A MINKOWSKI-TYPE INEQUALITY FOR HYPERSURFACES IN THE ANTI-DESITTER-SCHWARZSCHILD MANIFOLD

SIMON BRENDLE, PEI-KEN HUNG, AND MU-TAO WANG

Abstract. We prove a sharp inequality for hypersurfaces in the n-dimensional Anti-deSitter-Schwarzschild manifold for general $n \geq 3$. This inequality generalizes the classical Minkowski inequality [19] for surfaces in the three dimensional Euclidean space. The proof relies on a new monotonicity formula for inverse mean curvature flow, and uses a geometric inequality established in [4].

1. Introduction

The classical Minkowski inequality for a closed convex surface Σ in $\mathbb{R}^3$ states that

$$\int_\Sigma H \, d\mu \geq \sqrt{16\pi \, |\Sigma|},$$

where H is the mean curvature, the trace of the second fundamental form, and $|\Sigma|$ is the area of Σ. For a convex hypersurface Σ in $\mathbb{R}^n$, we have

$$\int_\Sigma H \, d\mu \geq (n-1) \, |S^{n-1}|^{\frac{1}{n-1}} \, |\Sigma|^{\frac{n-2}{n-1}}.$$

This was generalized to a mean convex and star-shaped surface using the method of inverse mean curvature flow (cf. [13]). Very recently, Huisken [14] showed that the assumption that Σ is star-shaped can be replaced by the assumption that Σ is outward-minimizing. Gallego and Solanes [9] have obtained a generalization of Minkowski's inequality to the hyperbolic three space; however, this result does not seem to be sharp.

In this paper, we extend Minkowski's inequality to the case of surfaces in the Anti-deSitter Schwarzschild manifold. Let us recall the definition of the Anti-deSitter-Schwarzschild manifold. We fix a real number $m > 0$, and let s_0 denote the unique positive solution of the equation $1 + s_0^2 - m \, s_0^{2-n} = 0$. We then consider the manifold $M = S^{n-1} \times [s_0, \infty)$ equipped with the Riemannian metric

$$\bar{g} = \frac{1}{1 + s^2 - m \, s^{2-n}} \, ds \otimes ds + s^2 \, g_{S^{n-1}},$$

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where $g_{S^{n-1}}$ is the standard round metric on the unit sphere S^{n-1}. The sectional curvatures of $(M, \bar{g})$ approach -1 near infinity, so $\bar{g}$ is asymptotically hyperbolic. Moreover, the scalar curvature of $(M, \bar{g})$ equals $-n(n-1)$. The boundary $\partial M = S^{n-1} \times \{s_0\}$ is referred to as the horizon.

The Anti-deSitter Schwarzschild spaces are examples of static spaces. If we define

$$f = \sqrt{1 + s^2 - m\, s^{2-n}}, \tag{1}$$

then the function f satisfies

$$(\bar{\Delta} f)\, \bar{g} - \bar{D}^2 f + f\, \mathrm{Ric} = 0. \tag{2}$$

Taking the trace in (2) gives $\bar{\Delta} f = n f$.

In general, a Riemannian metric is called static if it satisfies (2) for some positive function f. The condition (2) guarantees that the Lorentzian warped product $-f^2\, dt \otimes dt + \bar{g}$ is a solution of Einstein's equations.

We now state the main result of this paper:

Theorem 1. *Let Σ be a compact mean convex, star-shaped hypersurface Σ in the AdS-Schwarzschild space, and let Ω denote the region bounded by Σ and the horizon ∂M. Then*

$$\begin{aligned} &\int_\Sigma f\, H\, d\mu - n(n-1) \int_\Omega f\, d\mathrm{vol} \\ &\geq (n-1)\, |S^{n-1}|^{\frac{1}{n-1}} \left(|\Sigma|^{\frac{n-2}{n-1}} - |\partial M|^{\frac{n-2}{n-1}} \right). \end{aligned}$$

Moreover, equality holds if and only if Σ is a coordinate sphere, i.e. $\Sigma = S^{n-1} \times \{s\}$ for some number $s \in [s_0, \infty)$.

If we send $m \to 0$, then $s_0 \to 0$ and the AdS-Schwarzschild metric reduces to hyperbolic metric

$$g = \frac{1}{1 + s^2}\, ds \otimes ds + s^2\, g_{S^{n-1}}.$$

Moreover, the static potential becomes $f = \sqrt{1 + s^2} = \cosh r$, where r denotes the geodesic distance from the origin.

The rescaled metrics $m^{-\frac{2}{n-2}}\, \bar{g}$ converge to the standard Schwarzschild metric

$$g = \frac{1}{1 - s^{2-n}}\, ds \otimes ds + s^2\, g_{S^{n-1}}$$

as $m \to 0$. Furthermore, the static potential f converges to the static potential of the standard Schwarzschild manifold. Hence, Theorem 1 implies a sharp Minkowski-type inequality for surfaces in the Schwarzschild manifold.

Theorem 2. *Let Σ be a compact mean convex hypersurface Σ in the hyperbolic space $\mathbb{H}^n$ which is star-shaped with respect to the origin, and let Ω*

denote the region bounded by Σ. *Then*

$$\int_\Sigma (f\, H - (n-1)\, \langle \bar\nabla f, \nu\rangle)\, d\mu$$
$$\geq (n-1)\, |S^{n-1}|^{\frac{1}{n-1}}\, |\Sigma|^{\frac{n-2}{n-1}}.$$

Moreover, equality holds if and only if Σ *is a geodesic sphere centered at the origin.*

After this paper was submitted for publication, we learned of a preprint by L. Lopes de Lima and F. Girão [16], where a related inequality for hypersurfaces in hyperbolic space is proved.

When the surface Σ is very close to the origin, Theorem 2 reduces to the classical Minkowski inequality in $\mathbb{R}^n$.

The classical Minkowski inequality in $\mathbb{R}^n$ has important applications in general relativity, see [12]. In particular, the total mean curvature integral appears in the definition of the Brown-York mass and Liu-Yau mass (cf. [17], [18]). Our motivation came from the work [23] in which a generalization of the positivity of Brown-York and Liu-Yau mass was considered when the reference space is a hyperbolic space. It was observed in [23] that the mean curvature integral should be replaced by a weighted one in order to recover the right expression of mass (see [23], Theorem 1.4). The weighting factor is related to the coordinate functions of the embedding of a hyperboloid into the Minkowski space. The time component of the embedding can be chosen to be $\cosh r$ which is the same as the static potential here. In fact, the same weighting factor was considered in [22] where another quasilocal mass with the hyperbolic space as reference was studied. We remark that the total mass for asymptotically hyperbolic manifolds has been considered by many authors, see e.g. [1], [6], [7], [20], [24], [25].

An important tool in our proof is the inverse mean curvature flow which has some amazing connections to general relativity as well. It was first employed by Huisken and Ilmanen [15] to prove the Riemannian Penrose inequality in general relativity. Bray and Neves [3] used it to obtain a classification theorem of three-manifolds by the Yamabe invariant. Neves [21] studied the inverse mean curvature flow on asymptotically hyperbolic spaces in connection to the Penrose inequality on such spaces.

We now give an outline of the proof of Theorem 1. We start from a given mean convex, star shaped hypersurface Σ_0, and evolve it by the inverse mean curvature flow. We show that the inverse mean curvature flow exists for all time, and that the evolving surfaces Σ_t remain star shaped for all $t \geq 0$. Moreover, we estimate the mean curvature and second fundamental form of Σ_t. More precisely, we prove that $|h_i^j - \delta_i^j| \leq O(t^2 e^{-\frac{2}{n-1}\, t})$. We note that the extra factor of t^2 can be removed, but we will not need this stronger estimate.

