

Limit of quasilocal mass at spatial infinity

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Abstract

We study the limit of quasilocal mass defined in [4] and [5] for a family of spacelike 2-surfaces in spacetime. In particular, we show the limit coincides with the ADM mass at spatial infinity. The limit for coordinate spheres of a boosted slice of the Schwarzschild solution is computed explicitly and shown to give the expected energy-momentum four-vector.

1 Review of the definition of quasilocal energy

^{1 2} In [4] and [5], we define a notion of quasilocal mass for spacelike 2-surfaces in a spacetime. Given an isometric embedding of a 2-surface into $\mathbb{R}^{3,1}$ and a future timelike unit vector (observer) in $\mathbb{R}^{3,1}$, we associated a quasilocal energy with respect to a canonical gauge. Minimizing among the reference data gives the quasilocal mass and the quasilocal energy-momentum four-vector. We prove that the mass has the important positivity property and it vanishes for surfaces in $\mathbb{R}^{3,1}$. The expression for the mass is nevertheless rather nonlinear and complicated. In this article, we show that for a family of surfaces going out to spatial infinity, the expression indeed gets “linearized” and gives a well-defined energy-momentum four-vector.

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First of all, we recall the definition of quasilocal energy in [4]. Let Σ be a spacelike 2-surface in a time-orientable spacetime N . Consider a reference isometric embedding $\Sigma \hookrightarrow \mathbb{R}^{3,1}$. Fix a future timelike unit vector t_0^ν in $\mathbb{R}^{3,1}$. We decompose t_0^ν along $\Sigma \subset \mathbb{R}^{3,1}$ into $t_0^\nu = N_0 u_0^\nu + N_0^\nu$ in which N_0 is the lapse function, N_0^ν is the shift vector, and u_0^ν is the future timelike unit normal vector field along $\Sigma \subset \mathbb{R}^{3,1}$ determined by this decomposition. We also take the spacelike outward pointing unit normal v_0^ν that is orthogonal to u_0^ν along $\Sigma \subset \mathbb{R}^{3,1}$. (u_0^ν, v_0^ν) is the reference gauge for $\Sigma \subset \mathbb{R}^{3,1}$ with respect to t_0^ν . To compute the quasilocal energy, we also need the canonical gauge $(\bar{u}^\nu, \bar{v}^\nu)$ along $\Sigma \subset N$. \bar{u}^ν is characterized as the unique future timelike unit normal vector field along $\Sigma \subset N$ such that

$$h_\nu \bar{u}^\nu = (h_0)_\nu u_0^\nu, \quad (1.1)$$

where h^ν is the mean curvature vector of $\Sigma \subset N$ and h_0^ν is the mean curvature vector of $\Sigma \subset \mathbb{R}^{3,1}$. \bar{v}^ν is the spacelike unit normal vector that is orthogonal to \bar{u}^ν and satisfies $\bar{v}^\nu h_\nu < 0$. Take a spacelike hypersurface $\Omega_0 \subset \mathbb{R}^{3,1}$ spanned by $\Sigma \subset \mathbb{R}^{3,1}$ and v_0^ν , and a spacelike hypersurface $\bar{\Omega} \subset N$ spanned by $\Sigma \subset N$ and \bar{v}^ν . Let k_0 be the mean curvature of Σ with respect to Ω_0 and \bar{k} be the mean curvature of Σ with respect to $\bar{\Omega}$. Also denote by $(K_0)_{\mu\nu}$ and $\bar{K}_{\mu\nu}$ the extrinsic curvatures of Ω_0 and $\bar{\Omega}$, respectively. These data depend only on the gauges along Σ but not on the hypersurfaces. Quasilocal energy in the canonical gauge (see equation (6) in [4]) is defined to be

$$\frac{1}{8\pi} \int_\Sigma (k_0 - \bar{k}) N_0 - (v_0^\mu (K_0)_{\mu\nu} - \bar{v}^\mu \bar{K}_{\mu\nu}) N_0^\nu. \quad (1.2)$$

We shall rewrite the quasilocal energy in terms of the mean curvature gauge. In order to do so, we adopt a different set of notations from [5]. Set $T_0 = t_0^\nu$, $H_0 = h_0^\nu$, $H = h^\nu$, $\check{e}_3 = v_0^\nu$, $\check{e}_4 = u_0^\nu$, $\bar{e}_3 = \bar{v}^\nu$, $\bar{e}_4 = \bar{u}^\nu$. Denote by $X : \Sigma \rightarrow \mathbb{R}^{3,1}$ the position vector of the isometric embedding and by $\tau = \langle X, T_0 \rangle$ the restriction of the time function associated with T_0 . $T_0 = \sqrt{1 + |\nabla \tau|^2} \check{e}_4 - \nabla \tau$ and thus $N_0 = \sqrt{1 + |\nabla \tau|^2}$ and $N_0^\nu = -\nabla \tau$. The quasilocal energy becomes

$$\frac{1}{8\pi} \int_\Sigma (-\langle H_0, \check{e}_3 \rangle + \langle H, \bar{e}_3 \rangle) \sqrt{1 + |\nabla \tau|^2} - (\langle \nabla_{-\nabla \tau}^{\mathbb{R}^{3,1}} \check{e}_4, \check{e}_3 \rangle - \langle \nabla_{-\nabla \tau}^N \bar{e}_4, \bar{e}_3 \rangle). \quad (1.3)$$

