RIGIDITY OF TIME-FLAT SURFACES IN THE MINKOWSKI SPACETIME

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Abstract. A time-flat condition on spacelike 2-surfaces in spacetime is considered here. This condition is analogous to constant torsion condition for curves in three dimensional space and has been studied in [2, 4, 5, 12, 13]. In particular, any 2-surface in a static slice of a static spacetime is time-flat. In this article, we address the question in the title and prove several local and global rigidity theorems for such surfaces in the Minkowski spacetime.

1. Introduction

The geometry of spacelike 2-surfaces in spacetime plays a crucial role in general relativity. Penrose's singularity theorem predicts future singularity formation from the existence of a trapped surface. A black hole is quasi-locally described by a marginally outer trapped surface. These conditions can be expressed in terms of the mean curvature vector field H of the 2-surface. H is the unique normal vector field determined by the variation of the area functional and is ultimately connected to the warping of spacetime in the vicinity of the 2-surface. It is thus not surprising that several definitions of quasi-local mass in general relativity are closely related to the mean curvature vector field. In particular, both the Hawking mass [7] and the Brown-York-Liu-Yau mass [3, 8] involve the norm of the mean curvature vector field $|H|$. In the new definition of quasi-local mass in [12, 13], in addition to $|H|$, the direction of the mean curvature vector field is also utilized. When the mean curvature vector field is spacelike everywhere on Σ (thus $|H| > 0$), H defines a connection one-form α_H of the normal bundle (see Definition 2 for the precise definition of α_H). The quasi-local mass in [12, 13] is defined in terms of the induced metric σ on the surface, $|H|$, and α_H. In particular, the condition

$$div_\sigma(\alpha_H) = 0 \tag{1.1}$$

implies that the isometric embedding of Σ into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ is an optimal embedding in the sense of [12, 13]. Recently, Bray and Jauregui [2] discovered a very interesting monotonicity property of the Hawking mass along surfaces that satisfy the condition (1.1). Such surfaces are said to be "time-flat" in [2] and include all 2-surfaces in a time-symmetric initial data

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set. A natural rigidity question raised by Bray [1] is "Must such a surface in the Minkowski spacetime lie in a totally geodesic $\mathbb{R}^3$?" Due to the existence of non-flat curves in $\mathbb{R}^3$ with constant torsion, some global condition needs to be imposed in order for the rigidity property to hold. In this article, we prove several global and local rigidity theorems for time-flat surfaces in the Minkowski spacetime under various conditions. The local rigidity theorem holds in any higher dimensional Minkowski spacetime:

Theorem 4. *Suppose Σ is a mean convex hypersurface which lies in a totally geodesic $\mathbb{R}^n$ in the $n+1$ dimensional Minkowski spacetime $\mathbb{R}^{n,1}$, then Σ is locally rigid as a time-flat $n-1$ dimensional submanifold in $\mathbb{R}^{n,1}$. In other words, any infinitesimal deformation of Σ that preserves the time-flat condition must be a deformation in the $\mathbb{R}^n$ direction, a deformation that is induced by a Lorentz transformation of $\mathbb{R}^{n,1}$, or a combination of these two types of deformations.*

We also proved two global rigidity theorems:

Theorem 5. *Suppose Σ is a time-flat 2-surface in $\mathbb{R}^{3,1}$ such that $\alpha_H = 0$ and Σ is a topological sphere, then Σ lies in a totally geodesic hyperplane.*

Theorem 6. *Suppose Σ is a time-flat 2-surface in $\mathbb{R}^{3,1}$ such that*

1. *The induced metric on Σ is axially symmetric and of positive Gaussian curvature.*
2. *Σ can be written as the graph of an axially symmetric function τ over a convex surface in $\mathbb{R}^3$.*

Then Σ lies in a total geodesic hyperplane in $\mathbb{R}^{3,1}$.

Theorem 4 follows from applying the Reilly formula to the linearized equation (3.7). Theorem 6 uses a mean curvature comparison lemma derived in [5] and the minimizing property of critical points of the Wang-Yau energy. Theorem 4 is proved in §3, Theorem 5 is proved in §4, and Theorem 6 is proved in §5.

2. Geometry of spacelike 2-surface in spacetime

Let N be a time-oriented spacetime. Denote the Lorentzian metric on N by $\langle \cdot, \cdot \rangle$ and covariant derivative by ∇^N. Let Σ be a closed space-like two-surface embedded in N. Denote the induced Riemannian metric on Σ by σ and the gradient and Laplace operator of σ by ∇ and Δ, respectively.

