Proving A Manifold To Be Hyperbolic Once It Has Been Approximated To Be So

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Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
in the Graduate Schools of Arts and Sciences

COLUMBIA UNIVERSITY

2005

ABSTRACT

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Let M be a 3-manifold whose boundary consists of tori. The computer program SnapPea [19], created by Jeff Weeks, can approximate whether or not M is a complete hyperbolic manifold. However, until now, there has been no way to determine from this approximation if M is truly hyperbolic and complete. This paper provides two methods for proving that a manifold is complete hyperbolic based on the approximations of SNAP [7], a program that includes the functionality of SnapPea plus other features. The approximation is done by triangulating M, identifying consistency and completeness equations as described by Neumann and Zagier [13], and Benedetti and Petronio [3] with respect to this triangulation, and then trying to solve the system of equations using Newton's Method [20]. This produces an approximate, not actual solution. Assume the triangulation has n tetrahedra. There are n relevant equations, $f_1(z_1, \ldots, z_n) = 0, \ldots, f_n(z_1, \ldots, z_n) = 0$, in n variables. Let $a_1, \ldots, a_n$ be an approximate solution to the equations. Define $b_i = f_i(a_1, \ldots, a_n)$ for $1 \leq i \leq n$ and $f : \mathbb{C}^n \to \mathbb{C}^n$ such that $f(z_1, \ldots, z_n) = (f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$, so $f(a_1, \ldots, a_n) = (b_1, \ldots, b_n) \in \mathbb{C}^n$ is very close to $(0, \ldots, 0) \in \mathbb{C}^n$. The first method applies the concepts inherent in the proof of the Inverse Func-

tion Theorem[21] to see if there is a neighborhood of $(a_1, \ldots, a_n) \in \mathbb{C}^n$ that f maps homeomorphically onto a neighborhood of $(b_1, \ldots, b_n) \in \mathbb{C}^n$ that contains $(0, \ldots, 0)$. If so, there is a solution to the equations and we have guaranteed that M is complete hyperbolic [3]. The second method applies the Kantorovich Theorem [8] to f with the same goal of testing for a solution to the equations. Using these methods, every manifold in the SnapPea cusped census is definitively proven to have a complete hyperbolic structure.

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Acknowledgements

First of all, I want to thank my advisor, Walter Neumann, for all his help and support since my first day at Columbia. He is truly a superb human being. I am also very grateful to Joan Birman for her encouragement and advice over the years, as well as for alerting me to the existence of the Kantorovich Theorem.

I am very appreciative of Chris Leininger's contribution to my examples. I want to thank all three of the above, as well as Lee Mosher and Linda Keen for being on my thesis defense committee. Thanks also to Ilya Kofman and Abhijit Champanerkar for the many discussions.

I especially want to thank the Columbia University mathematics department for giving me the opportunity to return after a hiatus of almost 35 years.

A big thanks also to Terrance, Delores, and Laurent for all their assistance over the years.

I reserve my final thanks for my family. My children have encouraged me along the way. My husband, Harvey, has been outstanding. He has been a constant support, always believing in me, and uplifting me through the hard times, as well as sharing the good times.

To my grandchildren Cameron, Charlotte and Jonah

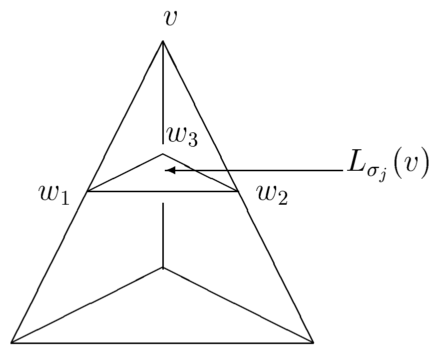

Figure 1: The Tetrahedron σ_j

1 Introduction

Since the determination that M is complete hyperbolic is dependent on there being a solution to a set of equations, we shall first review the development of these equations. Every orientable complete hyperbolic manifold of finite volume is obtained from an ideally triangulated one by Dehn surgery on some of its cusps. This fact is documented by Neumann [13], based on a Thurston preprint [16], so we first examine N, a non-compact 3-manifold that is the interior of a compact one whose boundary consists of k tori. N can be realized as a gluing of n tetrahedra, $\sigma_1, \ldots, \sigma_n$, having k vertices after gluing, with a *conic neighborhood* of each vertex removed [3]. A *conic neighborhood of the vertex,* v, is described as follows. Let v be a vertex and σ_j a tetrahedron that v belongs to. Take the second barycentric subdivision of the edges of σ_j containing v and let w_1, w_2 and w_3 be the closest vertices to v for these edges with respect to this subdivision. See Figure 1.