Suppose the mean curvature vector H_0 of Σ in $\mathbb{R}^{3,1}$ is spacelike. Let $e_3^{H_0} = \frac{-H_0}{|H_0|}$ be the unit vector in the direction of $-H_0$ and $e_4^{H_0}$ the future-directed

time-like unit normal vector with $\langle e_3^{H_0}, e_4^{H_0} \rangle = 0$. The relation between the two gauges is

$$e_3^{H_0} = \cosh \theta_0 \check{e}_3 + \sinh \theta_0 \check{e}_4, \text{ and } e_4^{H_0} = \sinh \theta_0 \check{e}_3 + \cosh \theta_0 \check{e}_4$$

for some $\theta_0 \in \mathbb{R}$. Since $\Delta\tau = -\langle H_0, T_0 \rangle$, we derive

$$\sinh \theta_0 = \frac{-\Delta\tau}{|H_0| \sqrt{1 + |\nabla\tau|^2}}. \quad (1.4)$$

Therefore,

$$\langle \nabla_{\nabla\tau}^{\mathbb{R}^{3,1}} \check{e}_4, \check{e}_3 \rangle = -\nabla\theta_0 \cdot \nabla\tau + \langle \nabla_{\nabla\tau}^{\mathbb{R}^{3,1}} e_4^{H_0}, e_3^{H_0} \rangle.$$

The canonical gauge condition (1.1)

$$\langle H_0, \check{e}_4 \rangle = \langle H, \bar{e}_4 \rangle$$

implies $e^H = \frac{-H}{|H|}$ is given by

$$e_3^H = \cosh \theta \bar{e}_3 + \sinh \theta \bar{e}_4 \text{ with } \sinh \theta = \frac{-\Delta\tau}{|H| \sqrt{1 + |\nabla\tau|^2}}.$$

Expression (1.3) can now be rewritten in terms of the mean curvature gauge.

To summarize, let $\Sigma \subset N$ be a spacelike 2-surface in a spacetime N and let $X : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ be a reference isometric embedding of Σ into the Minkowski space. For any given future timelike constant unit vector $T_0 \in \mathbb{R}^{3,1}$, the time function on Σ is denoted by $\tau = -\langle X, T_0 \rangle$. Let H be the mean curvature vector of Σ in N , we assume H is spacelike. Let J be the future timelike unit normal vector field along Σ in N which is dual to H along the light cone in the normal bundle of Σ in N . Denote by H_0 and J_0 the corresponding data on the isometric embedding in $\mathbb{R}^{3,1}$. Again, H_0 is assume to be spacelike in $\mathbb{R}^{3,1}$. The quasilocal energy of Σ with respect to the pair (X, T_0) is given by

$$\begin{aligned} E(\Sigma, X, T_0) &= \frac{1}{8\pi} \int_{\Sigma} \sqrt{|H_0|^2(1 + |\nabla\tau|^2) + (\Delta\tau)^2} - \sqrt{|H|^2(1 + |\nabla\tau|^2) + (\Delta\tau)^2} \\ &\quad - \Delta\tau \left[\sinh^{-1} \left(\frac{\Delta\tau}{\sqrt{1 + |\nabla\tau|^2} |H_0|} \right) - \sinh^{-1} \left(\frac{\Delta\tau}{\sqrt{1 + |\nabla\tau|^2} |H|} \right) \right] \\ &\quad - \langle \nabla_{\nabla\tau}^{\mathbb{R}^{3,1}} \frac{J_0}{|H_0|}, \frac{H_0}{|H_0|} \rangle + \langle \nabla_{\nabla\tau}^N \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_{\Sigma}, \end{aligned} \quad (1.5)$$

where $\Delta\tau$ is the Laplacian of τ on Σ (with respect to the induced metric), and ∇^N and $\nabla^{\mathbb{R}^{3,1}}$ are the covariant derivatives on N and $\mathbb{R}^{3,1}$, respectively, and $\nabla\tau$ is the gradient of τ on Σ (with respect to the induced metric again), considered as a tangent vector field on Σ . In the expressions for the last two integrands, we push forward $\nabla\tau$ by the embeddings and identify it as vector fields along Σ in $\mathbb{R}^{3,1}$ and N , respectively.

2 General formula for the limit of quasilocal energy

Fix $R > 0$ and suppose Σ_r , $R < r < \infty$, is a family of closed 2-surfaces in N , and X_r is a family of isometric embeddings of Σ_r into $\mathbb{R}^{3,1}$. In the following theorem, we derive an expression for the limit of $E(\Sigma_r, X_r, T_0)$.

Theorem 2.1 *Suppose the mean curvature vectors of Σ_r and of the image of X_r in $\mathbb{R}^{3,1}$ are both spacelike for $r > R_0$ and $\frac{|H|}{|H_0|} \rightarrow 1$ as $r \rightarrow \infty$. Then the limit of $E(\Sigma_r, X_r, T_0)$ as $r \rightarrow \infty$ (if exists) is the same as the limit of*

$$\frac{1}{8\pi} \int_{\Sigma_r} -\langle T_0, \frac{J_0}{|H_0|} \rangle (|H_0| - |H|) - \langle \nabla_{\nabla\tau}^{\mathbb{R}^{3,1}} \frac{J_0}{|H_0|}, \frac{H_0}{|H_0|} \rangle + \langle \nabla_{\nabla\tau}^N \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_r.$$

Proof. We compute

$$\Delta\tau = -H_0 \cdot T_0 = |H_0| \langle e_3^{H_0}, T_0 \rangle \quad (2.1)$$

and

$$|\nabla\tau|^2 = -1 + \langle e_4^{H_0}, T_0 \rangle^2 - \langle e_3^{H_0}, T_0 \rangle^2 \quad (2.2)$$

where $e_3^{H_0} = \frac{-H_0}{|H_0|}$ and $e_4^{H_0} = \frac{J_0}{|H_0|}$ is the future timelike unit normal dual to $e_3^{H_0}$ along the image of X in $\mathbb{R}^{3,1}$.