We next consider the quantity

$$Q(t) = |\Sigma_t|^{-\frac{n-2}{n-1}} \left(\int_{\Sigma_t} f\, H\, d\mu - n(n-1) \int_\Omega f\, d\mathrm{vol} + (n-1)\, s_0^{n-2}\, |S^{n-1}| \right),$$

where f is the static potential defined above. It turns out that $Q(t)$ is monotone decreasing along the inverse mean curvature flow. The proof of this monotonicity property uses the fact that $(M, \bar{g})$ is static. We also use the inequality

$$(n-1) \int_{\Sigma_t} \frac{f}{H}\, d\mu \geq n \int_{\Omega_t} f\, d\mathrm{vol} + s_0^n\, |S^{n-1}|$$

(cf. [4]). This inequality was used in [4] to prove a generalization of Alexandrov's theorem (see also [5]).

Finally, we study the limit of $Q(t)$ as $t \to \infty$. The roundness estimate for Σ_t is not strong enough to calculate the limit of $Q(t)$, and we expect that the limit of $Q(t)$ depends on the choice of the initial surface Σ_0. A similar issue arose in the work of Neves [21], where the limit of the Hawking mass was studied. However, we are able to give a lower bound for the limit of $Q(t)$. Using our estimate for the second fundamental form of Σ_t, we show that

$$\begin{aligned} Q(t) &\geq (n-1) \left(\int_{S^{n-1}} \lambda^{n-1}\, d\mathrm{vol}_{S^{n-1}} \right)^{-\frac{n-2}{n-1}} \\ &\quad \cdot \left(\frac{1}{2} \int_{S^{n-1}} \lambda^{n-4}\, |\nabla \lambda|^2_{g_{S^{n-1}}}\, d\mathrm{vol}_{S^{n-1}} + \int_{S^{n-1}} \lambda^{n-2}\, d\mathrm{vol}_{S^{n-1}} \right) - o(1), \end{aligned} \tag{3}$$

where λ is a positive function on S^{n-1} which depends on t. In order to estimate the right hand side in (3), we use a sharp version of the Sobolev inequality on S^{n-1} due to Beckner [2]. Using this inequality, we obtain

$$\liminf_{t \to \infty} Q(t) \geq (n-1)\, |S^{n-1}|^{\frac{1}{n-1}}.$$

Since $Q(t)$ is monotone decreasing, we conclude that $Q(0) \geq (n-1)\, |S^{n-1}|^{\frac{1}{n-1}}$. From this, Theorem 1 follows immediately.

2. Star-shaped hypersurfaces in the AdS-Schwarzschild manifold

Lemma 3. *By a change of variable, the AdS-Schwarzschild metric can be rewritten as*

$$g = dr \otimes dr + \lambda(r)^2\, g_{S^{n-1}}$$

where $\lambda(r)$ satisfies the ODE

$$\lambda'(r) = \sqrt{1 + \lambda^2 - m\lambda^{2-n}} \tag{4}$$

and the asymptotic expansion

$$\lambda(r) = \sinh(r) + \frac{m}{2n}\, \sinh^{-n+1}(r) + O(\sinh^{-n-1}(r)).$$

Proof. We define

$$r(s) = \int_{s_0}^{s} \frac{1}{\sqrt{1+t^2-mt^{2-n}}}\,dt - b,$$

where

$$b = \int_{s_0}^{\infty} \Big(\frac{1}{\sqrt{1+t^2-mt^{2-n}}} - \frac{1}{\sqrt{1+t^2}}\Big)\,dt - \int_0^{s_0} \frac{1}{\sqrt{1+t^2}}\,dt.$$

With this understood, the metric g can be written as $g = dr \otimes dr + \lambda(r)^2\, g_{S^{n-1}}$, where $\lambda(r(s)) = s$.

The function $r(s)$ can be rewritten as

$$\begin{aligned} r(s) &= \int_0^{s} \frac{1}{\sqrt{1+t^2}}\,dt - \int_s^{\infty} \Big(\frac{1}{\sqrt{1+t^2-mt^{2-n}}} - \frac{1}{\sqrt{1+t^2}}\Big)\,dt \\ &= \operatorname{arsinh}(s) - \int_s^{\infty} \Big(\frac{m}{2}\,t^{-n-1} + O(t^{-n-3})\Big)\,dt \\ &= \operatorname{arsinh}(s) - \frac{m}{2n}\,s^{-n} + O(s^{-n-2}). \end{aligned}$$

Hence, by Taylor expansion, we have

$$\begin{aligned} \sinh(r(s)) &= s - \frac{m}{2n}\,s^{-n+1} + O(s^{-n-1}) \\ &= s - \frac{m}{2n}\,\sinh^{-n+1}(r(s)) + O(\sinh^{-n-1}(r(s))). \end{aligned}$$

From this, the assertion follows. □

We calculate the asymptotic expansion of Riemannian curvature tensors in the next lemma.

Lemma 4. *Let e_α, $\alpha = 1, 2, \dots, n$ be a orthonormal frame and $R_{\alpha\beta\gamma\mu}$ is the Riemannian curvature tensor of the AdS-Schwarzschild metric. Then*

$$R_{\alpha\beta\gamma\mu} = -\delta_{\beta\mu}\delta_{\alpha\gamma} + \delta_{\beta\gamma}\delta_{\alpha\mu} + O(e^{-nr}) \tag{5}$$

and

$$\bar{D}_\rho R_{\alpha\beta\gamma\mu} = O(e^{-nr}). \tag{6}$$

Moreover, the Ricci tensor satisfies

$$\operatorname{Ric}(\partial_r, \partial_r) = -(n-1) - m\,\frac{(n-1)(n-2)}{2}\,\sinh^{-n}(r) + O(e^{-(n+2)r})$$

and

$$\lambda^{-2}\operatorname{Ric}(\partial_{\theta^i}, \partial_{\theta^j}) = \Big(-(n-1) + m\,\frac{n-2}{2}\,\sinh^{-n}(r)\Big)\,\sigma_{ij} + O(e^{-(n+2)r}),$$

where $\sigma_{ij} = g_{S^{n-1}}(\partial_{\theta^i}, \partial_{\theta^j})$.

Proof. Each level set of r is a round sphere with induced metric $\lambda(r)^2\, g_{S^{n-1}}$ and second fundamental form $\lambda(r)\,\lambda'(r)\,g_{S^{n-1}}$. Applying the Gauss equation, we compute

$$R(\partial_{\theta^i}, \partial_{\theta^j}, \partial_{\theta^k}, \partial_{\theta^l}) = \lambda(r)^2\,(1-\lambda'(r)^2)\,(\sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk}).$$

Since the level set of r is umbilic, from the Codazzi equation, we derive

$$R(\partial_{\theta^i}, \partial_{\theta^j}, \partial_{\theta^k}, \partial_r) = 0.$$

The remaining components of the curvature tensors are

$$\begin{aligned} R(\partial_{\theta^i}, \partial_r, \partial_{\theta^j}, \partial_r) &= \langle (\bar\nabla_i \bar\nabla_r - \bar\nabla_r \bar\nabla_i)\partial_r, \partial_{\theta^j} \rangle \\ &= -\langle \bar\nabla_r \bar\nabla_i \partial_r, \partial_{\theta^j} \rangle \\ &= -\Big\langle \bar\nabla_r \Big(\frac{\lambda'}{\lambda} \, \partial_{\theta^i} \Big), \partial_{\theta^j} \Big\rangle \\ &= -\lambda(r)\, \lambda''(r)\, \sigma_{ij}. \end{aligned}$$

From this, (5) and (6) follow easily.