Definition 1.1 • $\mathbf{L_{\sigma_j}(v)}$ = triangle having vertices w_1, w_2 *and* w_3 as above with respect to v *and* σ_j

- $\mathbf{L(v)} = \bigcup_{\substack{v \text{ vertex of } \sigma_j \\ 1 \leq j \leq n}} L_{\sigma_j}(v)$

  $L(v)$ is called the **link** *of v*

- $\mathbf{U_{\sigma_j}(v)}$ = tetrahedron having vertices v, w_1, w_2 *and* w_3

- **conic neighborhood of v** $= \bigcup_{\substack{v \text{ vertex of } \sigma_j \\ 1 \leq j \leq n}} U_{\sigma_j}(v)$.

Every vertex is identified with a cusp of N, and its link is a torus. These truncated tetrahedra resulting from the removal of the conic sections can now be treated as ideal hyperbolic tetrahedra, so there exists a hyperbolic structure on $N \backslash$1-skeleton of N. In order for N to have a hyperbolic structure, there must be consistency across the 1-skeleton. The conditions for this to happen are embodied in the consistency equations and will be described in detail in Section 2, "Identifying the Equations".

Completeness applies to the cusps. Once a hyperbolic structure is identified, it induces a *similarity structure* (i.e., a $(\mathbb{C}, \text{Aff}(\mathbb{C}))$ structure) on each of the k tori, $T_1, \ldots, T_k$. If the similarity structure of a torus identified with a cusp is Euclidean, N will be complete at that cusp [3]. This occurs when the image of the holonomy of the similarity structure for the torus consists entirely of translations, or equivalently, has at least one non-trivial translation [3]. A *holonomy* of a similarity structure for a torus, T, is a map θ such that $\theta : \pi_1(T) \to \text{Aff}(\mathbb{C})$ [3]. The conditions for the image of θ to consist entirely of translations are presented by the completeness equations which will also be discussed in Section 2, "Identifying The Equations".

Once we establish the conditions for cusps of N to be complete, we turn our attention to the manifold M, obtained from N by Dehn surgery on some of the cusps. Assume h cusps remain unsurgered, so there are $k - h$ surgered cusps. M

must satisfy the consistency equations; however, there are now only h cusps that must be shown to be complete, so we only need the completeness equations referring to these h cusps. The remaining $k-h$ surgered cusps must result from Dehn surgery with co-prime coefficients (p_i, q_i) for $1 \leq i \leq k-h$ where (p_i, q_i) and the holonomy of the similarity structure of T_i are joined in one equation [3].

Once the equations needed to prove a manifold complete hyperbolic are identified, we set up the machinery to test whether a solution exists. There are two types of tests, and they occur in Section 3, "How to Test for a Solution." The method described there concludes the proof of the following theorem, which is our main result, where the first inequality is established in Section 3.1 and the second in Section 3.2.

Theorem 1.2 *Let M be a manifold and assume there are n tetrahedra in the triangulation of M according to SnapPea. There are n equations, $\{f_i(z) = 0 \mid f_i : \mathbb{C}^n \to \mathbb{C}\}$ for $1 \leq i \leq n$, whose simultaneous solution will guarantee that M is complete hyperbolic. If SnapPea finds an approximate geometric solution to these equations, let $a = (a_1, \ldots, a_n)$ be an approximate geometric solution generated by SNAP on the SnapPea manifold file for M. Let $b_i = f_i(a)$ for $1 \leq i \leq n$ and $f : \mathbb{C}^n \to \mathbb{C}^n$ with $f(z) = (f_1(z), \ldots, f_n(z))$, so $f(a) = b = (b_1, \ldots, b_n)$. Then there are identifiable δ, η and $L > 0$ such that there is a genuine solution to the equations, making M complete hyperbolic when at least one of the following inequalities are true:*

$$
\begin{aligned}
1)\ |b| &< \frac{\eta\delta}{4} \\
2)\ |b| &\leq \frac{1}{2L|f'(a)^{-1}|^2}.
\end{aligned}
$$

We devote the final section to examples. Every manifold in the cusped census of SnapPea has been examined and the results are reported in Section 4, “Examples.” However, for detailed discussion, six examples are presented. There are simple ones, such as the figure 8 knot and Whitehead link complements, as well as Dehn surgery on both of them. There are also two complicated link complements, one with 4 cusps and 32 tetrahedra and the other with 11 cusps and 57 tetrahedra. In uncomplicated cases, it is sometimes possible to show that a knot or link complement has a complete hyperbolic structure using means other than the SnapPea approximation. Thurston has proven that the figure 8 knot complement has a complete hyperbolic structure, and shown when a (p, q) Dehn filling has the same property [18]. Neumann and Reid have done the same for Dehn fillings of the Whitehead link [12]. However, when it comes to complicated knots and links, until now, it may have been impossible to definitively determine whether this structure exists. For several years Leininger has withheld publication of his paper devoted to the links in the last two examples [9] because he could not prove that their complements have a complete hyperbolic structure. The paper can now be released using the method presented here. So far, every manifold that has an approximate solution with respect to a geometric triangulation in SnapPea that has been tested by this method has been verified to have a complete hyperbolic structure.