Rationalize the expression

$$\sqrt{|H_0|^2(1 + |\nabla\tau|^2) + (\Delta\tau)^2} - \sqrt{|H|^2(1 + |\nabla\tau|^2) + (\Delta\tau)^2}$$

as

$$(|H_0| - |H|) \frac{(|H_0| + |H|)(1 + |\nabla\tau|^2)}{\sqrt{|H_0|^2(1 + |\nabla\tau|^2) + (\Delta\tau)^2} + \sqrt{|H|^2(1 + |\nabla\tau|^2) + (\Delta\tau)^2}}.$$

By assumption $\frac{|H|}{|H_0|} \rightarrow 1$ at infinity, the limit as $r \rightarrow \infty$ is thus the same as the limit of

$$\frac{1}{8\pi} \int_{\Sigma_r} \frac{\langle e_4^{H_0}, T_0 \rangle^2 - \langle e_3^{H_0}, T_0 \rangle^2}{-\langle e_4^{H_0}, T_0 \rangle} (|H_0| - |H|) dv_{\Sigma_r}. \quad (2.3)$$

Next we study the term

$$-\Delta\tau \left[\sinh^{-1} \left(\frac{\Delta\tau}{\sqrt{1 + |\nabla\tau|^2} |H_0|} \right) - \sinh^{-1} \left(\frac{\Delta\tau}{\sqrt{1 + |\nabla\tau|^2} |H|} \right) \right]$$

by rewriting it as

$$-\Delta\tau \left[\sinh^{-1} \left(\frac{\Delta\tau}{\sqrt{1 + |\nabla\tau|^2} |H_0|} \right) - \sinh^{-1} \left(\frac{\Delta\tau}{\sqrt{1 + |\nabla\tau|^2} |H_0|} \frac{|H_0|}{|H|} \right) \right].$$

Note that

$$\frac{\sinh^{-1} A - \sinh^{-1}(A(1+x))}{x} \rightarrow \frac{-A}{\sqrt{1+A^2}}$$

as $x \rightarrow 0$. With $x = \frac{|H_0|}{|H|} - 1 \rightarrow 0$, the limit of the second term is thus the same as the limit of

$$\frac{1}{8\pi} \int_{\Sigma_r} \frac{\langle e_3^{H_0}, T_0 \rangle^2}{-\langle e_4^{H_0}, T_0 \rangle} (|H_0| - |H|) dv_r. \quad (2.4)$$

The theorem is proved by combining (2.3) and (2.4). \square

Suppose the image of the isometric embedding lies X_r in $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$, then $e_4^{H_0} = \frac{J_0}{|H_0|}$ is a constant vector and the $\langle \nabla_{\nabla\tau}^{\mathbb{R}^{3,1}} \frac{J_0}{|H_0|}, \frac{H_0}{|H_0|} \rangle$ term vanishes. In this case, $e_3^{H_0}$ coincide with the outward unit normal of the embedding in \mathbb{R}^3 .

Corollary 2.1 *Suppose the reference isometric embedding is in $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ and $\frac{|H|}{|H_0|} \rightarrow 1$ as $r \rightarrow \infty$, then the limit of the quasilocal energy with respect to $T_0 = (\sqrt{1 + |a|^2}, a^1, a^2, a^3)$ with $|a|^2 = \sum_{i=1}^3 (a^i)^2$ is*

$$(\sqrt{1 + |a|^2}) \frac{1}{8\pi} \int_{\Sigma_r} |H_0| - |H| dv_r + \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla_{\nabla\tau}^N \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_r. \quad (2.5)$$

Suppose the isometric embedding for Σ_r is given by $X_r = (X^1, X^2, X^3) : \Sigma \rightarrow \mathbb{R}^3$ and consider $X^i, i = 1, 2, 3$ as functions on Σ_r . Thus $\nabla \tau = -\sum_{i=1}^3 a^i \nabla X^i$ and we obtain a limiting quasilocal energy-momentum four-vector (e, p_1, p_2, p_3) as the limit of

$$\begin{aligned} e &= \lim_{r \rightarrow \infty} \frac{1}{8\pi} \int_{\Sigma_r} |H_0| - |H| dv_r \\ p_i &= \lim_{r \rightarrow \infty} \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla_{-\nabla X^i}^N \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_r, i = 1, 2, 3. \end{aligned} \quad (2.6)$$

3 Relating to ADM energy-momentum

Let (M, g_{ij}, p_{ij}) be an asymptotically flat hypersurface in a spacetime N . Thus there exists a compact set $K \subset M$ such that $M \setminus K$ is diffeomorphic to a union of complements of balls in \mathbb{R}^3 (ends) such that $g_{ij} = \delta_{ij} + a_{ij}$ with $a_{ij} = O(\frac{1}{r})$, $\partial_k(a_{ij}) = O(\frac{1}{r^2})$, $\partial_l \partial_k(a_{ij}) = O(\frac{1}{r^3})$, and $p_{ij} = O(\frac{1}{r^2})$, $\partial_k(p_{ij}) = O(\frac{1}{r^3})$ on each end of $M \setminus K$.

