=0}\sigma_k\left(t\chi+\underline{\chi}\right).\end{aligned}\tag{A.6}$$

More generally, we have

$$\begin{aligned}P_{r,s}(\chi,\underline{\chi}) &= \sigma_{(r+s)}(\underbrace{\chi,\cdots,\chi}_{r},\underbrace{\underline{\chi},\cdots,\underline{\chi}}_{s})=\frac{s!}{(r+s)!}\,\frac{d^r}{dt^r}\Big|_{t=0}\sigma_{r+s}\left(t\chi+\underline{\chi}\right)\\ &= \frac{1}{\binom{r+s}{r}}\left(\frac{1}{r!}\frac{d^r}{dt^r}\Big|_{t=0}\sigma_{r+s}(t\chi+\underline{\chi})\right).\end{aligned}$$

Equation (A.5) is verified by a sequence of direct computations:

$$\begin{aligned}\sum_{a,b}\sigma_{ab}\frac{\partial P_{r,s}(\chi,\underline{\chi})}{\partial\chi_{ab}} &= \frac{1}{\binom{r+s}{r}}\frac{1}{r!}\frac{d^r}{dt^r}\Big|_{t=0}\sum_{a,b}\sigma_{ab}\frac{\partial\sigma_{r+s}(t\chi+\underline{\chi})}{\partial\chi_{ab}}\\ &= \frac{1}{\binom{r+s}{r}}\frac{1}{r!}\frac{d^r}{dt^r}\Big|_{t=0}\left[(n-(r+s))\,\sigma_{r+s-1}(t\chi+\underline{\chi})t\right]\\ &= \frac{1}{\binom{r+s}{r}}\frac{1}{r!}\left(n-(r+s)\right)r\frac{d^{r-1}}{dt^{r-1}}\Big|_{t=0}\left[\sigma_{r+s-1}(t\chi+\underline{\chi})\right]\\ &= \frac{n-(r+s)}{\binom{r+s}{r}}\binom{r+s-1}{r-1}P_{r-1,s}(\chi,\underline{\chi})\\ &= \frac{(n-(r+s))\,r}{r+s}P_{r-1,s}(\chi,\underline{\chi}),\end{aligned}$$

where we use equation $\sum_{i=1}^{n-1}\frac{\partial\sigma_k(\lambda)}{\partial\lambda_i}=(n-k)\sigma_{k-1}(\lambda)$ to get the second identity above. ☐

We now briefly review Gårding's inequality for polarized elementary symmetric functions and apply it to deduce a version of Newton-MacLaurin inequality for $P_{r,s}(\chi,\underline{\chi})$ which is used several times in the previous sections. First, we recall the definition of the positive cone for σ_k:

Definition A.4. *For $1 \le k \le n-1$, let Γ_k be a cone in $\mathbb{R}^{n-1}$ defined by*

$$\Gamma_k = \ \{\lambda \in \mathbb{R}^{n-1} \ : \ \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\},$$

where

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the k-th symmetric function. An $(n-1)\times(n-1)$ symmetric matrix W is said to belong to Γ_k if its spectrum $\lambda(W) \in \Gamma_k$.

According to the standard theory for elementary symmetric functions from the hyperbolic polynomials point of view (see Corollary 13.1 and Proposition 13.3 in [12]), we know that Γ_k is the positive cone for both σ_k and its polarization $\sigma_{(k)}$, i.e.,

$$\sigma_{(k)}(W^1, \cdots, W^k) > 0, \quad \text{for } W^i \in \Gamma_k \text{ with } i = 1, \cdots, k.$$

In view of the relation (A.2), we also have

$$P_{r,s}(\chi, \underline{\chi}) > 0,, \quad \text{for } \chi, \underline{\chi} \in \Gamma_{r+s}. \tag{A.7}$$

The following lemma is a special case of a theorem of Gårding for hyperbolic polynomials, which can be found in [11] (or see Appendix of [13] or [16]).

Lemma A.5. *For any $W^i \in \Gamma_k$ or $-\Gamma^k$, $i = 1, \cdots, k$, we have*

$$\sigma^2_{(k)}\left(W^1, W^2, W^3, \cdots, W^k\right) \ge \sigma_{(k)}\left(W^1, W^1, W^3, \cdots, W^k\right)\sigma_{(k)}\left(W^2, W^2, W^3, \cdots, W^k\right).$$

The equality holds if and only if W^1 and W^2 are multiples of each other.