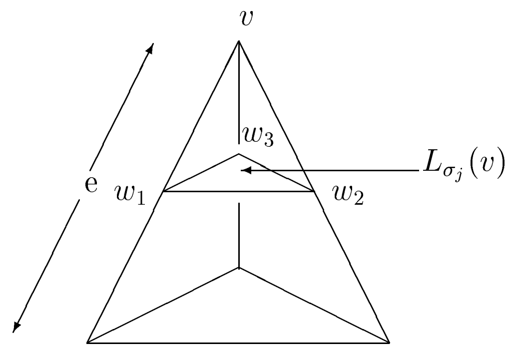

Figure 2: Edge e of the Tetrahedron σ_j

2 Identifying the Equations

Let σ_j be an ideal hyperbolic tetrahedron as described in Section 1, "Introduction", and pick an edge e such that $w_1 \in e$ and prior to truncation, e ended in the vertex v, as in Figure 2. Then $L_{\sigma_j}(v)$, the triangle with vertices w_1, w_2 and w_3 naturally has a similarity structure as the triangle in $\mathbb{C}$ with vertices 0, 1 and z (see Figure 3) [17, 13, 3], and the dihedral angle at e will be $\arg(z)$. Clearly, z must be in $\mathbb{C}_+$, the upper half plane in $\mathbb{C}$. The *modulus of* $L_{\sigma_j}(v)$ *with respect to* w_1 is z, so that the inner angle of the triangle at w_1 is $\arg(z)$. The *modulus of* σ_j *at edge* e is z. The only other modulii at the other edges of σ_j will be either $1-(1/z)$ or $1/(1-z)$, so z uniquely describes σ_j in the upper half plane. There are six edges with opposite edges having the same modulus [13, 3, 15]. See Figure 4.

2.1 Consistency Equations

In order for N to be hyperbolic, if e is an edge of N, the tetrahedra gluing together at e must close up around e. That is, the product of all the edge modulii associated with e (different modulus for each tetrahedron e belongs to) must be $e^{2\pi i}$, assuring that the

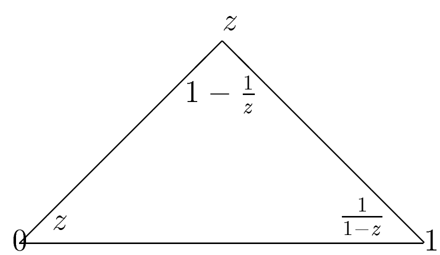

Figure 3: The Triangle Similar to $L_{\sigma_j}(v)$

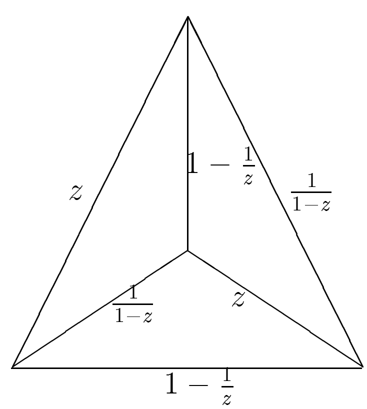

Figure 4: Modulii Associated To Edges of the Tetrahedron σ_j

sum of the arguments is precisely 2π. Any of the three distinct edge modulii of a tetrahedron, σ_j, can expressed as $\pm z_j^{r'_j}(1-z_j)^{r''_j}$ with $(r'_j, r''_j) \in \{(1,0),(-1,1),(0,-1)\}$, so the gluing requirement at edge e is

$$\prod_{j=1}^{n} z_j^{r'_j}(1-z_j)^{r''_j} = \pm 1,$$

where $r'_j = r''_j = 0$ if σ_j does not contain e. A tetrahedron can have more than one edge glued at e so r'_j and r''_j can take values between -2 and 2. The Euler characteristic of N is zero, so it can be shown that N has n edges [13]. Thus, the n edge equations can be expressed as

$$\prod_{j=1}^{n} z_j^{r'_{ij}}(1-z_j)^{r''_{ij}} = \pm 1 \qquad (i = 1, \ldots, n). \tag{1}$$