The ADM energy momentum (Arnowitt-Deser-Misner) of an end of M is the four vector (E, P_1, P_2, P_3) where

$$E = \lim_{r \rightarrow \infty} \frac{1}{16\pi} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) \nu^i dv_r$$

is the total energy and

$$P_k = \lim_{r \rightarrow \infty} \frac{1}{16\pi} \int_{S_r} 2(p_{ik} - \delta_{ik} p_{jj}) \nu^i dv_r$$

is the total momentum. Here S_r is a coordinate sphere of radius r on the end and ν is the outward unit normal of S_r .

The positive mass theorem (Schoen-Yau [3], Witten [6]) asserts that under the dominant energy condition, the four-vector (E, P_1, P_2, P_3) is future timelike, i.e.

$$E \geq 0 \text{ and } -E^2 + P_1^2 + P_2^2 + P_3^2 \leq 0.$$

In the following, we prove that for coordinate spheres of radius r , the limit of the quasilocal energy momentum (2.6) is the same as the ADM energy-momentum.

Theorem 3.1 *Suppose S_r is the coordinate sphere of radius r in an end of an asymptotically flat three-manifold (M, g_{ij}, p_{ij}) and (E, P_1, P_2, P_3) is the ADM energy-momentum four vector of this end, then*

$$\lim_{r \rightarrow \infty} E(S_r, X_r, T_0) = \sqrt{1 + |a|^2} E + \sum_{i=1}^3 a^i P_i$$

where X_r is the (unique) isometric embedding of S_r into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ and $T_0 = (\sqrt{1 + |a|^2}, a^1, a^2, a^3)$ is an arbitrary constant timelike unit vector.

Proof. Denote by e_0 the future timelike unit normal of the hypersurface M and ν the unit outward normal of the coordinate sphere S_r . Let (y^1, y^2, y^3) be the asymptotically flat coordinates on the end. S_r is given by $(y^1)^2 + (y^2)^2 + (y^3)^2 = r^2$ and we denote the embedding of S_r into M by Y . Since $p_{ij} = O(\frac{1}{r^2})$, we have $\langle H, e_0 \rangle = O(\frac{1}{r^2})$. It is known that $\langle H, \nu \rangle = \frac{2}{r} + O(\frac{1}{r^2})$ (see for example [1]). Since $H = \langle H, \nu \rangle \nu - \langle H, e_0 \rangle e_0$, we estimate

$$|H| - |\langle H, \nu \rangle| = O(\frac{1}{r^3}).$$

Therefore,

$$\lim_{r \rightarrow \infty} \int_{S_r} |H_0| - |H| dv_r = \lim_{r \rightarrow \infty} \int_{S_r} |H_0| - |\langle H, \nu \rangle| dv_r,$$

i.e, the Brown-York energy and the Liu-Yau energy have the same limit at spatial infinity. It is known that (see for example [1] and the reference therein) the Brown-York energy approaches the ADM energy E at spatial infinity.

Now it suffices to prove

$$\sum_{i=1}^3 a^i P_i = \sum_{i=1}^3 a^i p_i = \frac{1}{8\pi} \int_{\Sigma_r} \langle \nabla_{\nabla \tau}^N \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_r.$$

By definition, the ADM momentum is

$$\sum_{i=1}^3 a^i P_i = \frac{1}{8\pi} \int_{S_r} p(a^i \frac{\partial}{\partial y^i}, \nu) - (tr p) \langle a^i \frac{\partial}{\partial y^i}, \nu \rangle dv_r.$$

We decompose $a^i \frac{\partial}{\partial y^i} = (a^i \frac{\partial}{\partial y^i})^\top + \langle a^i \frac{\partial}{\partial y^i}, \nu \rangle \nu$ and the integrand becomes

$$p((a^i \frac{\partial}{\partial y^i})^\top, \nu) + \langle a^i \frac{\partial}{\partial y^i}, \nu \rangle (p(\nu, \nu) - (trp)).$$

By the definition of the mean curvature vector H , we obtain $p(\nu, \nu) - (trp) = \langle H, e_0 \rangle$. Therefore the ADM momentum term is

$$\sum_{i=1}^3 a^i P_i = \frac{1}{8\pi} \int_{S_r} \langle \nabla_{(a^i \frac{\partial}{\partial y^i})^\top}^N e_0, \nu \rangle + \langle H, e_0 \rangle \langle a^i \frac{\partial}{\partial y^i}, \nu \rangle dv_r. \quad (3.1)$$

Now we turn to the limit of the quasilocal energy momentum. We can express the normal vector fields H and J in terms of ν and e_0 as

$$H = \langle H, \nu \rangle \nu - \langle H, e_0 \rangle e_0 \text{ and } J = -\langle H, \nu \rangle e_0 + \langle H, e_0 \rangle \nu.$$

