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Isometric Emdeddings and Quasi-local Mass

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Abstract

In this note, we report and discuss some recent results on the quasi-local mass in general relativity and the relations to the isometric embedding problems.

Keywords and Phrases: Isometric embedding, Quasi-local mass, Positive mass theorem.

1. Introduction

I would like to start with the positive mass theorem which is one of the few general relativity theorems that are well-known among mathematicians and enormously useful to a geometric-analyst like myself. Let N^4 be a space-time, i.e. a four-manifold with a Lorentzian metric $g_{\alpha\beta}$ of signature $(-+++)$ that satisfies the *Einstein equation*:

$$R_{\alpha\beta} - \frac{s}{2} g_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

where $R_{\alpha\beta}$ and s are the Ricci curvature and the Ricci scalar curvature of $g_{\alpha\beta}$, respectively. G is the gravitational constant and $T_{\alpha\beta}$ is the energy-momentum tensor of matter density. The metric $g_{\alpha\beta}$ defines space-like, time-like and null vectors on the tangent space of N^4 accordingly.

A "dominant energy condition", which corresponds to a positivity condition on the matter density $T_{\alpha\beta}$, is expected to be satisfied on any realistic space-time. It means the following: for any time-like vector e_0, $T(e_0, e_0) \geq 0$ and $T(e_0, \cdot)$ is a non-space-like co-vector. We shall assume our space-time N^4 satisfies the dominant energy condition. Consider a

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space-like hypersurface (M^3, g_{ij}, p_{ij}) in N^4 where g_{ij} is the induced (Riemannian) metric and p_{ij} is the second fundamental form with respect to a time-like unit normal vector field e_0 of M. The dominant energy condition together with the compatibility conditions for submanifolds imply

$$\mu \geq |J| \tag{1.1}$$

where

$$\mu = \frac{1}{2}(R - p_{ij}p^{ij} + (p^k_k)^2)$$

and

$$J^i = D_j(p^{ij} - p^k_k g^{ij}).$$

Here R is the scalar curvature of M^3. We shall assume our (M^3, g_{ij}, p_{ij}) (so called an initial data set) satisfies (2.1).

An important special case is when $p_{ij} = 0$ (so called time-symmetric case) and the dominant energy condition implies M is simply a Riemannian manifolds with non-negative scalar curvature.

The positive mass theorems proved by Schoen-Yau [SY1, SY2, SY3] (later a different proof by Witten [WI]) states:

Theorem 1 *Let (M^3, g_{ij}, p_{ij}) be a complete three manifold that satisfies (2.1). Suppose M is asymptotically flat: i.e. there exists a compact set $K \subset M$ such that $M \backslash K$ is diffeomorphic to a union of complements of balls in $\mathbb{R}^3$ (called ends) such that $g_{ij} = \delta_{ij} + a_{ij}$ on $M \backslash K$ with $a_{ij} = O(\frac{1}{r})$, $\partial_k(a_{ij}) = O(\frac{1}{r^2})$, $\partial_l\partial_k(a_{ij}) = O(\frac{1}{r^3})$, and $p_{ij} = O(\frac{1}{r^2})$, $\partial_k(p_{ij}) = O(\frac{1}{r^3})$.*

Then the ADM mass (Arnowitt-Deser-Misner) of M is positive, i.e.

$$E \geq |P| \tag{1.2}$$

where

$$E = \lim_{r \to \infty} \frac{1}{16\pi G} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) d\Omega^i$$

is the total energy and

$$P_k = \lim_{r \to \infty} \frac{1}{16\pi G} \int_{S_r} 2(p_{ik} - \delta_{ik} p_{jj}) d\Omega^i$$

is the total momentum. Here S_r is the coordinate sphere of radius r on each end.

We notice that the conclusion of the theorem is equivalent to the four-vector (E, P_1, P_2, P_3) is future-directed time-like, i.e.

$$E \geq 0 \text{ and } -E^2 + P_1^2 + P_2^2 + P_3^2 \leq 0.$$

The asymptotic flat condition is indeed a gauge condition so M can be compared to the flat space $\mathbb{R}^3$. The essence of the positive mass theorem is that positive local matter density (2.1) measured pointwise should imply positive total energy momentum (2.2) measured at infinity. In contrast, the "quasi-local mass" corresponds to the measurement of mass of in-between scales.

2. Quasi-local mass of two-surfaces in space-time

Suppose Σ is a closed space-like two-surface in a space-time N^4. We assume Σ^2 bounds a space-like hypersurface M^3 in N^4. The definition of a notion of quasi-local mass m_Σ demands that (see [EA], [CY])

(1) $m_\Sigma \geq 0$ under the dominant energy condition.

(2) $m_\Sigma = 0$ if and only if Σ is in the Minkowski spacetime.