Moreover, we have

$$\begin{aligned} \mathrm{Ric}(\partial_r, \partial_r) &= -(n-1)\, \frac{\lambda''(r)}{\lambda(r)} \\ &= -(n-1) - m\, \frac{(n-1)(n-2)}{2}\, \sinh^{-n}(r) + O(e^{-(n+2)r}). \end{aligned}$$

As the scalar curvature is $-n(n-1)$, the expression of $\mathrm{Ric}(\partial_{\theta^i}, \partial_{\theta^j})$ follows. □

Let $\theta = \{\theta^j\}_{j=1,2,\ldots,n-1}$ be a coordinate system on S^{n-1} and ∂_{θ^j} be the corresponding coordinate vector field in M. A star-shaped hypersurface $\Sigma \subset M$ can be parametrized by

$$\Sigma = \{ (r(\theta), \theta) : \; \theta \in S^{n-1} \}$$

for a smooth function r on S^{n-1}. We next define a new function $\varphi : S^{n-1} \to \mathbb{R}$ by

$$\varphi(\theta) = \Phi(r(\theta)),$$

where $\Phi(r)$ is a positive function satisfying $\Phi'(r) = \frac{1}{\lambda(r)}$.

Let $\varphi_i = \nabla_i \varphi$ and $\varphi_{ij} = \nabla_j \nabla_i \varphi$ denote the covariant derivatives of φ with respect to the round metric $g_{S^{n-1}}$. Moreover, let

$$v = \sqrt{1 + |\nabla \varphi|^2_{S^{n-1}}}.$$

In the next lemma, we express the metric and second fundamental form of Σ in terms of covariant derivatives of φ as in [8]:

Proposition 5. *Let g_{ij} be the induced metric on Σ and h_{ij} be the second fundamental form in term of the coordinates θ^j. Then*

$$g_{ij} = \lambda^2 \, (\sigma_{ij} + \varphi_i \varphi_j)$$

and

$$h_{ij} = \frac{\lambda}{v} \, \big(\lambda' \, (\sigma_{ij} + \varphi_i \varphi_j) - \varphi_{ij} \big).$$

Proof. A basis of tangent vector fields of Σ is of the form $r_j\partial_r + \partial_{\theta^j}$. We compute

$$\begin{aligned} g_{ij} &= \langle r_i\partial_r + \partial_{\theta^i}, r_j\partial_r + \partial_{\theta^j}\rangle \\ &= \lambda^2(r)\sigma_{ij} + r_i r_j \\ &= \lambda^2(r)(\sigma_{ij} + \varphi_i\varphi_j). \end{aligned}$$

The unit normal vector ν is given by

$$\nu = \frac{1}{v}\Big(\partial_r - \frac{r^j}{\lambda^2}\partial_{\theta^j}\Big).$$

Thus, the second fundamental form is given by

$$\begin{aligned} h_{ij} &= -\langle \bar\nabla_{r_i\partial_r + \partial_{\theta^i}}(r_j\partial_r + \partial_{\theta^j}), \nu\rangle \\ &= -\Big\langle (r_{ij} - \lambda\lambda')\,\partial_r + \frac{\lambda'}{\lambda} r_j\,\partial_{\theta^i} + \frac{\lambda'}{\lambda} r_i\,\partial_{\theta^j}, \nu\Big\rangle \\ &= \frac{1}{v}\Big(\lambda\lambda'\,\sigma_{ij} + \frac{2\lambda'}{\lambda} r_i\, r_j - r_{ij}\Big) \\ &= \frac{\lambda}{v}\big(\lambda'\,(\sigma_{ij} + \varphi_i\varphi_j) - \varphi_{ij}\big), \end{aligned}$$

where $\bar\nabla$ denotes the Levi-Civita connection in the ambient AdS-Schwarzschild manifold. □

3. The inverse mean curvature flow

Let Σ_0 be a mean convex star-shaped hypersurface in M which is given by an embedding

$$F_0 : S^{n-1} \to M$$

Let $F_t : S^{n-1} \to M$, $t \in [0, T)$, be the solution of inverse mean curvature flow with initial data F_0. In other words,

$$\frac{\partial F}{\partial t} = \frac{1}{H}\,\nu, \tag{7}$$

where ν is the unit outer normal vector and H is the mean curvature. We shall call (7) the parametric form of the flow. The evolution of the mean curvature is given by

$$\frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - 2\,\frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} - \frac{\mathrm{Ric}(\nu,\nu)}{H}. \tag{8}$$

We can write Σ_0 as the graph of a function $\tilde r_0$ defined on the unit sphere:

$$\Sigma_0 = \{(\tilde r_0(\theta), \theta) : \theta \in S^{n-1}\}.$$

If each Σ_t is star-shaped, it can be parametrized them as the graph

$$\Sigma_t = \{(\tilde r(\theta, t), \theta) : \theta \in S^{n-1}\}.$$

In this case, the inverse mean curvature flow can be written as a parabolic PDE for $\tilde{r}$. As long as the solution of (7) exists and remains star-shaped, it is equivalent to

$$\frac{\partial \tilde{r}}{\partial t} = \frac{v}{H}, \tag{9}$$

where v is defined as above.

The equation (9) will be referred as the non-parametric form of the inverse mean curvature flow. Notice that the velocity vector of (7) is always normal, while the velocity vector of (9) is in the direction of ∂_r. To go from one to the other, we take the difference which is a (time-dependent) tangential vector field and compose the flow of the reparametrization associated with the tangent vector field.

Notice that associated with $\tilde{r}$, we define

$$\varphi(\theta, t) := \Phi(\tilde{r}(\theta, t)),$$

where $\Phi(r)$ is a positive function satisfying $\Phi'(r) = \frac{1}{\lambda(r)}$. Then φ satisfies

$$\frac{\partial \varphi}{\partial t} = \frac{v}{\lambda H}. \tag{10}$$

In the sequel, we use the non-parametric form to derive C^0 and C^1 estimates of $\tilde{r}$. Some of theses estimates can be found in [8] or [11] (see also [10]). For completeness, we derive all the estimates here.

Lemma 6. *Let $\bar{r} = \sup_{S^{n-1}} \tilde{r}(\cdot, t)$ and $\underline{r}(t) = \inf_{S^{n-1}} \tilde{r}(\cdot, t)$. Then*

$$\lambda(\bar{r}(t)) \leq e^{\frac{1}{n-1}\, t}\, \lambda(\bar{r}(0))$$

and

$$\lambda(\underline{r}(t)) \geq e^{\frac{1}{n-1}\, t}\, \lambda(\underline{r}(0)).$$

Proof. Recall that

$$\frac{\partial \tilde{r}}{\partial t} = \frac{v}{H}.$$

Moreover, we have

$$H = \frac{(n-1)\lambda'}{\lambda v} - \frac{\tilde{\sigma}^{ij}}{\lambda v}\, \varphi_{ij},$$

where $\tilde{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2}$. At the point where the function $\tilde{r}(\cdot, t)$ attains its maximum, we have $H \geq \frac{(n-1)\lambda'}{\lambda}$. This implies

$$\frac{d}{dt} \bar{r}(t) \leq \frac{\lambda(\bar{r}(t))}{(n-1)\lambda'(\bar{r}(t))},$$

hence

$$\frac{d}{dt} \lambda(\bar{r}(t)) \leq \frac{\lambda(\bar{r}(t))}{n-1}.$$

From this, the first statement follows. The second statement follows similarly. $\square$