They are referred to as the consistency equations. The existence of a solution is sufficient to make N hyperbolic. We rewrite them as log equations because they are easier to use this way and it reflects the fact that the sum of the arguments of the modulii at each edge is exactly 2π [11].

$$\sum_{j=1}^{n}(r'_{ij}\log(z_j) + r''_{ij}\log(1-z_j)) = c_i\pi i \qquad c_i \in \mathbb{Z} \qquad (i = 1, \ldots, n) \tag{2}$$

Let $\mathbf{R}$, $\mathbf{C}$ and $\overline{\mathbf{R}}$ be the following matrices.

$$\mathbf{R} = \begin{pmatrix} r'_{11} & \ldots & r'_{1n} & r''_{11} & \ldots & r''_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ r'_{n1} & \ldots & r'_{nn} & r''_{n1} & \ldots & r''_{nn} \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -c_1 \\ \vdots \\ -c_n \end{pmatrix} \quad \overline{\mathbf{R}} = (\mathbf{R}, \mathbf{C})$$

Proposition 2.1 *If rank $\overline{\mathbf{R}} = p$, then the space of solutions to the consistency equations can be defined by exactly p consistency equations.*

Proof. Let rank $\overline{\mathbf{R}} = p \le n$, so, without loss of generality, we can assume the first p rows of $\overline{\mathbf{R}}$ are linearly independent. For $s > p$, there exist $\lambda_i^s \in \mathbb{C}$ for $1 \le i \le p$ such that

$$r'_{sj} = \sum_{i=1}^{p} \lambda_i^s r'_{ij} \qquad r''_{sj} = \sum_{i=1}^{p} \lambda_i^s r''_{ij} \qquad c_s = \sum_{i=1}^{p} \lambda_i^s c_i.$$

Assume we have a solution $z = (z_1, \ldots, z_n)$ to the first p consistency equations. Then

$$\sum_{j=1}^{n} (r'_{ij} \log(z_j) + r''_{ij} \log(1 - z_j)) - c_i \pi i = 0 \qquad (i = 1, \ldots, p).$$

Thus,

$$\sum_{i=1}^{p} \lambda_i^s (\sum_{j=1}^{n} (r'_{ij} \log(z_j) + r''_{ij} \log(1 - z_j)) - c_i \pi i) = 0.$$

Hence,

$$\sum_{j=1}^{n} \Big((\sum_{i=1}^{p} \lambda_i^s r'_{ij}) \log(z_j) + (\sum_{i=1}^{p} \lambda_i^s r''_{ij}) \log(1 - z_j) \Big) - (\sum_{i=1}^{p} \lambda_i^s c_i) \pi i = 0.$$

This is the same as

$$\sum_{j=1}^{n} (r'_{sj} \log(z_j) + r''_{sj} \log(1 - z_j)) - c_s \pi i = 0.$$

Therefore, the last $n - p$ consistency equations are determined by the first p, so we only need the first p equations to determine hyperbolicity. ■

In [13, 3] it is proven that for a complete hyperbolic manifold, rank $\mathbf{R} = n - k$. However, we need to prove hyperbolicity. Neumann's work in Combinatorics of Triangulations and the Chern-Simons Invariant for Hyperbolic 3-Manifolds [11] tells us, without a priori knowledge of hyperbolicity, that rank $\mathbf{R} = n - k$, and $\mathbf{C}$ is determined by $\mathbf{R}$, so rank $\overline{\mathbf{R}} = n - k$. This will be explained in Section 2.3, "Matrix Rank". Then, by the above proposition, we only need $n - k$ consistency equations to determine hyperbolicity.