Given any two tangent vector X and Y of Σ, the second fundamental form of Σ in N is given by $\mathrm{II}(X,Y) = (\nabla^N_X Y)^\perp$ where $(\cdot)^\perp$ denotes the projection onto the normal bundle of Σ. The mean curvature vector is the trace of the second fundamental form, or $H = tr_\Sigma \mathrm{II} = \sum_{a=1}^2 \mathrm{II}(e_a, e_a)$ where $\{e_1, e_2\}$ is an orthonormal basis of the tangent bundle of Σ.

The normal bundle is of rank two with structure group $SO(1,1)$ and the induced metric on the normal bundle is of signature $(-,+)$. Since the Lie algebra of $SO(1,1)$ is isomorphic to $\mathbb{R}$, the connection form of the normal bundle is a genuine 1-form that depends on the choice of the normal frames. The curvature of the normal bundle is then given by an exact 2-form which reflects the fact that any $SO(1,1)$ bundle is topologically trivial. Connections of different choices of normal frames differ by an exact form. We define (see [13])

Definition 1. *Let e_3 be a space-like unit normal along Σ, the connection one-form determined by e_3 is defined to be*

$$\alpha_{e_3} = \langle \nabla^N_{(\cdot)} e_3, e_4 \rangle \tag{2.1}$$

where e_4 is the future-directed time-like unit normal that is orthogonal to e_3.

Definition 2. *Suppose the mean curvature vector field H of Σ in N is a spacelike vector field. The connection one-form in mean curvature gauge is*

$$\alpha_H = \langle \nabla^N_{(\cdot)} e_3, e_4 \rangle,$$

where $e_3 = -\frac{H}{|H|}$ and e_4 is the future-directed timelike unit normal that is orthogonal to e_3.

3. Local rigidity of mean convex hypersurfaces in $\mathbb{R}^n \subset \mathbb{R}^{n,1}$

The local rigidity theorem holds in higher dimensional Minkowski spacetime as well. Let Σ be a closed embedded spacelike codimension-2 submanifold in Minkowski spacetime $\mathbb{R}^{n,1}$ and σ be its induced metric. Suppose that the mean curvature vector H of Σ is spacelike. Let $e_n = -\frac{H}{|H|}$ and e_{n+1} be the unit future timelike normal that is orthogonal to e_n. Let $\alpha_H = \langle \nabla_{(\cdot)} e_n, e_{n+1} \rangle$ be the connection one-form on the normal bundle of Σ determined by the mean curvature gauge.

Definition 3. *We say Σ is time-flat if $div_\sigma(\alpha_H) = 0$.*

The local rigidity problem can be formulated as follows. Suppose Σ is time-flat and is given by an embedding X. Suppose V is a smooth vector field along Σ such that the image of $X(s) = X + sV$ is infinitesimally time-flat in the sense the derivatives of $div_\sigma(\alpha_H)$ along the image with respect to s is zero when $s = 0$. Do all such V correspond to trivial deformations?

It is easy to see that submanifolds lying in a totally geodesic slice is time-flat. We assume $\partial\Omega = \Sigma \subset \{t = 0\} = \mathbb{R}^n$. It is clear that any deformation in the $\mathbb{R}^n$ direction preserves the time-flat condition. On the other hand, a Lorentz transformation preserves the geometry of Σ and thus preserves the time-flat condition.

Let ∇, Δ denote the covariant derivative and Laplacian of the induced metric σ. Let h_{ab}, h be the second fundamental form and mean curvature of $\Sigma \subset \mathbb{R}^n$ with respect to the outward unit normal ν.