(3) The limit of m_Σ on large coordinates spheres of asymptotically flat hypersurfaces should approach the ADM mass.

(4) The definition should be compatible with standard definition such as the mass of horizons in the Schwarzchild space-time.

The quasi-local mass is supposed to be closely related to the formation of black holes according to the hoop conjecture of Throne.

Various definitions for the quasi-local mass have been proposed. Each one is useful is a certain perspective. For example, the Hawking mass has a very nice monotonicity property under the inverse mean curvature flow and plays a crucial role in the proof of the Riemannian Penrose conjecture (Huisken-Ilmanen [HI], Bray [BR]). However, as of now there has not been a quasi-local mass definition that satisfies all the above conditions.

In this article, we shall focus on quasi-local mass defined by the following comparison principle: anchor the intrinsic geometry (the induced metric) by isometric embeddings and compare other extrinsic geometries. In the rest of this section, we review the extrinsic geometry of a space-like surface in a space-time.

Let Σ be a surface in a Lorentzian four-manifold N^4. Fix a local coordinates system u^1, u^2 on Σ. We assume Σ^2 is space-like so the induced metric $g_{ij} = \langle \frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^j} \rangle$ is Riemannian. The normal bundle is of rank two with structure group $SO(1,1)$ and the induced metric on the normal bundle is of signature $(-,+)$. Since the Lie algebra of $SO(1,1)$ is isomorphic to $\mathbb{R}$, the connection form of the normal bundle is a genuine 1-form that depends on the choice of the normal frames. Connections of different choices of normal frames differ by a closed form. The curvature of the normal bundle is then given by an exact 2-form which reflects the fact that any $SO(1,1)$ bundle is topologically trivial. The second fundamental form is given by $A(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}) = (\nabla^N_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i})^\perp$ where $(\cdot)^\perp$ denotes the projection onto the normal bundle of Σ. The mean curvature vector $\vec{H} = g^{ij} A(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j})$ is the trace of A. Given any normal vector (space-like, time-like, or null), we may consider the mean curvature in the direction of ν given by $-\langle \vec{H}, \nu \rangle$.

3. Definitions relying on isometric embeddings

Historically, the first such a definition was proposed by Brown and York [BY1, BY2] (see also [EP, KI, LA]). For a compact surface that bounds an initial data set M^3 in a space-time N^4, they consider

$$\int_\Sigma H_0 - \int_\Sigma H_M$$

where H_0 is the mean curvature of an isometric embedding of Σ into $\mathbb{R}^3$ and H_M is the mean curvature of Σ as the boundary of M^3.'

Brown and York deduce this expression from the Hamiltonian formulation of general relativity. $\int_\Sigma H_0$ corresponds to the reference term to be subtracted from the action. They actually consider the full quasi-local energy momentum in which the connection form in the gauge defined by M is also involved. The Brown-York mass has nice asymptotic properties in the sense that it approaches the ADM mass on large coordinates spheres of an asymptotically flat space.

That the Brown-York mass is well-defined relies on the following isometric embedding theorem (proposed by Weyl and proved independently by Nirenberg and Pogorelov) which states:

Theorem 2 *Suppose g is Riemannian metric on S^2 with positive Gaussian curvature. Then there exists an embedding of S^2 into $\mathbb{R}^3$ so that the induced metric is the same as g. The embedding is unique up to rigid motions in $\mathbb{R}^3$.*

As such any quantity acquired through the isometric embedding is canonically associated with the metric g.

The total mean curvature $\int_\Sigma H$ plays an interesting role in the proof of rigidity theorem, i.e. any two convex surface in $\mathbb{R}^3$ with the same induced metric are different by a rigid motion.

Suppose $X : \Sigma \to \mathbb{R}^3$ and $X^* : \Sigma \to \mathbb{R}^3$ are two isometric embedding, a formula of Herglotz shows

$$\int_\Sigma H - \int_\Sigma H^* = \int_\Sigma \det(h_{ij} - h^*_{ij}) X \cdot \nu$$

where ν is the outward unit normal of $X(\Sigma)$.

Now an algebraic lemma shows that $\det h_{ij} = \det h^*_{ij} = K > 0$ implies

$$\det(h_{ij} - h^*_{ij}) \leq 0.$$

By symmetric $\int_\Sigma H = \int_\Sigma H^*$ so $h_{ij} = h^*_{ij}$. The last equality implies the embeddings X and X^* are different by just a rigid motion in $\mathbb{R}^3$.

4. Positivity of quasi-local mass

Shi and Tam [ST] proved the following result which corresponds to the positivity of Brown-York mass in the time-symmetric ($p_{ij} = 0$) case.