We compute

$$\langle \nabla_{\nabla\tau}^N \frac{J}{|H|}, \frac{H}{|H|} \rangle = -\nabla\tau \cdot \nabla \sinh^{-1}(\frac{\langle H, e_0 \rangle}{|H|}) - \langle \nabla_{\nabla\tau}^N e_0, \nu \rangle.$$

Integrating by parts gives

$$\frac{1}{8\pi} \int_{S_r} \langle \nabla_{\nabla\tau}^N \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_r = \frac{1}{8\pi} \int_{S_r} -\langle \nabla_{\nabla\tau}^N e_0, \nu \rangle + \Delta\tau \sinh^{-1}(\frac{\langle H, e_0 \rangle}{|H|}) dv_r.$$

Plug in (2.1), the second integrand on the right hand side becomes

$$\langle e_3^{H_0}, T_0 \rangle |H_0| \sinh^{-1}(\frac{\langle H, e_0 \rangle}{|H|}).$$

Recall the asymptotics

$$\langle H, e_0 \rangle = O(\frac{1}{r^2}), |H| = \frac{2}{r} + O(\frac{1}{r^2}), |H_0| = \frac{2}{r} + O(\frac{1}{r^2})$$

and $\sinh^{-1} x \sim x$ if $x \ll 1$. We see the limit of $\frac{1}{8\pi} \int_{S_r} \langle \nabla_{\nabla\tau}^N \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_r$ is the same as

$$\lim_{r \rightarrow \infty} \frac{1}{8\pi} \int_{S_r} -\langle \nabla_{\nabla\tau}^N e_0, \nu \rangle + \langle H, e_0 \rangle \langle e_3^{H_0}, T_0 \rangle dv_r. \quad (3.2)$$

Now we can compare (3.1) and (3.2). Write out the tangential part of $a^i \frac{\partial}{\partial y^i}$,

$$(a^i \frac{\partial}{\partial y^i})^\top = \langle a^i \frac{\partial}{\partial y^i}, \frac{\partial Y}{\partial u^a} \rangle \sigma^{ab} \frac{\partial Y}{\partial u^b} = a^i g_{ij} \frac{\partial Y^j}{\partial u^a} \sigma^{ab} \frac{\partial Y}{\partial u^b}.$$

On the other hand, as $\tau = -a^i X^i$, and the push-forward of $\nabla \tau$ becomes

$$\nabla \tau = -a^i \frac{\partial X^i}{\partial u^a} \sigma^{ab} \frac{\partial Y}{\partial u^b}.$$

The isometric embeddings satisfy (see for example [1])

$$|X^i - Y^i| = O(1) \text{ and } |e_3^{H_0} - \nu| = O(\frac{1}{r}).$$

From these, we deduce that (3.2) is the same as the limit of the right hand side of (3.1) and the theorem is proved. \square

4 Explicit computation in a boosted slice of Schwarzschild's solution

In this section, we compute the limit of quasilocal energy-momentum for coordinate spheres of a boosted slice of Schwarzschild's solution.

4.1 Asymptotics of the geometry of coordinate spheres

Let (y^0, y^1, y^2, y^3) be the isotropic coordinates of Schwarzschild's solution in which the spacetime metric is of the form:

$$G_{\alpha\beta} dy^\alpha dy^\beta = -\frac{1}{F^2} (dy^0)^2 + \frac{1}{G^2} \sum_{i=1}^3 (dy^i)^2$$

with

$$F^2 = \frac{(1 + \frac{M}{2\rho})^2}{(1 - \frac{M}{2\rho})^2}, \quad G^2 = \frac{1}{(1 + \frac{M}{2\rho})^4}$$

and $\rho^2 = \sum_{i=1}^3 (y^i)^2$. Given $\gamma > 0$ and β which satisfy $\gamma^2 - \beta^2 \gamma^2 = 1$, consider the surface Σ_{r_0} defined by

$$\gamma y^0 - \beta \gamma y^3 = 0$$

and

$$(y^1)^2 + (y^2)^2 + (\gamma y^3 - \beta \gamma y^0)^2 = r_0^2$$

with $r_0 \rightarrow \infty$. With the coordinate change $(y^0)' = \gamma y^0 - \beta \gamma y^3$ and $(y^3)' = \gamma y^3 - \beta \gamma y^0$, these surfaces are coordinate spheres of radius r_0 in the asymptotically flat slice $\gamma y^0 - \beta \gamma y^3 = 0$

We parametrize the 2-surfaces Σ_{r_0} by

$$\begin{aligned} y^0 &= \beta \gamma r_0 \cos \theta \\ y^1 &= r_0 \sin \theta \sin \phi \\ y^2 &= r_0 \sin \theta \cos \phi \\ y^3 &= \gamma r_0 \cos \theta. \end{aligned}$$

Denote the embedding of Σ_{r_0} into Schwarzschild's solution by $Y = (y^0, y^1, y^2, y^3)$. In terms of local coordinates $u^1 = \theta$ and $u^2 = \phi$ on the surface, the induced metric on Σ_{r_0} is

$$\sigma_{ab} = r_0^2 \left[1 + \frac{2M}{\rho} (1 + 2\beta^2 \gamma^2 \sin^2 \theta) \right] d\theta^2 + r_0^2 \left(1 + \frac{2M}{\rho} \right) \sin^2 \theta d\phi^2 + O(r_0), \quad (4.1)$$

and

$$\sqrt{\det \sigma_{ab}} = r_0^2 |\sin \theta| \left[1 + \frac{2M}{\rho} (1 + \beta^2 \gamma^2 \sin^2 \theta) \right] + O(r_0). \quad (4.2)$$