Theorem 4. *Suppose Σ is a mean convex hypersurface which lies in a totally geodesic $\mathbb{R}^n$ in the $n+1$ dimensional Minkowski spacetime $\mathbb{R}^{n,1}$, then Σ is locally rigid as a time-flat $n-1$ dimensional submanifold in $\mathbb{R}^{n,1}$. In other words, any infinitesimal deformation of Σ that preserves the time-flat condition must be a deformation in the $\mathbb{R}^n$ direction, a deformation that is induced by a Lorentz transformation of $\mathbb{R}^{n,1}$, or a combination of these two types of deformations.*

Proof. In this proof, we denote α_H by α. Since $\delta(div_\sigma \alpha)$ depends linearly on infinitesimal deformation and any deformation in $\mathbb{R}^n$ corresponds to trivial deformations, it suffices to consider deformations in the time direction. Let $V = f\frac{\partial}{\partial t}$ for a smooth function f defined on Σ be such an infinitesimal deformation and $X(s) = (\tau(s), X^1(s), \dots, X^n(s))$ be the corresponding deformation. Since we only vary the surface in the time direction, $X^i(s) = X^i(0)$ for $i = 1, \dots, n$, and $\frac{\partial}{\partial s}\big|_{s=0}\tau(s) = f$.

We start by computing the variation of $div_\sigma \alpha$. The induced metrics satisfy

$$\sigma(s)_{ab} = \sigma_{ab} - \frac{\partial \tau(s)}{\partial u^a}\frac{\partial \tau(s)}{\partial u^b}. \tag{3.1}$$

Since $\tau(0) = 0$, $\delta\sigma = 0$. Let Δ_s be the Laplacian of the induced metric on $X(s)$. We have

$$H = (\Delta_s \tau(s), \Delta_s X^1, \dots, \Delta_s X^n). \tag{3.2}$$

$\delta\sigma = 0$ implies the infinitesimal variation of Laplacian is zero. Therefore, we have

$$\delta H = (\Delta f)\frac{\partial}{\partial t} \tag{3.3}$$

$$\delta |H|^2 = 2\langle \delta H, -h e_n\rangle = 0 \tag{3.4}$$

$$\delta e_n = -\frac{\delta H}{h} + \frac{\delta |H|}{h^2} H = -\frac{\Delta f}{h}\frac{\partial}{\partial t} \tag{3.5}$$

From $0 = \delta\langle e_{n+1}, \frac{\partial}{\partial u^a}\rangle = \delta\langle e_{n+1}, e_n\rangle$, we have

$$\delta e_{n+1} = \nabla f - \frac{\Delta f}{h} e_n \tag{3.6}$$

Putting these facts together, we get

$$\begin{aligned}
(\delta\alpha)_a &= \delta\langle D_a e_n, e_{n+1}\rangle \\
&= \langle D_{\frac{\partial f}{\partial u^a}\frac{\partial}{\partial t}} e_n, e_{n+1}\rangle + \langle D_a\left(-\frac{\Delta f}{h}\right)\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\rangle + \langle D_a e_n, \nabla f + \Delta f e_n\rangle \\
&= \nabla_a\left(\frac{\Delta f}{h}\right) + h_{ab}\nabla^b f
\end{aligned}$$

and

$$\delta(div_\sigma \alpha) = \Delta\left(\frac{\Delta f}{h}\right) + \nabla^a\left(h_{ab}\nabla^b f\right). \tag{3.7}$$

We remark that the linearization of this operator was also derived in [4, 9]. Let u solve the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \Sigma \end{cases}$$

We recall the Reilly formula on $\mathbb{R}^n$

$$\int_\Omega |D^2 u|^2 = -\int_\Sigma \left(h^{ab}\nabla_a f\nabla_b f + 2\Delta f e_n(u) + h(e_n(u))^2\right). \tag{3.8}$$

This was used in [9] to derive minimizing property of the Wang-Yau quasi-local energy. On the other hand, multiply $\delta(div_\sigma \alpha) = 0$ by f and integrate over Σ to get

$$\int_\Sigma \frac{(\Delta f)^2}{h} - h^{ab}\nabla_a f \nabla_b f = 0. \tag{3.9}$$

Adding (3.8) and (3.9) together and completing square, we obtain

$$\int_\Omega |D^2 u|^2 + \int_\Sigma \left(\frac{\Delta f}{\sqrt{h}} + \sqrt{h} e_n(u) \right)^2 = 0.$$

Hence u is a linear function up to a constant. □

4. Global rigidity for surfaces with $\alpha_H = 0$

The global rigidity holds true if one assumes in addition that $\alpha_H = 0$.