Lemma 7. *We have $H \leq n - 1 + O(e^{-\frac{2}{n-1}\, t})$.*

Proof. Note that $|\mathrm{Ric} + (n-1)\, g| \leq O(e^{-nr})$. This implies $|\mathrm{Ric} + (n-1)\, g| \leq O(e^{-\frac{n}{n-1}\, t})$ on Σ_t. Using (8) and the inequality $|A|^2 \geq \frac{1}{n-1}\, H^2$, we obtain

$$\frac{d}{dt} H_{max}^2 \leq -\frac{2}{n-1}\, H_{max}^2 + 2(n-1) + O(e^{-\frac{n}{n-1}\, t}).$$

This implies

$$H_{max}(t)^2 \leq (n-1)^2 + C\, e^{-\frac{2}{n-1}\, t}.$$

From this, the assertion follows easily. □

We next consider the function

$$F = \frac{\lambda H}{v} = \frac{(n-1)\lambda' - \tilde{\sigma}^{ij}\varphi_{ij}}{v^2}.$$

where $\tilde{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2}$. The non-parametric form of the equation is

$$\frac{\partial \varphi}{\partial t} = \frac{1}{F}. \tag{11}$$

First, we derive the evolution of the first space and time derivatives of φ.

Lemma 8. *The evolution equation of $\omega = \frac{1}{2}\, |\nabla\varphi|_{g_{S^{n-1}}}^2$ is*

$$\begin{aligned}\frac{\partial \omega}{\partial t} &= \frac{\tilde{\sigma}^{ij}}{v^2 F^2}\, \omega_{ij} - \frac{1}{F^2}\, \frac{\partial F}{\partial \varphi_i}\, \omega_i - \frac{2(n-2)\omega}{v^2 F^2} \\ &\quad - \frac{\tilde{\sigma}^{ij}}{v^2 F^2}\, \sigma^{kl}\, \varphi_{ik}\, \varphi_{jl} - \frac{2(n-1)\lambda\lambda''}{v^2 F^2}\, \omega.\end{aligned} \tag{12}$$

Proof. If we differentiate (11) with respect to $\varphi^k\, \nabla_k$, we get

$$\begin{aligned}\frac{\partial \omega}{\partial t} &= -\frac{1}{F^2}\, \Big(\frac{\partial F}{\partial \varphi_{ij}}\, \nabla_k \varphi_{ij}\, \varphi^k + \frac{\partial F}{\partial \varphi_i}\, \varphi_{ik}\, \varphi^k + \frac{(n-1)\lambda''}{v^2}\, r_k\, \varphi^k\Big) \\ &= -\frac{1}{F^2}\, \Big(-\frac{\tilde{\sigma}^{ij}}{v^2}\, \nabla_k \varphi_{ij}\, \varphi^k + \frac{\partial F}{\partial \varphi_i}\, \omega_i + \frac{2(n-1)\lambda\lambda''}{v^2}\, \omega\Big).\end{aligned}$$

We next observe

$$\begin{aligned}\omega_{ij} &= \varphi_{kij}\, \varphi^k + \varphi_{ki}\, \varphi^k{}_j \\ &= \varphi_{ijk}\, \varphi^k + (\delta_k^p \sigma_{ij} - \delta_j^p \sigma_{ik})\, \varphi_p\, \varphi^k + \varphi_{ki}\, \varphi^k{}_j \\ &= \varphi_{ijk}\, \varphi^k + \sigma_{ij}\, |\nabla\varphi|_{g_{S^{n-1}}}^2 - \varphi_i\, \varphi_j + \varphi_{ki}\, \varphi^k{}_j,\end{aligned}$$

where the covariant derivatives are taken with respect to $g_{S^{n-1}}$. Putting these facts together, we conclude

$$\begin{aligned}\frac{\partial \omega}{\partial t} &= \frac{\tilde{\sigma}^{ij}}{v^2 F^2}\, \omega_{ij} + \frac{1}{F^2}\, \frac{\partial F}{\partial \varphi_i}\, \omega_i - \frac{\tilde{\sigma}^{ij}}{v^2 F^2}\, (\sigma_{ij}\, |\nabla\varphi|_{g_{S^{n-1}}}^2 - \varphi_i \varphi_j) \\ &\quad - \frac{\tilde{\sigma}^{ij}}{v^2 F^2}\, \sigma^{kl}\, \varphi_{ik}\, \varphi_{jl} - \frac{2(n-1)\lambda\lambda''}{v^2 F^2}\, \omega.\end{aligned}$$

Since

$$\tilde{\sigma}^{ij}\, (\sigma_{ij}\, |\nabla\varphi|_{g_{S^{n-1}}}^2 - \varphi_i \varphi_j) = 2(n-2)\, \omega,$$

the equation (12) follows. □

Proposition 9. *We have* $|\nabla\varphi|_{g_{S^{n-1}}} = O(e^{-\frac{1}{n-1}t})$ *or, equivalently,* $|\nabla r|_{g_{S^{n-1}}} = O(1)$ *and* $|\nabla r|_g = O(e^{-\frac{1}{n-1}t})$.

Proof. Using Lemma 7, we obtain

$$\frac{2(n-1)\lambda\lambda''}{v^2 F^2} = \frac{2(n-1)\lambda\lambda''}{\lambda^2 H^2} \geq \frac{2}{n-1} - C\,e^{-\frac{2}{n-1}t}.$$

Using (12) and the maximum principle, we conclude

$$\frac{d}{dt}\omega_{max} \leq -\Big(\frac{2}{n-1} - C\,e^{-\frac{2}{n-1}t}\Big)\,\omega_{max},$$

where $\omega_{max} = \frac{1}{2}\sup_{S^{n-1}}|\nabla\varphi|^2_{g_{S^{n-1}}}$. Thus

$$\omega_{max}(t) = O(e^{-\frac{2}{n-1}t}).$$

This implies

$$|\nabla r|^2_{g_{S^{n-1}}} = \lambda^2\,|\nabla\varphi|^2_{g_{S^{n-1}}} = O(1)$$

and

$$|\nabla r|^2_g = \lambda^2\,|\nabla\varphi|^2_g = \Big(\sigma^{ij} - \frac{\varphi^i\,\varphi^j}{v^2}\Big)\,\varphi_i\,\varphi_j = \frac{|\nabla\varphi|^2_{g_{S^{n-1}}}}{1+|\nabla\varphi|^2_{g_{S^{n-1}}}} = O(e^{-\frac{2}{n-1}t}).$$

□

Proposition 10. *The function* $\dot\varphi = \frac{v}{\lambda H}$ *is uniformly bounded from above. In particular,* $H \geq c\,e^{-\frac{1}{n-1}t}$ *for some positive constant* c.

Proof. If we differentiate (11) with respect to t, we obtain

$$\frac{\partial\dot\varphi}{\partial t} = \frac{\tilde\sigma^{ij}}{v^2F^2}\,\dot\varphi_{ij} - \frac{1}{F^2}\,\frac{\partial F}{\partial\varphi_i}\,\dot\varphi_i - \frac{(n-1)\lambda\lambda''}{v^2F^2}\,\dot\varphi.$$

Hence, the assertion follows from the maximum principle. □

4. Estimates for the mean curvature and second fundamental form

In this section, we prove estimates for the mean curvature and second fundamental form. From now on, we will always work with the parametric form of the flow. We begin by computing the evolution equations for the function

$$\chi = \frac{1}{\langle\nu,\lambda\,\partial_r\rangle} = \frac{v}{\lambda}$$

and the function λ.