The mean curvature vector $H = H^\gamma \frac{\partial}{\partial y^\gamma}$ of Σ_{r_0} is by definition:

$$H^\gamma = \sigma^{ab} \left(\frac{\partial^2 y^\gamma}{\partial u^a \partial u^b} + \Gamma_{\alpha\beta}^\gamma \frac{\partial y^\alpha}{\partial u^a} \frac{\partial y^\beta}{\partial u^b} \right) (\delta_\beta^\gamma - \Pi_\beta^\gamma)$$

where $\Gamma_{\alpha\beta}^\gamma$ are the Christoffel symbols of the metric $G_{\alpha\beta}$ and $\Pi_\beta^\gamma = G_{\beta\alpha} \sigma^{ab} \frac{\partial y^\alpha}{\partial u^a} \frac{\partial y^\gamma}{\partial u^b}$ is the projection operator on the tangent space of Σ_{r_0} . The asymptotic expansion of $\Gamma_{\alpha\beta}^\gamma$ can be computed from the asymptotic expansion of $G_{\alpha\beta}$.

Denote by $\tilde{y}^\alpha = \frac{y^\alpha}{r_0}$ and $\tilde{\rho} = \frac{\rho}{r_0}$, which are both scaling invariant now. We shall use the following frames along Σ_{r_0} to express the mean curvature vector:

$$\begin{aligned} N &= \tilde{y}^\alpha \frac{\partial}{\partial y^\alpha}, \\ B &= \gamma \left(\frac{\partial}{\partial y^0} + \beta \frac{\partial}{\partial y^3} \right), \text{ and} \\ T &= \frac{\partial \tilde{y}^\alpha}{\partial \theta} \frac{\partial}{\partial y^\alpha}. \end{aligned}$$

We notice that T is a tangent vector field to Σ_{r_0} while N and B are only asymptotically normal in the sense that $\langle N, T \rangle = O(\frac{1}{r_0})$ and $\langle B, T \rangle = O(\frac{1}{r_0})$. We also have $\langle N, N \rangle = 1 + O(\frac{1}{r_0})$, $\langle B, B \rangle = -1 + O(\frac{1}{r_0})$, and $\langle T, T \rangle = 1 + O(\frac{1}{r_0})$.

A straightforward calculation gives

Lemma 4.1

$$H = \frac{-2}{r_0}N + \frac{1}{r_0^2}(\mathfrak{n}N + \mathfrak{t}T + \mathfrak{b}B) + O(\frac{1}{r_0^3})$$

with

$$\begin{aligned}\mathfrak{n} &= \frac{M}{\tilde{\rho}^3}(6 + 6\beta^2\gamma^2 + 2\beta^4\gamma^4\sin^2\theta\cos^2\theta), \\ \mathfrak{t} &= \frac{M}{\tilde{\rho}^3}(-8\tilde{\rho}^2)(\beta^2\gamma^2\sin\theta\cos\theta), \text{ and} \\ \mathfrak{b} &= \frac{M}{\tilde{\rho}^3}(2\beta\gamma^2\cos\theta)(\beta^2\gamma^2\sin^2\theta - 1).\end{aligned}$$

From here, we compute the norm of $|H|$,

Proposition 4.1

$$|H| = \frac{2}{r_0} + \frac{1}{r_0^2}\left[\frac{2M}{\tilde{\rho}}(1 + 2\beta^2\gamma^2\cos^2\theta) - \mathfrak{n}\right] + O(\frac{1}{r_0^3}).$$

Let J be the future-directed timelike normal vector that is dual to H along the light cone in the normal bundle.

Lemma 4.2 J is given by

$$\begin{aligned}&\frac{2}{r_0}B - \frac{1}{r_0^2}\left[\frac{-2M(1 + 2\beta^2\gamma^2\cos^2\theta + \gamma^2 + \beta^2\gamma^2)}{\tilde{\rho}} + \mathfrak{n}\right]B \\ &+ \frac{1}{r_0^2}\left[\frac{8M}{\tilde{\rho}}(\beta\gamma^2\sin\theta)T - (\mathfrak{b} + \frac{8M\beta\gamma^2\cos\theta}{\tilde{\rho}})N\right].\end{aligned}$$

Proof. The coefficients of J are determined by the following equations:

$$-\langle J, J \rangle = \langle H, H \rangle,$$

$$\langle J, H \rangle = 0$$

and

$$\langle J, T \rangle = 0.$$

□

With this explicit formula, we compute the coefficients of the connection form of the normal bundle in the mean curvature vector gauge:

Proposition 4.2

$$\langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H \rangle = \frac{1}{r_0^3} (2\mathfrak{b}' + \frac{8M}{\tilde{\rho}} \beta \gamma^2 \sin \theta) + O(\frac{1}{r_0^4})$$

and

$$\langle \nabla_{\frac{\partial Y}{\partial \phi}} J, H \rangle = O(\frac{1}{r_0^4}).$$

It turns out the second term does not contribute to the limit of the quasilocal energy.