Theorem 5. *Suppose Σ is a time-flat 2-surface in $\mathbb{R}^{3,1}$ such that $\alpha_H = 0$ and Σ is a topological sphere, then Σ lies in a totally geodesic $\mathbb{R}^3$*

Proof. Denote by $e_3 = -\frac{H}{|H|}$ and e_4 to be the unit future timelike normal that is orthogonal to e_3. The second fundamental form of Σ can be written as $h^3_{ab} e_3 - h^4_{ab} e_4$ and h^4_{ab} is trace-free. The Codazzi equation for h^4_{ab} reads

$$\nabla^a h^4_{ab} - \nabla_b tr_\sigma h^4 + (\alpha_H)^a h^4_{ab} - tr_\sigma h^4 (\alpha_H)_b = 0.$$

Since $\alpha_H = 0$ and h^4_{ab} is trace-free, this reduces to

$$\nabla^a h^4_{ab} = 0.$$

A divergence-free symmetric trace-free 2-tensor corresponds to a holomorphic quadratic differential, which must vanish since Σ is a topological sphere. In particular, h^3_{ab} and h^4_{ab} can be diagonalized simultaneously. By [14], it follows that Σ lies in a umbilical hypersurface in $\mathbb{R}^{3,1}$. If Σ lies in a totally geodesic $\mathbb{R}^3$ then the proof is finished. If Σ lies in a hyperbola in $\mathbb{R}^{3,1}$ then the position vector is a unit timelike normal of Σ and the connection form determined by the position vector is also zero. Hence, the angle between e_4 and the position vector is constant. It follows that Σ has constant mean curvature in the hyperbola. Hence, Σ is a round sphere in the hyperbola and thus the intersection of a totally geodesic $\mathbb{R}^3$ with the hyperbola. □

5. Wang–Yau quasi-local energy.

In the next section, the positivity of Wang–Yau quasi-local energy is used to prove the global rigidity of time-flat axially symmetric surfaces in $\mathbb{R}^{3,1}$. We recall the definition of Wang–Yau quasi-local energy in this section. Let Σ be a spacelike surface in a spacetime N. The definition of Wang–Yau quasi-local energy relies on the physical data on Σ which consist of the induced metric σ, the norm of the mean curvature vector $|H| > 0$, and the connection one-form α_H. Given the triple of physical data $\{\sigma, |H|, \alpha_H\}$, one assigns a quasi-local energy for each pair of isometric embedding X of Σ into $\mathbb{R}^{3,1}$ and future-directed unit timelike vector T_0 in $\mathbb{R}^{3,1}$. In terms of $\tau = -X \cdot T_0$, the quasi-local energy is defined to be

$$E(\Sigma,\tau) = \int_{\widehat{\Sigma}} \widehat{H} dv_{\widehat{\Sigma}} - \int_{\Sigma} \left[\sqrt{1+|\nabla\tau|^2} \cosh\theta |H| - \nabla\tau \cdot \nabla\theta - \alpha_H(\nabla\tau) \right] dv_{\Sigma}, \tag{5.1}$$

where ∇ and Δ are the covariant derivative and Laplace operator with respect to σ, and θ is defined by $\sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$. Finally, $\widehat{\Sigma}$ is the projection of $X(\Sigma)$ onto the complement of T_0 and $\widehat{H}$ is the mean curvature of $\widehat{\Sigma}$ in $\mathbb{R}^3$. In $E(\Sigma,\tau)$, the first argument Σ represents a physical surface in spacetime with the data $(\sigma, |H|, \alpha_H)$, while the second argument τ indicates an isometric embedding of the induced metric into $\mathbb{R}^{3,1}$ with time function τ with respect to T_0. As remarked in [5], no information is lost by choosing a fixed future-directed unit timelike vector T_0 and considering $E(\Sigma,\tau)$ as a energy functional on functions on Σ.

6. Global rigidity for axially symmetric and time-flat surfaces in $\mathbb{R}^{3,1}$.

In this section, we prove the following theorem for the global rigidity of time-flat axially symmetric surfaces in $\mathbb{R}^{3,1}$.

Theorem 6. *Suppose Σ is a time-flat 2-surface in $\mathbb{R}^{3,1}$ such that*

(1) *The induced metric on Σ is axially symmetric and of positive Gaussian curvature.*
(2) *Σ can be written as the graph of an axially symmetric function τ over a convex surface in $\mathbb{R}^3$.*

Then Σ lies in a total geodesic hyperplane in $\mathbb{R}^{3,1}$.

Proof. Let σ be the induced metric on Σ and K be the Gaussian curvature of σ. By condition (1) and the isometric embedding theorem of Nirenberg and Pogorelov [10, 11], there exists a unique isometric embedding of Σ into $\mathbb{R}^3$. Denote the image by Σ_0, which is a convex surface in $\mathbb{R}^3$.