Lemma 11. *We have*

$$\frac{\partial\log\chi}{\partial t} = \frac{\Delta\log\chi}{H^2} - \frac{1}{H^2}\,|\nabla\log\chi|^2 - \frac{|A|^2}{H^2} + \frac{1}{H^2}\,O(e^{-\frac{n}{n-1}t}) \tag{13}$$

and

$$
\begin{aligned}
\frac{\partial \log\lambda}{\partial t} &= \frac{\Delta\log\lambda}{H^2} - \frac{\lambda\lambda''}{\lambda'^2 H^2}\,|\nabla\log\lambda|^2 + 2\,\frac{|\nabla\log\lambda|^2}{H^2}\\
&\quad - \frac{1}{n-1}\Big(\frac{(n-1)\lambda'}{\lambda H}\Big)^2 + \frac{2}{(n-1)v}\Big(\frac{(n-1)\lambda'}{\lambda H}\Big).
\end{aligned} \tag{14}
$$

Proof. We first calculate the equation for χ. The vector field $\lambda\,\partial_r$ satisfies the property that

$$\bar\nabla_X(\lambda\,\partial_r) = \lambda'\,X. \tag{15}$$

Then

$$\frac{\partial\chi}{\partial t} = -\chi^2\Big(\frac{\langle\nabla H, \lambda\,\partial_r\rangle}{H^2} + \frac{\lambda'}{H}\Big).$$

Let ∂_i, $i = 1, 2, \dots n-1$ be coordinate vector fields on Σ_t, we derive

$$D_i\chi = -\chi^2\,h_i^k\,\langle\partial_k, \lambda\,\partial_r\rangle$$

and

$$
\begin{aligned}
D_j D_i\chi = &-2\chi\,\chi_j\,h_i^k\,\langle\partial_k, \lambda\,\partial_r\rangle - \chi^2\,D_j h_i^k\,\langle\partial_k, \lambda\,\partial_r\rangle\\
&+ \chi^2\,h_i^k\,h_{kj}\,\langle\nu, \lambda\,\partial_r\rangle - \chi^2\,h_i^k\,\langle\partial_k, \lambda'\,\partial_j\rangle.
\end{aligned}
$$

Using the Codazzi equations and Lemma 4, we obtain

$$\Delta\chi = 2\,\frac{|\nabla\chi|^2}{\chi} - \chi^2\,\lambda\,\langle\nabla H, \partial_r\rangle + \chi\,|A|^2 - \chi^2\,\lambda'\,H + \lambda\chi^2\,O(e^{-nr}).$$

Since $r = O(\frac{t}{n-1})$, the identity (13) follows.

We next derive (14). The parametric form of the equation implies

$$\frac{\partial r}{\partial t} = \frac{\langle\partial_r, \nu\rangle}{H} = \frac{1}{Hv}.$$

Using (15) and the identity $\bar\nabla r = \partial_r$, we derive

$$\Delta r = (n-1)\,\frac{\lambda'}{\lambda} - \frac{\lambda'}{\lambda}\,|\nabla r|^2 - \frac{H}{v}.$$

We thus have

$$\frac{\partial r}{\partial t} = \frac{\Delta r}{H^2} + \frac{2}{Hv} - (n-1)\,\frac{\lambda'}{\lambda H^2} + \frac{\lambda'}{\lambda H^2}\,|\nabla r|^2,$$

hence

$$\frac{\partial\lambda}{\partial t} = \frac{\Delta\lambda}{H^2} - \Big(\frac{\lambda\lambda'' - \lambda'^2}{\lambda H^2}\Big)\,|\nabla r|^2 - \frac{(n-1)\lambda'^2}{\lambda H^2} + 2\frac{\lambda'}{Hv}.$$

The identity (14) follows by taking log of the last equation. □

Proposition 12. *H is uniformly bounded from below globally in time.*

Proof. We derive

$$\frac{\partial \log H}{\partial t} = \frac{\Delta \log H}{H^2} - \frac{1}{H^2}\,|\nabla \log H|^2 - \frac{|A|^2}{H^2} + \frac{n-1}{H^2} + \frac{1}{H^2}\,O(e^{-\frac{n}{n-1}\,t}). \tag{16}$$

Combining (13) and (16), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(\log \chi - \log H) \leq \frac{1}{H^2}\,\Delta(\log\chi - \log H) - \frac{1}{H^2}\,|\nabla \log \chi|^2 + \frac{1}{H^2}\,|\nabla \log H|^2 \\ - \frac{n-1}{H^2}\,(1 - L\,e^{-\frac{n}{n-1}\,t})\end{aligned}$$

for some positive constant L which does not depend on t. Let us fix a real number $\tau > 0$ such that $L\,e^{-\frac{n}{n-1}\,\tau} < 1$. By Proposition 10, the function H is uniformly bounded from below on $[0,\tau]$. Using Lemma 7, we conclude that

$$\begin{aligned}\frac{\partial}{\partial t}(\log \chi - \log H) \leq \frac{1}{H^2}\,\Delta(\log\chi - \log H) - \frac{1}{H^2}\,|\nabla \log \chi|^2 + \frac{1}{H^2}\,|\nabla \log H|^2 \\ - \frac{1}{n-1} - C\,e^{-\frac{2}{n-1}\,t}\end{aligned}$$

for $t \geq \tau$. By the maximum principle, $\frac{\chi}{H} \leq O(e^{-\frac{1}{n-1}\,t})$. Since $\chi = \frac{v}{\lambda}$, we conclude that $\frac{1}{H} \leq O(1)$. □

We next study the second fundamental form of Σ_t. The second fundamental form satisfies the evolution equation

$$\begin{aligned}\frac{\partial h_i^j}{\partial t} &= \frac{\Delta h_i^j}{H^2} + \frac{|A|^2}{H^2}\,h_i^j - 2\,\frac{h_i^k h_k^j}{H} - 2\,\frac{H_i H^j}{H^3} \\ &\quad + \frac{2}{H^2}\,g^{kl}\,g^{sj}\,R_{miks}\,h_l^m - \frac{1}{H^2}\,g^{kl}\,g^{sj}\,R_{mksl}\,h_i^m - \frac{1}{H^2}\,g^{kl}\,R_{mkil}\,h^{mj} \\ &\quad + \frac{1}{H^2}\,\mathrm{Ric}(\nu,\nu)\,h_i^j - \frac{2}{H}\,g^{mj}\,R_{\nu i \nu m}.\end{aligned} \tag{17}$$

Combining (8) and (17), we obtain the following evolution equation for the tensor $M_i^j = H\,h_i^j$:

$$\begin{aligned}\frac{\partial M_i^j}{\partial t} &= \frac{\Delta M_i^j}{H^2} - 2\,\frac{D^k H\,D_k M_i^j}{H^3} - 2\,\frac{D_i H\,D^j H}{H^2} - 2\,\frac{M_i^k\,M_k^j}{H^2} \\ &\quad + \frac{2(n-1)M_i^j}{H^2} + \Big(\frac{|M|}{H^2} + 1\Big)\,O(e^{-\frac{n}{n-1}\,t}).\end{aligned} \tag{18}$$

Proposition 13. *The second fundamental form is uniformly bounded globally in time.*

Proof. Let μ be the maximal eigenvalue of the tensor $M_i^j = H\,h_i^j$. As H is uniformly bounded we have $|M| \leq C(\mu+1)$ for some constant C. From (18) we have

$$\frac{d\mu}{dt} \leq -\frac{2\mu^2}{H^2} + \frac{2(n-1)\mu}{H^2} + (\mu+1)\,O(e^{-\frac{n}{n-1}\,t}). \tag{19}$$

So μ is uniformly bounded from above. Again from uniform boundedness of H we know M_i^j and h_i^j are both uniformly bounded. □

Corollary 14. *The solution of the inverse mean curvature flow is defined on* $[0,\infty)$.