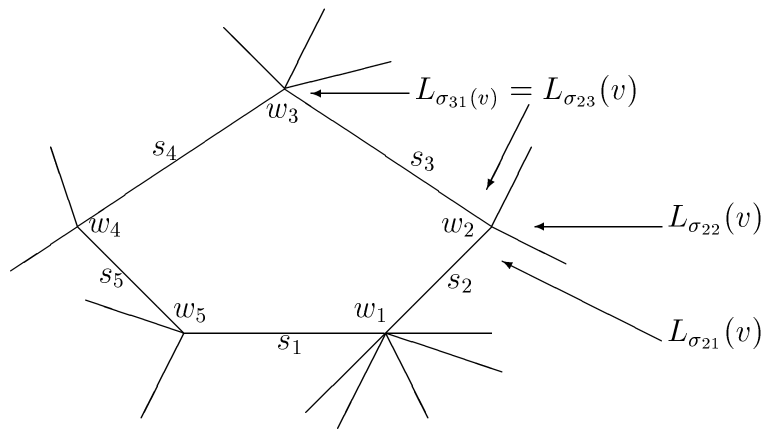

Figure 5: Simple Simplicial Loop, γ, on Torus T_i

2.2 Cusp Conditions

We now look at the k cusps of N. Details of the following discussion can be found in [3]. Let T_i be the torus associated with the i^{th} cusp. Select 2 simple oriented loops, m_i and l_i, on T_i, representing the 2 generators of the fundamental group of T_i. Furthermore, m_i and l_i can be chosen as simplicial loops with respect to T_i's triangulation. Such a loop is composed of segments where each segment is an edge of some triangle $L_{\sigma_q}(v) \subset L(v) = T_i$, as identified earlier when describing the triangulation of N. Let γ be any simple simplicial oriented loop on T_i consisting of d segments, $s_1, \ldots, s_d$, and d vertices, $w_1, \ldots, w_d$, where w_r is the vertex at the end of s_r as well as at the beginning of s_{r+1} for $1 \leq r \leq d-1$ and w_d is the vertex at the end of s_d and beginning of s_1. See Figure 5. We lift γ to $\mathbb{C} = \mathbb{R}^2$, the universal cover of T_i, starting at the beginning of s_1 and map it to $\mathbb{C}$ by way of the developing map [15, 18]. The resulting curve will consist of d straight segments, $\tilde{s}_1, \ldots, \tilde{s}_d$, joined at the vertices $\tilde{w}_r$ for $1 \leq r \leq d-1$, as in γ, except at $\tilde{w}_d$, which does not necessarily connect to the beginning of $\tilde{s}_1$. So it starts at the beginning of $\tilde{s}_1$ and ends at the end of $\tilde{s}_d$.

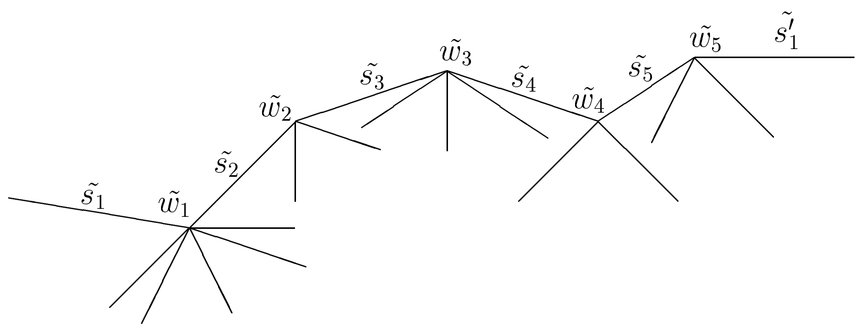

Figure 6: Developing Map Image of γ

Repeat the development map process, starting at the end of $\tilde{s}_d$ and let $\tilde{s'}_1$ be the first segment this time, so $\tilde{w}_d$ is the vertex between $\tilde{s}_d$ and $\tilde{s'}_1$. See Figure 6. Call this curve $\tilde{\gamma}$. Aff($\mathbb{C}$) can be regarded as $\mathbb{C} \rtimes \mathbb{C}^*$ with $(a, b) \in \mathbb{C} \rtimes \mathbb{C}^*$ such that it represents $a + bx$, an affine map of $\mathbb{C}$. The *dilation component of* (a, b) is b. Thus, if an oriented triangle in $\mathbb{C}$ has two edges $\tilde{e}_1$ and $\tilde{e}_2$ where $\tilde{e}_1$ ends in the vertex $\tilde{x}$, and $\tilde{e}_2$ begins at $\tilde{x}$, and the modulus of the triangle with respect to $\tilde{x}$ is y, then the one and only orientation preserving similarity of $\mathbb{C}$ that takes $\tilde{e}_1$ to $\tilde{e}_2$ has dilation component equal to $-y$. Remember, the modulus of the triangle with respect to $\tilde{x}$ is defined so that $\tilde{e}_1$ is identified with the edge from 0 to 1 and $\tilde{e}_2$ with the edge from 0 to y in the triangle with vertices $(0, 1, y)$. If $x_{r1}, \ldots, x_{rp_r}$ are the vertices of the p_r triangles, $L_{\sigma_{r1}}(v), \ldots, L_{\sigma_{rp_r}}(v)$, that touch γ at w_r, as in Figure 5, we get p_r corresponding triangles, $\tilde{L}_{\sigma_{r1}}(v), \ldots, \tilde{L}_{\sigma_{rp_r}}(v)$, touching $\tilde{\gamma}$ at $\tilde{w}_r$ with $\tilde{x}_{r1}, \ldots, \tilde{x}_{rp_r}$ the respective vertices of these triangles at $\tilde{w}_r$. The ordering is such that $\tilde{s}_r$ is the first edge of $\tilde{L}_{\sigma_{r1}}(v)$, and $\tilde{s}_{r+1}$ is the second edge of $\tilde{L}_{\sigma_{rp_r}}(v)$, at $\tilde{w}_r$ unless $r = d$, and then $\tilde{s'}_1$ is the second edge of $\tilde{L}_{\sigma_{dp_d}}(v)$. See Figure 6. If the corresponding triangle modulii at $\tilde{w}_r$ are $y_{r1}, \ldots, y_{rp_r}$, then the dilation component of the affine map that takes $\tilde{s}_r$ to $\tilde{s}_{r+1}$ is $-\prod_{i=1}^{p_r} y_{ri}$. Orientation is responsible for the "$-$" in the product. Hence, the affine map that takes $\tilde{s}_1$ to $\tilde{s'}_1$ has dilation component of $\prod_{r=1}^{d}(-1)\prod_{i=1}^{p_r} y_{ri} = (-1)^d \prod_{r=1}^{d}\prod_{i=1}^{p_r} y_{ri}$. Note that the modulus of $\tilde{L}_{\sigma_{ri}}(v)$ at $\tilde{x}_{ri}$ for $1 \leq i \leq p_r$ is the same as the modulus of $L_{\sigma_{ri}}(v)$ at x_{ri} for $1 \leq i \leq p_r$, and this latter modulus has already been identified as either z_j, $1/(1-z_j)$ or $1 - 1/z_j$ for some $1 \leq j \leq n$. Therefore, the dilation component of the affine map that takes $\tilde{s}_1$ to $\tilde{s}'_1$ is of the form