From condition (2) and equation (3.3) of [13], it follows

$$K + \frac{det(\nabla^2\tau)}{1+|\nabla\tau|^2} > 0.$$

This together with $K > 0$ implies that, for any $0 \le s \le 1$,

$$K + \frac{det(\nabla^2 s\tau)}{1+|\nabla s\tau|^2} > 0.$$

Hence, there is a unique isometric embedding of the surface into $\mathbb{R}^{3,1}$ with time function $s\tau$ for any $0 \le s \le 1$ by Theorem 3.1 of [13]. Denote the image by Σ_s and its mean curvature vector by H_s. Up to a Lorentz transformation, we can assume that $\Sigma_1 = \Sigma$. Let $E(\Sigma_0, s\tau)$ be the quasi-local energy for the surface Σ_0 with reference embedding Σ_s.

Lemma 3 of [5] gives the following inequality about the mean curvature vectors of Σ_s

$$|H_0| \ge |H_s| \tag{6.1}$$

for any $0 \le s \le 1$. Namely, for axially symmetric surfaces with the same induced metric in $\mathbb{R}^{3,1}$, the isometric embedding into $\mathbb{R}^3$ has the largest norm of the mean curvature vector.

We prove the theorem by combining equation (6.1) with an inequality obtained from the positivity of Wang–Yau quasi-local energy.

We treat Σ_0 in the totally geodesic $\mathbb{R}^3$ in $\mathbb{R}^{3,1}$ as a surface in a spacetime. Namely, it comes with the physical data $\{\sigma, |H_0|, 0\}$. We consider the Wang–Yau quasi-local energy for other isometric embedding of Σ_0 into $\mathbb{R}^{3,1}$. Fixing a future-directed unit timelike vector T_0 in $\mathbb{R}^{3,1}$, the quasi-local energy becomes an functional $E(\Sigma_0, f)$ for functions on Σ_0. Since the data $\{\sigma, |H_0|, 0\}$ comes from the surface Σ_0 embedded in $\mathbb{R}^{3,1}$, $E(\Sigma_0, 0) = 0$ and it is natural to expect that quasi-local energy for other isometric embedding is non-negative. Namely, one expects that $f = 0$ is the global minimum of the quasi-local energy functional $E(\Sigma_0, f)$. The following lemma shows that this holds in axially symmetric cases.

Lemma 7. *Let Σ_0 be any convex and axially symmetric surface in a totally geodesic $\mathbb{R}^3$ in $\mathbb{R}^{3,1}$ and K be the Gaussian curvature of the induced metric σ. Suppose τ is an axially symmetric function on Σ_0 such that*

$$K + \frac{det(\nabla^2 \tau)}{1 + |\nabla \tau|^2} > 0.$$

Then

$$E(\Sigma_0, \tau) \geq 0.$$

Namely, $f = 0$ is the global minimum of the quasi-local energy functional $E(\Sigma_0, f)$ among all axially symmetric time functions.

Proof. This follows from Lemma 2 of [5] and equation (6.1). □

By Proposition 3.2 of [12], $E(\Sigma_0, \tau)$ can be written as

$$\begin{aligned}E(\Sigma_0, \tau) = &\int_\Sigma \sqrt{(1 + |\nabla \tau|^2)|H_1|^2 + (\Delta \tau)^2} - \Delta \tau \sinh^{-1}(\frac{\Delta \tau}{|H_1|\sqrt{1 + |\nabla \tau|^2}}) - \alpha_{H_1}(\nabla \tau)\\ &- \int_\Sigma \sqrt{(1 + |\nabla \tau|^2)|H_0|^2 + (\Delta \tau)^2} - \Delta \tau \sinh^{-1}(\frac{\Delta \tau}{|H_0|\sqrt{1 + |\nabla \tau|^2}}) - \alpha_{H_0}(\nabla \tau).\end{aligned}$$

Since Σ is time-flat and $\alpha_{H_0} = 0$, it follows that

$$\begin{aligned}E(\Sigma_0, \tau) = &\int_\Sigma \sqrt{(1 + |\nabla \tau|^2)|H_1|^2 + (\Delta \tau)^2} - \Delta \tau \sinh^{-1}(\frac{\Delta \tau}{|H_1|\sqrt{1 + |\nabla \tau|^2}})\\ &- \int_\Sigma \sqrt{(1 + |\nabla \tau|^2)|H_0|^2 + (\Delta \tau)^2} - \Delta \tau \sinh^{-1}(\frac{\Delta \tau}{|H_0|\sqrt{1 + |\nabla \tau|^2}})\\ \geq &0\end{aligned} \tag{6.2}$$