Proof. We may assume $W^i \in \Gamma_k$ for all i because changing W^i to $-W^i$ does not change the desired inequality. Since $W^3, \ldots, W^k \in \Gamma_k$, the homogeneous polynomial $p(x) = \sigma_{(k)}(x, x, W^3, \ldots, W^k)$ is hyperbolic with respect to every element in Γ_k (see Appendix of [13]). The assertion follows by applying the usual Gårding's inequality to the complete polarization of p.

□

The above Gårding's inequality yields the following Newton-MacLaurin inequality for $P_{r,s}(\chi, \underline{\chi})$.

Lemma A.6. *If χ and $\underline{\chi}$ are both in Γ_{r+s-1} cone, we have*

$$P^2_{r-1,s}(\chi, \underline{\chi}) \ge c(n, r, s) P_{r,s}(\chi, \underline{\chi}) P_{r-2,s}(\chi, \underline{\chi}), \tag{A.8}$$

where $c(n, r, s) = \frac{\binom{n-1}{r-1+s}^2}{\binom{n-1}{r-2+s}\binom{n-1}{r+s}} = \frac{r+s}{r+s-1} \cdot \frac{n-(r+s)+1}{n-(r+s)}$. The equality holds if and only if $\chi = cI_{n-1}$ for some constant c.

Proof. When $P_{r,s}(\chi, \underline{\chi}) \leq 0$, the inequality (A.8) is trivial. We thus assume $P_{r,s}(\chi, \underline{\chi}) \geq 0$. Replacing k by $(r+s)$ in the above Gårding's inequality (A.8), we obtain

$$\begin{aligned} &\sigma^2_{(r+s)}\left(W^1, W^2, W^3, \cdots, W^r, W^{r+1}, \cdots, W^{r+s}\right) \\ \geq &\sigma_{(r+s)}\left(W^1, W^1, W^3, \cdots, W^r, W^{r+1}, \cdots, W^{r+s}\right)\sigma_{(r+s)}\left(W^2, W^2, W^3, \cdots, W^r, W^{r+1}, \cdots, W^{r+s}\right). \end{aligned}$$

Setting $W^1 = I_{n-1}, W^2 = \cdots = W^r = \chi$ and $W^{r+1} = \cdots = W^{r+s} = \underline{\chi}$ and rewriting in terms of the complete polarization (A.1), we derive

$$\begin{aligned} &\binom{n-1}{r+s}^2 \sigma^2_{(n-1)}(\underbrace{\chi, \cdots, \chi}_{r-1}, \underbrace{\underline{\chi}, \cdots, \underline{\chi}}_{s}, \underbrace{I_{n-1}, \cdots, I_{n-1}}_{(n-1)-(r-1+s)}) \\ \geq &\binom{n-1}{r+s}^2 \sigma_{(n-1)}(\underbrace{\chi, \cdots, \chi}_{r-2}, \underbrace{\underline{\chi}, \cdots, \underline{\chi}}_{s}, \underbrace{I_{n-1}, \cdots, I_{n-1}}_{(n-1)-(r-2+s)})\sigma_{(n-1)}(\underbrace{\chi, \cdots, \chi}_{r}, \underbrace{\underline{\chi}, \cdots, \underline{\chi}}_{s}, \underbrace{I_{n-1}, \cdots, I_{n-1}}_{(n-1)-(r+s)}) \end{aligned}$$

From Gårding's inequality, we also see that the equality holds if and only if χ and I_{n-1} are proportional.