4.2 Total mean curvature of isometric embedding

We consider the isometric embedding of a general axially symmetric metric into \mathbb{R}^3 . The metric is of the form

$$r_0^2 P^2(r_0, \theta) d\theta^2 + r_0^2 Q^2(r_0, \theta) \sin^2 \theta d\phi^2$$

with

$$P(r_0, \theta) = 1 + O(\frac{1}{r_0}), \text{ and } Q(r_0, \theta) = 1 + O(\frac{1}{r_0}).$$

Suppose the isometric embedding is given by

$$X = (u(r_0, \theta) \sin \phi, u(r_0, \theta) \cos \phi, v(r_0, \theta)).$$

Thus

$$\frac{\partial X}{\partial \theta} = (\frac{\partial u}{\partial \theta} \sin \phi, \frac{\partial u}{\partial \theta} \cos \phi, \frac{\partial v}{\partial \theta})$$

and

$$\frac{\partial X}{\partial \phi} = (u \cos \phi, -u \sin \phi, 0).$$

It is not hard to see

Lemma 4.3 *u and v are given by*

$$\left(\frac{\partial u}{\partial \theta}\right)^2 + \left(\frac{\partial v}{\partial \theta}\right)^2 = r_0^2 P^2$$

and

$$u^2 = r_0^2 Q^2 \sin^2 \theta.$$

Proposition 4.3 *The mean curvature of the isometric embedding of the metric $r_0^2 P^2(r_0, \theta) d\theta^2 + r_0^2 Q^2(r_0, \theta) \sin^2 \theta d\phi^2$ into \mathbb{R}^3 is given by*

$$H_0 = -\left(\frac{1}{r_0^3 P^3}\right)\left(\frac{\partial^2 u}{\partial \theta^2} \frac{\partial v}{\partial \theta} + \frac{\partial^2 v}{\partial \theta^2} \frac{\partial u}{\partial \theta}\right) + \left(\frac{1}{r_0^2 P Q \sin \theta}\right) \frac{\partial v}{\partial \theta}.$$

where

$$u(r_0, \theta) = r_0 Q(r_0, \theta) \sin \theta,$$

and

$$\left(\frac{\partial v}{\partial \theta}\right)^2 = r_0^2 P^2 - \left(\frac{\partial u}{\partial \theta}\right)^2 = r_0^2 [P^2 - \left(\frac{\partial Q}{\partial \theta} \sin \theta + Q \cos \theta\right)^2].$$

Now suppose

$$P = 1 + \frac{p}{r_0} + O\left(\frac{1}{r_0^2}\right), p = p(\theta)$$

and

$$Q = 1 + \frac{q}{r_0} + O\left(\frac{1}{r_0^2}\right), q = q(\theta).$$

The asymptotic expansion of the mean curvature is found to be

$$H_0 = \frac{2}{r_0} - \frac{1}{r^2} (2p + \frac{\cos \theta}{\sin \theta} (2q' - p') + q''). \quad (4.3)$$

Comparing with (4.1), we deduce that in our case

$$p = \frac{M}{\tilde{\rho}} (1 + 2\beta^2 \gamma^2 \sin^2 \theta) \text{ and } q = \frac{M}{\tilde{\rho}}.$$

u and v can be solved explicitly:

$$u = r_0 \sin \theta + \frac{M}{\tilde{\rho}} \sin \theta \text{ and } v = r_0 \cos \theta + \frac{M}{\tilde{\rho}} \cos \theta + 2M\beta\gamma \sinh^{-1}(\beta\gamma \cos \theta). \quad (4.4)$$

Plug in the expression of p and q into (4.3) and integrate by parts, we obtain

Proposition 4.4

$$\int_{\Sigma_{r_0}} H_0 dv_{r_0} = 8\pi r_0 + 2\pi M \int_0^{2\pi} \frac{1 + \beta^2 \gamma^2 \sin^2 \theta}{\tilde{\rho}} |\sin \theta| d\theta + O\left(\frac{1}{r_0}\right).$$

This calculation is compatible with Lemma 2.4 in [1].

4.3 Evaluating the quasilocal energy

We are ready to compute the limit of the Liu-Yau mass:

Proposition 4.5

$$\int_{\Sigma_{r_0}} (H_0 - |H|) dv_{r_0} = 8\pi \gamma M + O\left(\frac{1}{r_0}\right).$$

Proof. Combine Proposition 4.1 and Proposition 4.4, we obtain

$$\int_{\Sigma_{r_0}} (H_0 - |H|) dv_{r_0} = \pi M \int_0^{2\pi} \frac{2 + 4\beta^2 \gamma^2 - 6\beta^2 \gamma^2 \cos^2 \theta - 4\beta^4 \gamma^4 \cos^4 \theta}{\tilde{\rho}} |\sin \theta| d\theta + O\left(\frac{1}{r_0}\right).$$