$$\pm 1 \prod_{j=1}^{n} z_j^{\gamma'_j}(1-z_j)^{\gamma''_j}.$$

The *holonomy of the* $(\mathbb{C}, \mathit{Aff}(\mathbb{C}))$ *structure* on T_i is a map $\theta : \pi_1(T_i) \to \mathrm{Aff}(\mathbb{C})$ such that if $[\gamma]$ is the element of $\pi_1(T_i)$ represented by the loop γ, then θ takes $[\gamma]$ to the affine map that takes $\tilde{s_1}$ to $\tilde{s'_1}$. This is a homomorphism that is well defined up to conjugacy class within $\mathrm{Aff}(\mathbb{C})$. However, any two elements of $\mathrm{Aff}(\mathbb{C})$ within a conjugacy class have the same dilation component [3], so the map

$$\begin{aligned} \psi_i : \pi_1(T_i) &\to \mathbb{C}^* \qquad \text{such that} \\ [\gamma] &\to \pm 1 \prod_{j=1}^{n} z_j^{\gamma'_{ij}}(1-z_j)^{\gamma''_{ij}} \end{aligned}$$

is a well defined homomorphism. $\theta([\gamma])$ will be a translation if its dilation component is 1, so $\theta([\gamma])$ will be a translation when $\psi_i([\gamma]) = 1$.

We now look at loops m_i and l_i. For simplicity of notation, we also refer to the corresponding generators of $\pi_1(T_i)$ as m_i and l_i so

$$\psi_i(m_i) = \pm 1 \prod_{j=1}^{n} z_j^{m'_{ij}}(1-z_j)^{m''_{ij}} \tag{3}$$

$$\psi_i(l_i) = \pm 1 \prod_{j=1}^{n} z_j^{l'_{ij}}(1-z_j)^{l''_{ij}} \tag{4}$$

If the triangulation of T_i causes m_i to be a simplicial loop with d segments and d vertices, then its holonomy will be a non-trivial translation when $\psi_i(m_i) = 1$ and the sum of the arguments of the modulii at the d vertices of m_i is $d\pi$ [3]. Rewriting in log form, these requirements are expressed as

$$\sum_{j=1}^{n}(m'_{ij}\log(z_j) + m''_{ij}\log(1 - z_j)) = c_{mi}\pi i \qquad \text{with } c_{mi} \in \mathbb{Z}.$$

Similarly, one can identify the log equation which sets the condition for the holonomy of l_i to be a non-trivial translation. It can be expressed as

$$\sum_{j=1}^{n}(l'_{ij}\log(z_j) + l''_{ij}\log(1 - z_j)) = c_{li}\pi i \qquad \text{with } c_{li} \in \mathbb{Z}.$$