We now establish an improved lower bound for the mean curvature.

Proposition 15. *We have* $H = n-1+O(te^{-\frac{2}{n-1}t})$.

Proof. Combining (13), (14), and (16), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}(\log\chi+\log\lambda-\log H) \\
&= \frac{1}{H^2}\,\Delta(\log\chi+\log\lambda-\log H) \\
&+ \frac{1}{H^2}\,|\nabla\log H|^2 - \frac{1}{H^2}\,|\log\chi|^2 - \frac{\lambda\lambda''}{\lambda'^2 H^2}\,|\nabla\log\lambda|^2 + 2\,\frac{|\nabla\log\lambda|^2}{H^2} \\
&- \frac{n-1}{H^2} - \frac{1}{n-1}\Big(\frac{(n-1)\lambda'}{\lambda H}\Big)^2 + \frac{2}{(n-1)v}\Big(\frac{(n-1)\lambda'}{\lambda H}\Big) + O(e^{-\frac{n}{n-1}t}).
\end{aligned}$$

At a critical point of the function $\log\chi+\log\lambda-\log H$, we have $\nabla\log H = \nabla\log\chi+\nabla\log\lambda$. Using Proposition 9 and Proposition 13, we obtain

$$|\nabla\log\lambda|^2 = \frac{\lambda'^2}{\lambda^2}\,|\nabla r|^2 = O(e^{-\frac{2}{n-1}t})$$

and

$$\begin{aligned}
&|\nabla\log\chi|^2 + |\nabla\log\lambda|^2 - |\nabla\log H|^2 \\
&= -2\,\langle\nabla\log\chi,\nabla\log\lambda\rangle = 2\chi\,\lambda'\,h_{ij}\,r^j\,r^j = O(e^{-\frac{2}{n-1}t}).
\end{aligned}$$

Thus the gradient terms can be estimated by $O(e^{-\frac{2}{n-1}t})$. Moreover, we have

$$\begin{aligned}
&-\frac{n-1}{H^2} - \frac{1}{n-1}\Big(\frac{(n-1)\lambda'}{\lambda H}\Big)^2 + \frac{2}{(n-1)v}\Big(\frac{(n-1)\lambda'}{\lambda H}\Big) \\
&= \frac{2}{n-1} - \frac{2}{H} - \frac{1}{n-1}\Big(\frac{n-1}{H}-1\Big)^2 \\
&- \frac{1}{n-1}\Big(\frac{(n-1)\lambda'}{\lambda H}-1\Big)^2 - \frac{2}{n-1}\,\frac{(n-1)\lambda'}{\lambda H}\Big(1-\frac{1}{v}\Big) \\
&\leq \frac{2}{n-1} - \frac{2}{H} \\
&\leq \frac{2}{n-1} - \frac{2\chi\lambda}{H} + O(e^{-\frac{2}{n-1}t})
\end{aligned}$$

where we have used the fact that $\chi\lambda = v = 1+O(e^{-\frac{2}{n-1}t})$. Hence, if we put $\rho(t) = \sup_{S^{n-1}}\frac{\chi\lambda}{H}$, then

$$\frac{d}{dt}\log\rho(t) \leq \frac{2}{n-1} - 2\rho(t) + O(e^{-\frac{2}{n-1}t}).$$

This implies

$$\frac{d}{dt}\rho(t) \leq \frac{2}{n-1}\Big(\frac{1}{n-1}-\rho(t)\Big) + O(e^{-\frac{2}{n-1}t})$$

whenever $\rho(t) \geq \frac{1}{n-1}$. From this, we deduce that $\rho(t) \leq \frac{1}{n-1} + O(te^{-\frac{2}{n-1}\,t})$. Since $\chi\lambda = O(e^{-\frac{2}{n-1}\,t})$, we conclude that $\frac{1}{H} \leq \frac{1}{n-1} + O(te^{-\frac{2}{n-1}\,t})$. $\square$

Finally, we estimate second fundamental form more precisely.

Proposition 16. *We have* $|h_i^j - \delta_i^j| \leq O(t^2 e^{-\frac{2}{n-1}\,t})$.

Proof. As above, we denote by μ the maximal eigenvalue of $M_i^j = H\,h_i^j$. Using (19), we obtain

$$\begin{aligned} \frac{d\mu}{dt} &\leq -\frac{2\mu^2}{H^2} + \frac{2(n-1)\mu}{H^2} + (\mu+1)\,O(e^{-\frac{n}{n-1}\,t}) \\ &= -\frac{2(n-1)}{H^2}\,(\mu-n+1) - \frac{2}{H^2}\,(\mu-n+1)^2 + (\mu+1)\,O(e^{-\frac{n}{n-1}\,t}) \\ &\leq -\frac{2}{n-1}\,(\mu-n+1) + O(te^{-\frac{2}{n-1}\,t}), \end{aligned}$$

where in the third line we have used that μ is uniformly bounded and $H = n-1+O(te^{-\frac{2}{n-1}\,t})$. Thus,

$$\mu - n + 1 \leq O(t^2 e^{-\frac{2}{n-1}\,t}).$$

As $M_i^j = Hh_i^j$ and $H = n-1+O(te^{-\frac{2}{n-1}\,t})$, we conclude that the largest eigenvalue of the second fundamental form is less than $1+O(t^2 e^{-\frac{2}{n-1}\,t})$. Since $H = n-1+O(te^{-\frac{2}{n-1}\,t})$, the smallest eigenvalue of the second fundamental form is greater than $1-O(t^2 e^{-\frac{2}{n-1}\,t})$. $\square$

5. The monotonicity formula

As above, we consider a family of star-shaped surfaces Σ_t evolving by inverse mean curvature flow. We define

$$Q(t) = |\Sigma_t|^{-\frac{n-2}{n-1}} \bigg(\int_{\Sigma_t} f\,H\,d\mu - n(n-1) \int_\Omega f\,d\text{vol} + (n-1)\,s_0^{n-2}\,|S^{n-1}| \bigg),$$

where f is defined by (1).

We first evaluate the limit of $Q(t)$ as $t \to \infty$. To that end, we need the following auxiliary result:

Proposition 17. *For every positive function u on S^{n-1}, we have*

$$\begin{aligned} &\frac{1}{2} \int_{S^{n-1}} u^{n-4}\,|\nabla u|^2_{g_{S^{n-1}}}\,d\text{vol}_{S^{n-1}} + \int_{S^{n-1}} u^{n-2}\,d\text{vol}_{S^{n-1}} \\ &\geq |S^{n-1}|^{\frac{1}{n-1}} \bigg(\int_{S^{n-1}} u^{n-1}\,d\text{vol}_{S^{n-1}} \bigg)^{\frac{n-2}{n-1}}. \end{aligned}$$

Moreover, equality holds if and only if u is constant.

Proof. It follows from Theorem 4 in [2] that

$$\begin{aligned}&\frac{2}{(n-2)(n-1)} \int_{S^{n-1}} |\nabla w|^2_{g_{S^{n-1}}} \, d\mathrm{vol}_{S^{n-1}} + \int_{S^{n-1}} w^2 \, d\mathrm{vol}_{S^{n-1}} \\ &\geq |S^{n-1}|^{\frac{1}{n-1}} \left(\int_{S^{n-1}} w^{\frac{2(n-1)}{n-2}} \, d\mathrm{vol}_{S^{n-1}} \right)^{\frac{n-2}{n-1}}\end{aligned}$$

for every positive smooth function w. Hence, if we put $u = w^{\frac{2}{n-2}}$, we obtain

$$\begin{aligned}&\frac{n-2}{2(n-1)} \int_{S^{n-1}} u^{n-4} \, |\nabla u|^2_{g_{S^{n-1}}} \, d\mathrm{vol}_{S^{n-1}} + \int_{S^{n-1}} u^{n-2} \, d\mathrm{vol}_{S^{n-1}} \\ &\geq |S^{n-1}|^{\frac{1}{n-1}} \left(\int_{S^{n-1}} u^{n-1} \, d\mathrm{vol}_{S^{n-1}} \right)^{\frac{n-2}{n-1}}.\end{aligned}$$

From this, the assertion follows. ☐

Proposition 18. *We have* $\liminf_{t\to\infty} Q(t) \geq (n-1)\, |S^{n-1}|^{\frac{1}{n-1}}$.