Theorem 3 *Let M be a compact three-manifold with boundary and non-negative scalar curvature. Suppose the boundary of M, Σ, has positive Gaussian curvature and the mean curvature H_M of Σ with respect to the outward normal is positive, then*

$$\int_\Sigma H_0 - \int_\Sigma H_M \geq 0$$

where H_0 is the mean curvature of the isometric embedding of Σ into $\mathbb{R}^3$ and equality holds if and only if M is flat.

Let Ω be the region in $\mathbb{R}^3$ that is bounded by the image of the isometric embedding of Σ. The proof uses a procedure invented by Bartnik [BA] that glues M with $\mathbb{R}^3 \backslash \Omega$ along Σ through the isometry and perturb the flat metric on $\mathbb{R}^3 \backslash \Omega$ by a parabolic partial differential equation so the mean curvatures agree along the joint and the scalar curvature remains zero. A monotonicity formula that relates the quasi-local mass and the ADM mass of the new manifold and eventually a positive mass theorem for manifolds with corners furnish the proof. This is a beautiful result regardless of its relation to the Brown-York mass. It demonstrates how the curvature of the ambient space influence the extrinsic geometry of submanifolds. It is interesting to notice that without attributing to the rigidity theorem, Shi-Tam's result indeed gives another different proof.

Liu and Yau [LY, LY2] defined a new quasi-local mass and proved the positivity in the general case with $(p_{ij} \neq 0)$.

Theorem 4 *Let Σ be space-like two-surface that bounds a space-like hypersurface in a space-time N^4 that satisfies the dominant energy condition. Suppose Σ, has positive Gaussian curvature, and the mean curvature vector $\vec{H}$ of Σ in N^4 is space-like, then*

$$\int_\Sigma H_0 - \int_\Sigma |\vec{H}| \geq 0$$

where $|\vec{H}|$ is the Lorentz norm of the mean curvature vector of Σ in N^4 and H_0 is the mean curvature of the isometric embedding of Σ into $\mathbb{R}^3$.

Notice that the Liu-Yau mass does not depend on the particular M bounded by Σ. An important new ingredient in the proof is the Jang's equation used in the proof of the positive mass theorem by Schoen and Yau [SY3]. The Jang's equation is a nonlinear elliptic equation which was designed to recognize if a space-like hypersurface is embedded in the Minkowski space. If one solves the Jang's equation on a hypersurface in the Minkowksi space, the solution should be a flat $\mathbb{R}^3$. In Schoen-Yau's proof, the solution of the Jang's equations gives a new three manifolds that retains the information of the p_{ij} and has less total mass. Liu-Yau solved the boundary problem for the Jang's equation and a boundary calculation of Yau [YA] in showed that the quasi-local mass is also non-increasing in this situation.

5. Questions to be answered after Liu-Yau's work

As elegant as the result of Liu-Yau is, there remain questions to be answered, in particular:

(1) One would like to define the quasi-local mass when the Gaussian curvature K is not positive? This is especially important for black hole collisions when, for examples, two black holes are enclosed by a dumbbell shaped surface whose Gaussian curvature is not positive.

(2) O'Murchadha et. al [OST] found examples of spacelike surfaces Σ in the Minkowski space whose Liu-Yau (and Brown-York) quasi-local mass are strictly positive. It seems the momentum information which corresponds the second fundamental form p_{ij} needs to be accounted for All the definitions of quasi-local mass uses isometric embeddings into $\mathbb{R}^3$ which sits totally geodescially in the Minkowski space $\mathbb{R}^{3,1}$. But the reference should consist of all space-like surfaces in $\mathbb{R}^{3,1}$.

(3) If we state the positive mass theorem as (μ, J^1, J^2, J^3) being future directed time-like implies (E, P_1, P_2, P_3) is a future directed time-like vector. We would expect the quasi-local energy momentum to be a future directed time-like vector as well. Although both Brown-York

and Liu-Yau define the quasi-local momentum in terms of the connection 1-form of a special gauge, they do not contribute in the positivity of the quasi-local energy momentum.

To be able to deal the first two questions, Yau and the author considered the simplest space-like non-flat hypersurface in the Minkowski space. Namely, the hyperbola $\mathbb{H}^3_{-\kappa^2}$ defined by $\{(x,y,z,t)|x^2+y^2+z^2-t^2=-\frac{1}{\kappa^2}\}$. The Lorentz metric on the Minkowski space induces the hyperbolic metric with constant sectional curvature $-\kappa^2$ on $\mathbb{H}^3_{-\kappa^2}$. In order to allow non-positive Gaussian curvature, we utilize the isometric embedding of Pogorelov:

Theorem 5 *If the Gauss curvature K of a Riemannian metric on S^2 satisfies $K > -\kappa^2$ then there exists an isometric embedding of S^2 into $\mathbb{H}^3_{-\kappa^2}$ that is convex. The embedding is unique up rigid motions of the hyperbolic space. .*

In carrying out the gluing construction and the proof of the positivity of the quasi-local mass, some interesting new phenomena occur. It turns out the total mean curvature expression need to be modified to account for the non-vanishing of p_{ij}.