When the holonomy of the affine structure on T_i has at least one non-trivial translation in its image, the affine structure is Euclidean [3]. But a Euclidean structure on T_i means that the i^{th} cusp is complete [3], so the completeness equations for all of the k cusps are

$$\sum_{j=1}^{n}(m'_{ij}\log(z_j) + m''_{ij}\log(1 - z_j)) - c_{mi}\pi i = 0 \qquad (i = 1, \ldots, k). \tag{5}$$

Now consider a hyperbolic manifold, N, with k cusps where h of the cusps are complete, so the above completeness equations hold only for $k - h + 1 \leq i \leq k$. Let T_i be a torus associated with one of the $k - h$ non-complete cusps. If p_i and q_i are co-prime integers, (p_i, q_i) Dehn filling can be performed on this cusp. In the literature, this process is frequently referred to as Dehn surgery, but it is really a filling. In this case, $p_i m_i + q_i l_i$ is the generator of $\pi_1(T_i)$ that is killed by Dehn filling. In order to extend the hyperbolic structure on N to the Dehn filling at this cusp, we need [13, 3]

$$p_i\Big(\sum_{j=1}^{n}(m'_{ij}\log(z_j) + m''_{ij}\log(1 - z_j)) - c_{mi}\pi i\Big) +$$

$$q_i\Big(\sum_{j=1}^{n}(l'_{ij}\log(z_j)+l''_{ij}\log(1-z_j))-c_{li}\pi i\Big)=2\pi i. \qquad (6)$$

That is

$$\sum_{j=1}^{n}\Big((p_i m'_{ij}+q_i l'_{ij})\log(z_j)+(p_i m''_{ij}+q_i l''_{ij})\log(1-z_j)\Big)=c_{si}\pi i \qquad \text{with } c_{si}\in\mathbb{Z}.$$

Therefore, if the equations

$$\sum_{j=1}^{n}\Big((p_i m'_{ij}+q_i l'_{ij})\log(z_j)+(p_i m''_{ij}+q_i l''_{ij})\log(1-z_j)\Big)=c_{si}\pi i \quad (i=1,\dots,k-h) \quad (7)$$

are satisfied, M, the manifold derived from N by Dehn filling on the $k-h$ cusps, will be hyperbolic near these cusps.

The last step in identifying the equations is the selection of the appropriate $n-k$ consistency equations. Let $s'_{ij}=p_i m'_{ij}+q_i l'_{ij}$ and $s''_{ij}=p_i m''_{ij}+q_i l''_{ij}$, and define the matrices $\mathbf{M}$, $\mathbf{L}$, $\mathbf{S}$ and $\mathbf{M_h}$ as $\mathbf{R}$ is defined on page 7 so that

$$\mathbf{M}=\begin{pmatrix} m'_{11} & \dots & m'_{1n} & m''_{11} & \dots & m''_{1n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ m'_{(k)1} & \dots & m'_{(k)n} & m''_{(k)1} & \dots & m''_{(k)n}\end{pmatrix}$$

$$\mathbf{L}=\begin{pmatrix} l'_{11} & \dots & l'_{1n} & l''_{11} & \dots & l''_{1n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ l'_{(k)1} & \dots & l'_{(k)n} & l''_{(k)1} & \dots & l''_{(k)n}\end{pmatrix}$$

$$\mathbf{S}=\begin{pmatrix} s'_{11} & \dots & s'_{1n} & s''_{11} & \dots & s''_{1n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ s'_{(k-h)1} & \dots & s'_{(k-h)n} & s''_{(k-h)1} & \dots & s''_{(k-h)n}\end{pmatrix}$$

$$\mathbf{M_h}=\begin{pmatrix} m'_{(k-h+1)1} & \dots & m'_{(k-h+1)n} & m''_{(k-h+1)1} & \dots & m''_{(k-h+1)n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ m'_{(k)1} & \dots & m'_{(k)n} & m''_{(k)1} & \dots & m''_{(k)n}\end{pmatrix}$$

Let

$$\mathbf{U} = \begin{pmatrix} \mathbf{S} \\ \mathbf{M_h} \end{pmatrix}$$

We will see that rank $\mathbf{U} = k$. We can select $n-k$ consistency equations so that their rows in $\mathbf{R}$ are linearly independent, and when concatenated with $\mathbf{U}$, give an $n \times (2n)$ matrix of rank n. The reasons for this are a consequence of [11], and will be explained in Section 2.3, "Matrix Rank". We will assume, without loss of generality, that the last $n-k$ out of n consistency equations are the ones we want.