The integral can be evaluate by the substitution $\beta \gamma \cos \theta = \sinh y$. \square
Now we turn to the momentum part. Suppose $T_0 = (\sqrt{1 + |a|^2}, a^1, a^2, a^3)$, $|a|^2 = \sum_{i=1}^3 (a^i)^2$ is a future timelike unit vector and the isometric embedding into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ is given by $X = (0, u \sin \phi, u \cos \phi, v)$. We know from (4.4) that $u = r \sin \theta + O(1)$ and $v = r \cos \theta + O(1)$. The gradient of τ is given by $\nabla \tau = \frac{\partial \tau}{\partial u^a} \sigma^{ab} \frac{\partial Y}{\partial u^b}$. We compute

$$\begin{aligned} \int_{\Sigma_{r_0}} \langle \nabla_{\nabla \tau} \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_{r_0} &= -a^1 \int_{\Sigma_{r_0}} \frac{1}{|H|^2} (u \sin \theta)' \sigma^{\theta\theta} \langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H \rangle dv_{r_0} \\ &\quad - a^2 \int_{\Sigma_{r_0}} \frac{1}{|H|^2} (u \cos \theta)' \sigma^{\theta\theta} \langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H \rangle dv_{r_0} \\ &\quad - a^3 \int_{\Sigma_{r_0}} \frac{1}{|H|^2} v' \sigma^{\theta\theta} \langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H \rangle dv_{r_0} + O\left(\frac{1}{r_0}\right). \end{aligned} \quad (4.5)$$

These integrals can be evaluated and we obtain

Proposition 4.6

$$\int_{\Sigma_{r_0}} \langle \nabla_{\nabla \tau}^N \frac{J}{|H|}, \frac{H}{|H|} \rangle dv_{r_0} = a^3 8\pi \beta \gamma M + O\left(\frac{1}{r_0}\right).$$

Proof. By Proposition 4.2, $\langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H \rangle$ is of the order $\frac{1}{r_0^3}$ while u and v are both of order r_0 , we have

$$\begin{aligned} & \int_{\Sigma_{r_0}} \frac{1}{|H|^2} (u \sin \theta)' \sigma^{\theta\theta} \langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H \rangle dv_{r_0} \\ &= \int_0^\pi \int_0^{2\pi} \frac{r_0^2}{4} (r_0 \sin^2 \theta)' \frac{1}{r_0^2} \langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H \rangle r_0^2 |\sin \theta| d\theta d\phi \\ &= \frac{\pi}{4} \int_0^{2\pi} (\sin^2 \theta)' [r_0^3 \langle \nabla_{\frac{\partial Y}{\partial \theta}} J, H \rangle] |\sin \theta| d\theta. \end{aligned}$$

Therefore the first integral on the right hand side of (4.5) is

$$-a^1 \frac{\pi}{4} \int_0^{2\pi} (\sin^2 \theta)' (2\mathbf{b}' + \frac{8M}{\tilde{\rho}} \beta \gamma^2 \sin \theta) |\sin \theta| d\theta$$

where

$$\mathbf{b} = \frac{M}{\tilde{\rho}^3} (2\beta \gamma^2 \cos \theta) (\beta^2 \gamma^2 \sin^2 \theta - 1),$$

and the second one is

$$-a^2 \frac{\pi}{4} \int_0^{2\pi} (\sin \theta \cos \theta)' (2\mathbf{b}' + \frac{8M}{\tilde{\rho}} \beta \gamma^2 \sin \theta) |\sin \theta| d\theta.$$

Both integrate to zero as they are of the form $\int_0^{2\pi} (\cos \theta) F(\cos^2 \theta) |\sin \theta| d\theta$ or $\int_0^{2\pi} (\sin \theta) F(\cos^2 \theta) |\sin \theta| d\theta$ for some smooth function F of $\cos^2 \theta$. The last integral becomes

$$a^3 \frac{\pi}{4} \int_0^{2\pi} \sin \theta (2\mathbf{b}' + \frac{8M}{\tilde{\rho}} \beta \gamma^2 \sin \theta) |\sin \theta| d\theta$$

which can be simplified as

$$a^3 4\pi M \beta \gamma^2 \int_0^\pi \frac{\sin \theta}{\tilde{\rho}^3} d\theta.$$

Using the same substitution $\beta \gamma \cos \theta = \sinh y$, the integral is

$$a^3 8\pi \beta \gamma M.$$

□

Therefore the limit of the quasilocal energy (1.5) is

$$(\sqrt{1 + |a|^2})\gamma M + a^3\beta\gamma M.$$

Recall that $\gamma^2 - \beta^2\gamma^2 = 1$. Minimizing this expression among all $T_0 = (\sqrt{1 + |a|^2}, a^1, a^2, a^3)$, we see the minimum is achieved at $(\gamma, 0, 0, -\beta\gamma)$ and the minimum value is M . The limit of the quasilocal energy-momentum is thus $M(\gamma, 0, 0, -\beta\gamma)$.

References

- [1] X.-Q. Fan, Y. Shi, and L.-F. Tam, *Large-sphere and small-sphere limits of the Brown-York mass*. Comm. Anal. Geom. **17** (2009), no. 1, 37-72.
- [2] C.-C. M. Liu and S.-T. Yau, *Positivity of quasilocal mass*, Phys. Rev. Lett. **90** (2003), no. 23, 231102.
- [3] R. Schoen and S.-T. Yau, *Positivity of the total mass of a general space-time*, Phys. Rev. Lett. **43** (1979), no. 20, 1457–1459.
- [4] M.-T. Wang and S.-T. Yau, *Quasilocal mass in general relativity*. Phys. Rev. Lett. **102** (2009), no. 2, no. 021101.
- [5] M.-T. Wang and S.-T. Yau, *Isometric embeddings into the Minkowski space and new quasi-local mass*. Comm. Math. Phys. **288** (2009), 919-942.
- [6] E. Witten, *A new proof of the positive energy theorem*. Comm. Math. Phys. **80** (1981), no. 3, 381–402.