On the other hand, using (A.1) and (A.2) again,

$$\begin{aligned} \sigma_{(n-1)}(\underbrace{\chi, \cdots, \chi}_{r-1}, \underbrace{\underline{\chi}, \cdots, \underline{\chi}}_{s}, \underbrace{I_{n-1}, \cdots, I_{n-1}}_{(n-1)-(r-1+s)}) &= \frac{1}{\binom{n-1}{r-1+s}}\sigma_{(r-1+s)}(\underbrace{\chi, \cdots, \chi}_{r-1}, \underbrace{\underline{\chi}, \cdots, \underline{\chi}}_{s}) \\ &= \frac{1}{\binom{n-1}{r-1+s}} P_{r-1,s}(\chi, \underline{\chi}), \\ \sigma_{(n-1)}(\underbrace{\chi, \cdots, \chi}_{r-2}, \underbrace{\underline{\chi}, \cdots, \underline{\chi}}_{s}, \underbrace{I_{n-1}, \cdots, I_{n-1}}_{(n-1)-(r-2+s)}) &= \frac{1}{\binom{n-1}{r-2+s}}\sigma_{(r-2+s)}(\underbrace{\chi, \cdots, \chi}_{r-2}, \underbrace{\underline{\chi}, \cdots, \underline{\chi}}_{s}) \\ &= \frac{1}{\binom{n-1}{r-2+s}} P_{r-2,s}(\chi, \underline{\chi}), \\ \sigma_{(n-1)}(\underbrace{\chi, \cdots, \chi}_{r}, \underbrace{\underline{\chi}, \cdots, \underline{\chi}}_{s}, \underbrace{I_{n-1}, \cdots, I_{n-1}}_{(n-1)-(r+s)}) &= \frac{1}{\binom{n-1}{r+s}}\sigma_{(r+s)}(\underbrace{\chi, \cdots, \chi}_{r}, \underbrace{\underline{\chi}, \cdots, \underline{\chi}}_{s}) \\ &= \frac{1}{\binom{n-1}{r+s}} P_{r,s}(\chi, \underline{\chi}) \end{aligned}$$

Thus, we reach that

$$P^2_{r-1,s}(\chi, \underline{\chi}) \geq \frac{\binom{n-1}{r-1+s}^2}{\binom{n-1}{r-2+s}\binom{n-1}{r+s}} P_{r-2,s}(\chi, \underline{\chi}) \cdot P_{r,s}(\chi, \underline{\chi}).$$

□

Appendix B. The Existence of Conformal Killing-Yano forms

In this appendix, we show the existence of conformal Killing-Yano form for a class of warped-product manifold. We recall the following equivalent definition of conformal Killing-Yano p-forms using the twistor equation [31, Definition 2.1].

Definition B.1. *A $p-$form Q on an $n-$dimensional pseudo-Riemannian manifold (V, g) is said to be a conformal Killing-Yano form if Q satisfies the twistor equation*

$$D_X Q - \frac{1}{n+1} X \lrcorner dQ + \frac{1}{n-p+1} g(X) \wedge d^* Q = 0 \tag{B.1}$$

for all tangent vector X.

The main result of the appendix is the following existence theorem.

Theorem B.2. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be two open sets. Let G be a warped-product metric on $U \times V$ of the form*

$$R^2(y)\sigma_{ab}(x)dx^a dx^b + g_{ij}(y)dy^i dy^j.$$

Then $Q = R^{n+1}(y)\sqrt{\det \sigma_{ab}}dx^1 \wedge \cdots \wedge dx^n$ and $*Q = R(y)\sqrt{\det g_{ij}}dy^1 \wedge \cdots \wedge dy^m$ are both conformal Killing-Yano forms.

Proof. By [31, Lemma 2.3], the Hodge star-operator $*$ maps conformal Killing-Yano $p-$form into conformal Killing-Yano $(n+m-p)-$form. It suffices to verify that Q satisfies the twistor equation. Let $\omega^\alpha, \alpha = 1, \ldots, n+m$ be a local orthonormal coframe for G such that $\omega^1, \ldots, \omega^n$ is an orthonormal coframe for $R^2(y)\sigma_{ab}(x)dx^a dx^b$ on each slice $U \times \{c_1\}$ and $\omega^{n+1}, \ldots, \omega^{n+m}$ is an orthonormal coframe for $g_{ij}(y)dy^i \wedge dy^j$ on each slice $\{c_2\} \times V$. Let E_α be the dual frame to ω^α. If we write $\Omega = \omega^1 \wedge \cdots \wedge \omega^n$, then $Q = R\Omega$. From the structure equations

$$\begin{aligned} d\omega^a &= -\omega^a{}_b \wedge \omega^b - \omega^a{}_{n+i} \wedge \omega^{n+i} = dR \wedge \sigma^a - R\gamma^a{}_b \wedge \sigma^b \\ d\omega^{n+i} &= -\omega^{n+i}_{\ \ b} \wedge \omega^b - \omega^{n+i}_{\ \ n+j} \wedge \omega^{n+j}, \end{aligned}$$

we solve for the connection 1-forms

$$\omega^a{}_{n+i} = \frac{E_{n+i}(R)}{R}\omega^a, \quad \omega^a{}_b = \gamma^a{}_b$$

where $\gamma^a{}_b$ are the connection 1-forms with respect to the metric $\sigma_{ab}(x)dx^a dx^b$.