Proof. Using the inequalities

$$\begin{aligned}&f = \lambda + O(e^{-\frac{1}{n-1}\,t}), \\ &H - n + 1 = O(te^{-\frac{2}{n-1}\,t}), \\ &\sqrt{\det g} = (\lambda^{n-1} + O(e^{\frac{n-3}{n-1}\,t}))\, \sqrt{\det g_{S^{n-1}}},\end{aligned}$$

we obtain

$$\int_{\Sigma_t} f\,(H-n+1)\, d\mu = \int_{S^{n-1}} \lambda^n\,(H-n+1)\, d\mathrm{vol}_{S^{n-1}} + O(te^{\frac{n-4}{n-1}\,t}). \tag{20}$$

By Proposition 5, the metric and second fundamental form on Σ_t are given by

$$g_{ij} = \lambda^2\,(\sigma_{ij} + \varphi_i \varphi_j)$$

and

$$h_{ij} = \frac{\lambda'}{\lambda v}\, g_{ij} - \frac{\lambda}{v}\, \varphi_{ij}.$$

Here, σ_{ij} is the round metric on S^{n-1} and φ_{ij} is the Hessian of φ with respect to $g_{S^{n-1}}$. By Proposition 16, we have $|h-g|_g \leq O(t^2 e^{-\frac{2}{n-1}\,t})$. This implies

$$\left| h - \frac{\lambda'}{\lambda v}\, g \right|_g \leq O(t^2 e^{-\frac{2}{n-1}\,t}),$$

hence

$$\left| h - \frac{\lambda'}{\lambda v}\, g \right|_{g_{S^{n-1}}} \leq O(t^2).$$

From this, we deduce that $|D^2\varphi|_{g_{S^{n-1}}} \leq O(t^2\, e^{-\frac{t}{n-1}})$, where $D^2\varphi$ denotes the Hessian of φ with respect to $g_{S^{n-1}}$. Using Proposition 9, we obtain

$$\tilde{\sigma}^{ij}\, \varphi_{ij} = \Delta_{S^{n-1}}\varphi + O(t^2 e^{-\frac{3}{n-1}}).$$

This implies

$$\begin{aligned} H &= \frac{(n-1)\lambda'}{\lambda v} - \frac{1}{\lambda v}\,\tilde{\sigma}^{ij}\,\varphi_{ij} \\ &= \frac{(n-1)\lambda'}{\lambda v} - \frac{1}{\lambda v}\,\Delta_{S^{n-1}}\varphi + O(t^2 e^{-\frac{4}{n-1}\,t}). \end{aligned}$$

Since $\lambda' = \lambda + \frac{1}{2}\,\lambda^{-1} + O(e^{-\frac{2}{n-1}\,t})$ and $\frac{1}{v} = 1 - \frac{1}{2}\,|\nabla\varphi|^2_{g_{S^{n-1}}} + O(e^{-\frac{4}{n-1}\,t})$, we conclude that

$$H = n-1 + \frac{n-1}{2\lambda^2} - \frac{n-1}{2}\,|\nabla\varphi|^2_{g_{S^{n-1}}} - \frac{1}{\lambda}\,\Delta_{S^{n-1}}\varphi + O(e^{-\frac{3}{n-1}\,t}).$$

Substituting this identity into (20), we obtain

$$\begin{aligned} &\int_{\Sigma_t} f\,(H-n+1)\,d\mu \\ &= \int_{S^{n-1}} \Big(\frac{n-1}{2}\,\lambda^{n-2} - \frac{n-1}{2}\,\lambda^n\,|\nabla\varphi|^2_{g_{S^{n-1}}} - \lambda^{n-1}\,\Delta_{S^{n-1}}\varphi\Big)\,d\mathrm{vol}_{S^{n-1}} + O(e^{\frac{n-3}{n-1}\,t}) \\ &= \int_{S^{n-1}} \Big(\frac{n-1}{2}\,\lambda^{n-2} - \frac{n-1}{2}\,\lambda^n\,|\nabla\varphi|^2_{g_{S^{n-1}}} + (n-1)\,\lambda^{n-2}\,\langle\nabla\lambda,\nabla\varphi\rangle_{S^{n-1}}\Big)\,d\mathrm{vol}_{S^{n-1}} \\ &+ O(e^{\frac{n-3}{n-1}\,t}). \end{aligned}$$

By Proposition 9, we have $|\nabla\varphi|_{g_{S^{n-1}}} \leq O(e^{-\frac{1}{n-1}}\,t)$. Since $\nabla\lambda = \lambda\lambda'\,\nabla\varphi$, it follows that $|\nabla\lambda - \lambda^2\,\nabla\varphi|_{g_{S^{n-1}}} \leq O(e^{-\frac{1}{n-1}}\,t)$. This implies

$$\begin{aligned} &\int_{\Sigma_t} f\,(H-n+1)\,d\mu \\ &= \int_{S^{n-1}} \Big(\frac{n-1}{2}\,\lambda^{n-2} + \frac{n-1}{2}\,\lambda^{n-4}\,|\nabla\lambda|^2_{g_{S^{n-1}}}\Big)\,d\mathrm{vol}_{S^{n-1}} + O(e^{\frac{n-3}{n-1}\,t}). \end{aligned} \tag{21}$$

On the other hand, the static potential satisfies

$$\begin{aligned} f - \langle\bar\nabla f,\nu\rangle &\geq f - |\bar\nabla f| \\ &= \sqrt{1+\lambda^2 - m\lambda^{2-n}} - (\lambda + \frac{m(n-2)}{2}\lambda^{-n+1}) \\ &= \frac{1}{2}\,\lambda^{-1} + O(e^{-t}). \end{aligned}$$

This gives

$$(n-1)\int_{\Sigma_t} (f - \langle\bar\nabla f,\nu\rangle)\,d\mu \geq \frac{n-1}{2}\int_{S^{n-1}} \lambda^{n-2}\,d\mathrm{vol}_{S^{n-1}} - O(1). \tag{22}$$

Adding (21) and (22), we obtain

$$\begin{aligned}&\int_{\Sigma_t} (f\,H - (n-1)\,\langle \bar{\nabla} f, \nu \rangle)\,d\mu \\ &\geq \frac{n-1}{2} \int_{S^{n-1}} \lambda^{n-4}\,|\nabla \lambda|^2_{g_{S^{n-1}}}\,d\mathrm{vol}_{S^{n-1}} \\ &+ (n-1) \int_{S^{n-1}} \lambda^{n-2}\,d\mathrm{vol}_{S^{n-1}} - O(e^{\frac{n-3}{n-1}\,t}).\end{aligned}$$

Moreover,

$$|\Sigma_t| = \int_{S^{n-1}} \lambda^{n-1}\,d\mathrm{vol}_{S^{n-1}} + O(e^{\frac{n-3}{n-1}\,t}).$$