Combining equation (6.1) and (6.2), it follows that $|H_0| = |H_1|$ and $E(\Sigma_0, \tau) = 0$. From the proof of Lemma 2 of [5], it follows that $E(\Sigma_0, s\tau) = 0$ for all s. In particular,

$$\partial_s^2 \big|_{s=0} E(\Sigma_0, s\tau) = 0$$

Applying Theorem 1.3 and Lemma 2.1 of [9] to the critical point $f = 0$ of the quasi-local energy functional $E(\Sigma_0, f)$, one concludes that τ is a linear combination of the coordinate functions of Σ_0 up to a constant. It follows that Σ lies in a totally geodesic $\mathbb{R}^3$ in $\mathbb{R}^{3,1}$. □

A corollary of Theorem 6 is a uniqueness theorem for isometric embeddings of an axially symmetric metric into $\mathbb{R}^{3,1}$.

Corollary 8. *Let σ be an axially-symmetric metric on S^2 with positive Gaussian curvature. Suppose X is a time-flat isometric embedding of σ into $\mathbb{R}^{3,1}$ such that the image can be written as the graph of an axially-symmetric function over a convex surface in $\mathbb{R}^3$. Then X must lie in a totally geodesic hyperplane.*

In view of the classical rigidity theorem of isometric embeddings [6] for metrics of positive Gaussian curvature into $\mathbb{R}^3$, the following conjecture is a natural extension by imposing the additional time-flat condition to make the problem a well-determined system:

Conjecture 9. *Let σ be a metric on S^2 with positive Gaussian curvature. Suppose X is a time-flat isometric embedding of σ into $\mathbb{R}^{3,1}$ such that the image can be written as the graph over a convex surface in $\mathbb{R}^3$. Then the image of X must lie in a totally geodesic hyperplane.*

References

[1] H. Bray, *Personal communications.*

[2] H. Bray and J. Jauregui, *preprint.*

[3] J. D. Brown and J. W. York, *Quasilocal energy in general relativity,* Mathematical aspects of classical field theory (Seattle, WA, 1991), 129-142, Contemp. Math. **132**, Amer. Math. Soc., Providence, RI, 1992.

[4] P. Chen, M.-T. Wang, and S.-T. Yau, *Evaluating quasilocal energy and solving optimal embedding equation at null infinity,* Comm. Math. Phys. **308** (2011), no.3, 845–863.

[5] P. Chen, M.-T. Wang ,and S.-T. Yau, *Minimizing properties of critical points of quasi-local energy,* to appear in Comm. Math. Phys. arXiv:1302.5321

[6] S. Cohn-Vossen, *Unstarre geschlossene Flächen,* (German) Math. Ann. **102** (1930), no. 1, 10–29.

[7] S. W. Hawking, *Gravitational radiation in an expanding universe*, J. Math. Phys., 9, 598– 604, (1968).

[8] C.C. Liu and S.T. Yau, *Positivity of quasilocal mass,* Phys. Rev. Lett. 90, 231102 (2003).

[9] P. Miao, L.-F. Tam and N.Q. Xie, *Critical points of Wang-Yau quasi-local energy.* Ann. Henri Poincaré. 12 (5): 987-1017, 2011

[10] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large,* Comm. Pure Appl. Math. **6**, (1953), 337–394.

[11] A. V. Pogorelov, *Regularity of a convex surface with given Gaussian curvature,* (Russian) Mat. Sbornik N.S. **31**(73), (1952). 88–103.

[12] M.-T. Wang and S.-T. Yau, *Quasilocal mass in general relativity,* Phys. Rev. Lett. **102** (2009), no. 2, no. 021101.

[13] M.-T. Wang and S.-T. Yau, *Isometric embeddings into the Minkowski space and new quasi-local mass,* Comm. Math. Phys. **288** (2009), no. 3, 919–942.

[14] S. T. Yau, Submanifolds with constant mean curvature. I, II, Amer. J. Math. **96** (1974), 346–366; ibid. **97** (1975), 76–100.