We compute each term in the twistor equation.

$$D_X Q = X(R)\Omega + R\nabla_X \Omega = X(R)\Omega - \sum_{i=1}^m E_{n+i}(R)\omega^{n+i} \wedge (X \lrcorner \Omega).$$

$$\begin{aligned}
X \lrcorner d\Omega &= X \lrcorner \sum_{i=1}^{m} \left(-\omega^{1}{}_{n+i} \wedge \omega^{n+i} \wedge \omega^{2} \wedge \cdots \wedge \omega^{n} + \omega^{1} \wedge \omega^{2}{}_{i} \wedge \omega^{n+i} \wedge \cdots \wedge \omega^{n} - \cdots\right) \\
&= X \lrcorner \sum_{i=1}^{m} n \left(\frac{E_{n+i}(R)}{R} \omega^{n+i} \wedge \Omega\right) \\
&= n \sum_{i=1}^{m} \frac{E_{n+i}(R)}{R} X \lrcorner (\omega^{n+i} \wedge \Omega) \\
&= n \sum_{i=1}^{m} \frac{E_{n+i}(R)}{R} \left(\omega^{n+i}(X)\Omega - \omega^{n+i} \wedge (X \lrcorner \Omega)\right)
\end{aligned}$$

This implies that

$$\begin{aligned}
X \lrcorner dQ &= X \lrcorner (dR \wedge \Omega + R d\Omega) \\
&= X(R)\Omega - dR \wedge (X \lrcorner \Omega) + n \sum_{i=1}^{m} E_{n+i}(R) \left(\omega^{n+i}(X)\Omega - \omega^{n+i} \wedge (X \lrcorner \Omega)\right).
\end{aligned}$$

On the other hand, $d^*Q = 0$ since R only depends on y. Putting these facts together, we verify that Q satisfies the twistor equation

$$\begin{aligned}
&D_X Q - \frac{1}{n+1} X \lrcorner dQ + \frac{1}{m+1} g(X) \wedge d^*Q \\
&= X(R)\Omega - \sum_{i=1}^{m} E_{n+i}(R) \omega^{n+i} \wedge (X \lrcorner \Omega) \\
&\quad - \frac{1}{n+1}\Big(X(R)\Omega - dR \wedge (X \lrcorner \Omega) - n \sum_{i=1}^{m} E_{n+i}(R) \left(\omega^{n+i}(X)\Omega - \omega^{n+i} X \lrcorner \Omega\right)\Big) \\
&= \frac{n}{n+1} X(R)\Omega - \frac{1}{n+1} E_{n+i}(R) \omega^{n+i} \wedge (X \lrcorner \Omega) \\
&\quad + \frac{1}{n+1} dR \wedge (X \lrcorner \Omega) - \frac{n}{n+1} E_{n+i}(R) \omega^{n+i}(X)\Omega \\
&= 0.
\end{aligned}$$

We use the fact that R only depends on y in the last equality. $\square$

We have the following existence result, generalizing the fact that $r dr \wedge dt$ is a conformal Killing-Yano two-form on the Minkowski and Schwarzschild spacetime.

Corollary B.3. *Let (V, g) be a warped product manifold with*

$$g = g_{tt}(t,r)dt^2 + 2g_{tr}(t,r)dtdr + g_{rr}(t,r)dr^2 + r^2 (g_N)_{ab} dx^a dx^b \tag{B.2}$$

where (N, g_N) is an $(n-1)-$dimensional Riemannian manifold. Then the two-form

$$Q = r \sqrt{\left|\det \begin{pmatrix} g_{tt} & g_{tr} \\ g_{rt} & g_{rr} \end{pmatrix}\right|}\ dr \wedge dt$$

is a conformal Killing-Yano two-form on (V, g).