Using Proposition 17, we conclude that

$$\liminf_{t \to \infty} |\Sigma_t|^{-\frac{n-2}{n-1}} \int_{\Sigma_t} (f\,H - (n-1)\langle \bar{\nabla} f, \nu \rangle)\,d\mu \geq (n-1)\,|S^{n-1}|^{\frac{1}{n-1}},$$

hence

$$\liminf_{t \to \infty} |\Sigma_t|^{-\frac{n-2}{n-1}} \left(\int_{\Sigma_t} f\,H\,d\mu - n(n-1) \int_{\Omega_t} f\,d\mathrm{vol} \right) \geq (n-1)\,|S^{n-1}|^{\frac{1}{n-1}}.$$

This completes the proof. □

Finally, we show that $Q(t)$ is monotone along the flow:

Proposition 19. *The quantity $Q(t)$ is monotone decreasing in t.*

Proof. The evolution of the mean curvature is given by

$$\frac{\partial}{\partial t} H = -\Delta\Big(\frac{1}{H}\Big) - \frac{1}{H}\,(|A|^2 + \mathrm{Ric}(\nu, \nu)).$$

This implies

$$\frac{\partial}{\partial t}(f\,H) = -f\,\Delta\Big(\frac{1}{H}\Big) - \frac{f}{H}\,(|A|^2 + \mathrm{Ric}(\nu, \nu)) + \langle \bar{\nabla} f, \nu \rangle.$$

Using the identity $\Delta f = \bar{\Delta} f - (D^2 f)(\nu,\nu) - H\,\langle \bar{\nabla} f, \nu\rangle$, we obtain

$$
\begin{aligned}
\frac{d}{dt}\Big(\int_{\Sigma_t} f\,H\,d\mu\Big) &= -\int_{\Sigma_t} f\,\Delta\Big(\frac{1}{H}\Big)\,d\mu - \int_{\Sigma_t} \frac{f}{H}\,(|A|^2 + \mathrm{Ric}(\nu,\nu))\,d\mu \\
&\quad + \int_{\Sigma_t} (\langle \bar{\nabla} f, \nu\rangle + f\,H)\,d\mu \\
&= -\int_{\Sigma_t} \frac{1}{H}\,\Delta f\,d\mu - \int_{\Sigma_t} \frac{f}{H}\,(|A|^2 + \mathrm{Ric}(\nu,\nu))\,d\mu \\
&\quad + \int_{\Sigma_t} (\langle \bar{\nabla} f, \nu\rangle + f\,H)\,d\mu \\
&= -\int_{\Sigma_t} \frac{1}{H}\,(\bar{\Delta} f - (D^2 f)(\nu,\nu))\,d\mu \qquad (23) \\
&\quad - \int_{\Sigma_t} \frac{f}{H}\,(|A|^2 + \mathrm{Ric}(\nu,\nu))\,d\mu \\
&\quad + \int_{\Sigma_t} (2\,\langle \bar{\nabla} f, \nu\rangle + f\,H)\,d\mu \\
&= -\int_{\Sigma_t} \frac{f}{H}\,|A|^2 + \int_{\Sigma_t} (2\,\langle \bar{\nabla} f, \nu\rangle + f\,H)\,d\mu \\
&\leq \int_{\Sigma_t} \Big(2\,\langle \bar{\nabla} f, \nu\rangle + \frac{n-2}{n-1}\,f\,H\Big)\,d\mu.
\end{aligned}
$$

Using the divergence theorem, we obtain

$$
\begin{aligned}
\int_{\Sigma_t} \langle \bar{\nabla} f, \nu\rangle\,d\mu &= \int_{\Omega_t} \bar{\Delta} f\,d\mathrm{vol} + \frac{(n-2)m + 2s_0^n}{2}\,|S^{n-1}| \\
&= n\int_{\Omega_t} f\,d\mathrm{vol} + \frac{(n-2)m + 2s_0^n}{2}\,|S^{n-1}|.
\end{aligned}
$$

Moreover, it was shown in [4] that

$$
(n-1)\int_{\Sigma_t} \frac{f}{H}\,d\mu \geq n\int_{\Omega_t} f\,d\mathrm{vol} + s_0^n\,|S^{n-1}|.
$$

Putting these facts together, we conclude that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\int_{\Sigma_t} f\,H\,d\mu - n(n-1)\int_{\Omega_t} f\,d\mathrm{vol}\Big) \\
&\leq \int_{\Sigma_t} \Big(2\,\langle \bar{\nabla} f, \nu\rangle\,d\mu + \frac{n-2}{n-1}\,f\,H - n(n-1)\,\frac{f}{H}\Big)\,d\mu \\
&\leq \frac{n-2}{n-1}\int_{\Sigma_t} f\,H\,d\mu - n(n-2)\int_{\Omega_t} f\,d\mathrm{vol} \\
&+ ((n-2)m + 2s_0^n)\,|S^{n-1}| - n\,s_0^n\,|S^{n-1}| \\
&= \frac{n-2}{n-1}\Big(\int_{\Sigma_t} f\,H\,d\mu - n(n-1)\int_{\Omega_t} f\,d\mathrm{vol} + (n-1)\,s_0^{n-2}\,|S^{n-1}|\Big).
\end{aligned}
$$

Thus, we conclude that $\frac{d}{dt} Q(t) \leq 0$, and equality holds when the surfaces Σ_t are coordinate spheres. □

Corollary 20. *We have*

$$\int_{\Sigma_0} f\, H\, d\mu - n(n-1) \int_{\Omega_0} f\, d\text{vol} \geq (n-1)\, |S^{n-1}|^{\frac{1}{n-1}} \big(|\Sigma_0|^{\frac{n-2}{n-1}} - |\partial M|^{\frac{n-2}{n-1}}\big).$$

Proof. Since $Q(t)$ is monotone decreasing, we have

$$Q(0) \geq \liminf_{t \to \infty} Q(t) \geq (n-1)\, |S^{n-1}|^{\frac{1}{n-1}}.$$

This implies

$$\int_{\Sigma_0} f\, H\, d\mu - n(n-1) \int_{\Omega_0} f\, d\text{vol} \geq (n-1)\, |S^{n-1}|^{\frac{1}{n-1}}\, |\Sigma_0|^{\frac{n-2}{n-1}} - (n-1)\, s_0^{n-2}\, |S^{n-1}|.$$

Since $|\partial M| = s_0^{n-1}\, |S^{n-1}|$, the assertion follows. □

It remains to discuss the case of equality. Suppose that

$$\int_{\Sigma_0} f\, H\, d\mu - n(n-1) \int_{\Omega_0} f\, d\text{vol} = (n-1)\, |S^{n-1}|^{\frac{1}{n-1}} \big(|\Sigma_0|^{\frac{n-2}{n-1}} - |\partial M|^{\frac{n-2}{n-1}}\big).$$

In this case, the function $Q(t)$ is constant. In particular, we must have equality in (23). Consequently, the surface Σ_0 is umbilic. If the mass m is positive, it follows that Σ_0 is a coordinate sphere, as claimed. On the other hand, if the mass m vanishes, then Σ_0 must be a geodesic sphere centered at some point x_0. If x_0 is not the origin, then the function λ converges to a non-constant function on S^{n-1} after rescaling. Using the equality statement in Proposition 17, we conclude that $\liminf_{t \to \infty} Q(t) > (n-1)\, |S^{n-1}|^{\frac{1}{n-1}}$, contrary to our assumption. Thus, Σ_0 must be a geodesic sphere centered at the origin.

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Department of Mathematics, Stanford University, Stanford, CA 94305

Department of Mathematics, National Taiwan University, Taipei

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027