In the Shi-Tam and Liu-Yau case where isometric embedding into $\mathbb{R}^3$ is used, the new metric is asymptotically flat.

In this case, the key fact is that $\lim_{r\to\infty}\int_{\Sigma_r}(H_0-H)$ is the ADM mass of the new asymptotically flat metric.

Indeed, $|u-1| = O(\frac{1}{r})$ and

$$\int_{\Sigma_r}(H_0-H)=\int_{\Sigma_r}H_0(1-\frac{1}{u})$$

is approaching the total mass as $r\to\infty$ since ($|\Sigma_r| = O(r^2)$ and $H_0 = O(\frac{1}{r})$.

In the $\mathbb{H}^3_{-\kappa^2}$ case, the new metric is asymptotically hyperbolic, $|u-1| = O(e^{-3\kappa r})$ but the limit of the expression

$$\int_{\Sigma_r}(H_0-H)=\int_{\Sigma_r}H_0(1-\frac{1}{u})$$

is always zero since $|\Sigma_r| = O(e^{2\kappa r})$ and $H_0 = O(1)$. Indeed, a weighted total mean curvature is obtained and the weighting corresponds to the square norm of parallel spinors of the hyperplane connection for the spinor bundle. This in turn is intimated related to the geometry of Minkowski space.

The theorem in [WY] states

Theorem 6 *Let M be a compact three-manifold with boundary and assume the scalar curvature of M is bounded from below by $-6\kappa^2$. Suppose* *the Gaussian curvature the boundary of M, Σ, is greater than $-\kappa^2$ and the mean curvature H_M of Σ with respect to the outward normal is positive, then there exists a future-directed time-like vector-valued function $\widetilde{X} : \Sigma \to \mathbb{R}^{3,1}$ depending only on the metric of Σ and H_M such that*

$$\int_\Sigma (H_0 - H_M)\widetilde{X}$$

is a future-directed non-space-like vector where H_0 is the mean curvature of the image of the (essentially unique) isometric embedding of Σ into $\mathbb{H}^3_{-\kappa^2}$. Also equality holds if and only if M is hyperbolic.

This assumption on the scalar curvature $R \geq -6\kappa^2$ corresponds to requiring M to be an initial data set with $p_{ij} = \kappa g_{ij}$. $\widetilde{X}$ is a $\mathbb{R}^{3,1}$ valued function modified from the position vector of the isometric embedding by a parabolic partial differential equation satisfied by the parallel spinors of the hyperplane connection on the hyperbola. The four-vector $\int_\Sigma (H_0 - H_M)\widetilde{X}$ is equivariant with respect to isometric embeddings of Σ. One tends to conjecture that the conclusion remains true if $\widetilde{X}$ is replaced by the actual position vector of the embedding $X : \Sigma \to \mathbb{H}^3 \subset \mathbb{R}^{3,1}$. Shi and Tam [ST2] proved that the time-component of $\int_\Sigma (H_0 - H_M)X$is positive. Indeed they show $\int_\Sigma (H_0 - H_M)\cosh \kappa r \geq 0$ where r is the distance function to the point $(0, 0, 0, 1/\kappa)$ on $\mathbb{H}^3_{-\kappa^2}$.

This family of quasi-local mass depends on the parameter κ. A surface of positive Gaussian curvature can be isometrically embedded into any $\mathbb{H}^3_{-\kappa^2}$ and the relation among the associated quasi-local mass is not clear.

6. A new isometric embedding theorem and its implications

In order to anchor the definition of quasi-local mass, we need an existence and uniqueness theorem for isometric embedding problems into the Minkowski space. We also need a new expression for quasi-local mass to accommodate the momentum information. An embedding into the Minkowski space is determined by four coordinate variables $X = (x_0, x_1, x_2, x_3) : \Sigma \to \mathbb{R}^{3,1}$, but the isometric requirement only prescribed three functions E, F, and G in

$$dX \cdot dX = E du^2 + 2F du dv + G dv^2$$

for the first fundamental form.

We need one more condition to make the problem a well-determined system. The convexity condition in Weyl's embedding theorem needs to be replaced by some ellipticity condition that will guarantee the uniqueness.

In my ICCM talk, I shall report:

(1) An existence and uniqueness theorem for isometric embedding problems into $\mathbb{R}^{3,1}$.

(2) A generalization of Herglotz's formula to prove the rigidity theorem for isometric embeddings.

(3) A quasi-local mass expression that is positive under the dominant energy condition and vanishes if and only if the surface is in the Minkowski space.

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