Appendix C. Curvature tensors on Schwarzschild spacetime

We consider $(n+1)$-dimensional (exterior) Schwarzschild spacetime with the metric (1.2). The spacetime admits a conformal Killing-Yano tensor $Q = rdr \wedge dt$. Let Q^2 be the symmetric 2-tensor given by

$$(Q^2)_{\alpha\beta} = Q_\alpha{}^\gamma Q_{\gamma\beta}. \tag{C.1}$$

Lemma C.1. *The curvature tensor of Schwarzschild spacetime can be expressed as*

$$\begin{aligned}\bar{R}_{\alpha\beta\gamma\delta} = &\frac{2m}{r^n}\left(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\right) - \frac{n(n-1)m}{r^{n+2}}\left(\frac{2}{3}Q_{\alpha\beta}Q_{\gamma\delta} - \frac{1}{3}Q_{\alpha\gamma}Q_{\delta\beta} - \frac{1}{3}Q_{\alpha\delta}Q_{\beta\gamma}\right)\\ &- \frac{nm}{r^{n+2}}\left(\bar{g}\circ Q^2\right)_{\alpha\beta\gamma\delta}\end{aligned} \tag{C.2}$$

where $(\bar{g}\circ Q^2)_{\alpha\beta\gamma\delta} = \bar{g}_{\alpha\gamma}(Q^2)_{\beta\delta} - \bar{g}_{\alpha\delta}(Q^2)_{\beta\gamma} + \bar{g}_{\beta\delta}(Q^2)_{\alpha\gamma} - \bar{g}_{\beta\gamma}(Q^2)_{\alpha\delta}$

Proof. Denote $f^2 = 1 - \frac{2m}{r^{n-2}}$. Let $E_1, E_2, \ldots, E_{n+1}$ be the orthonormal frames for $\bar{g}$ with $E_{n+1} = \frac{1}{f}\frac{\partial}{\partial t}, E_n = f\frac{\partial}{\partial r}$ and $E_i, i = 1, \ldots, n-1$ tangent to the sphere of symmetry. We have

$$\begin{aligned}\bar{R}(E_{n+1}, E_n, E_{n+1}, E_n) &= -\frac{m(n-1)(n-2)}{r^n}\\ \bar{R}(E_{n+1}, E_i, E_{n+1}, E_j) &= \frac{m(n-2)}{r^n}\delta_{ij}\\ \bar{R}(E_n, E_i, E_n, E_j) &= -\frac{m(n-2)}{r^n}\delta_{ij}\\ \bar{R}(E_i, E_j, E_k, E_l) &= \frac{2m}{r^n}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\end{aligned}$$

Except for the symmetries of the curvature tensors, the other components are zero.

On the other hand, we have $Q(E_n, E_{n+1}) = r, (Q^2)(E_{n+1}, E_{n+1}) = -r^2$, and $(Q^2)(E_n, E_n) = r^2$. Let $b(Q) = \frac{2}{3}Q_{\alpha\beta}Q_{\gamma\delta} - \frac{1}{3}Q_{\alpha\gamma}Q_{\delta\beta} - \frac{1}{3}Q_{\alpha\delta}Q_{\beta\gamma}$. The following table lists the nonzero components for the $(0,4)$-tensors involved.

T	$\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}$	$b(Q)$	$(\bar{g}\circ Q^2)_{\alpha\beta\gamma\delta}$
$T(E_{n+1}, E_n, E_{n+1}, E_n)$	-1	r^2	$-2r^2$
$T(E_{n+1}, E_i, E_{n+1}, E_j)$	$-\delta_{ij}$	0	$-r^2\delta_{ij}$
$T(E_n, E_i, E_n, E_j)$	δ_{ij}	0	$r^2\delta_{ij}$
$T(E_i, E_j, E_k, E_l)$	$\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$	0	0

Suppose $\bar{R}_{\alpha\beta\gamma\delta} = A\frac{m}{r^n}\left(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\right) + B\frac{m}{r^{n+2}}\left(\frac{2}{3}Q_{\alpha\beta}Q_{\gamma\delta} - \frac{1}{3}Q_{\alpha\gamma}Q_{\delta\beta} - \frac{1}{3}Q_{\alpha\delta}Q_{\beta\gamma}\right) + C\frac{m}{r^{n+2}}(\bar{g}\circ Q^2)_{\alpha\beta\gamma\delta}$. We can solve for $A = 2, B = -n(n-1)$, and $C = -n$. □

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Mu-Tao Wang, Department of Mathematics, Columbia University, U.S.
E-mail address: mtwang@math.columbia.edu

Ye-Kai Wang, Department of Mathematics, Columbia University, U.S.
E-mail address: yw2293@columbia.edu

Xiangwen Zhang, Department of Mathematics, Columbia University, U.S.
E-mail address: xzhang@math.columbia.edu