In summary, we have

$n-k$ consistency equations,

$$\sum_{j=1}^{n}(r'_{ij}\log(z_j) + r''_{ij}\log(1-z_j)) - c_i\pi i = 0 \qquad (i = k+1, \ldots, n),$$

$k-h$ surgery equations,

$$\sum_{j=1}^{n}\Big((p_i m'_{ij} + q_i l'_{ij})\log(z_j) + (p_i m''_{ij} + q_i l''_{ij})\log(1-z_j)\Big) - c_{si}\pi i = 0 \quad (i = 1, \ldots, k-h),$$

and h completeness equations,

$$\sum_{j=1}^{n}(m'_{ij}\log(z_j) + m''_{ij}\log(1-z_j)) - c_{mi}\pi i = 0 \qquad (i = k-h+1, \ldots, k).$$

giving a total of n equations that must have a simultaneous solution to make a manifold complete hyperbolic.

2.3 Matrix Rank

In [11], Neumann has constructed a chain complex, $\mathcal{J}$, and described its homology. Using the terminology of Section 1, "Introduction," with respect to the triangulation of N and M, let K be the gluing of the n tetrahedra, $\sigma_1, \ldots, \sigma_n$. The modules of the chain complex are C_0, C_1 and J, where

1. $C_0 = \mathbb{Z}$ module generated by the k vertices of K. Each vertex will be associated with a cusp of N, and the torus that is the link of the vertex.

2. $C_1 = \mathbb{Z}$ module generated by $E_1, \ldots, E_n$, the n edges of K.

3. With regard to J, for each tetrahedron, σ_j, label the edges as $e_{j1}, \ldots, e_{j6}$ according to the associated parameters as:

$$\begin{array}{lll} e_{j1} = z & e_{j2} = \frac{1}{1-z} & e_{j3} = 1 - \frac{1}{z} \\ e_{j4} = z & e_{j5} = \frac{1}{1-z} & e_{j6} = 1 - \frac{1}{z} \end{array}$$

Let $J_{\sigma_j} = \mathbb{Z}$ module generated by the six edges of σ_j with the relations $e_{j\tau} - e_{j(\tau+3)} = 0$ for $1 \leq \tau \leq 3$ and $e_{j1} + e_{j2} + e_{j3} = 0$. Thus, opposite edges of the tetrahedron are represented by the same element of J_{σ_j}, and e_{j3} can be defined in terms of e_{j1} and e_{j2}. This means that e_{j1} and e_{j2} generate the $\mathbb{Z}$ module, J_{σ_j}. Let

$$J = \coprod_{1 \leq j \leq n} J_{\sigma_j}.$$

The chain complex sequence is

$$\mathcal{J}: \quad 0 \to C_0 \stackrel{\alpha}{\to} C_1 \stackrel{\beta}{\to} J \stackrel{\beta^*}{\to} C_1 \stackrel{\alpha^*}{\to} C_0 \to 0.$$

We have α, β, α^* and β^* defined as follows:

1. $\alpha : C_0 \to C_1$, where α takes a vertex to the sum of the edges containing the vertex, with an edge counted twice if both ends of the edge are at the vertex.

2. $\beta : C_1 \to J$ can be defined by letting

$$E_i \to \sum_{1 \leq j \leq n} \sum_{\substack{1 \leq \tau \leq 6 \\ E_i \text{ is identified with } e_{j\tau}}} e_{j\tau}$$

We have the sum $\sum_{\substack{1\leq\tau\leq 6 \\ E_i \text{ is identified with } e_{j\tau}}} e_{j\tau} \in J_{\sigma_j}$ because more than one edge of σ_j can be identified with E_i.

3. To define $\beta^* : J \to C_1$, note that for each σ_j, we have the edge set $\{e_{j1}, \ldots, e_{j6}\}$. Let $\rho : \{e_{j1}, \ldots, e_{j6}\} \to \{E_1, \ldots, E_n\}$ be such that $\rho(e_{j\tau}) = E_i$ when $e_{j\tau}$ is identified with the edge E_i. Then, let

$$\beta^*(e_{j\tau}) = \rho(e_{j(\tau+1)}) - \rho(e_{j(\tau+2)}) + \rho(e_{j(\tau+4)}) - \rho(e_{j(\tau+5)}) \qquad \text{(indices mod 6)}$$

That is, β^* takes $e_{j\tau}$ to the alternating sum of the edges of N identified with the edges of σ_j that touch $e_{j\tau}$.

4. $\alpha^* : C_1 \to C_0$, where α^* sends an edge, E_i, to the sum of its end points.

N is the interior of a compact manifold, $\overline{N}$, whose boundary is the union of the k tori, $T_1, \ldots, T_k$, that are the links of the vertices of K.

Lemma 2.2 *When tensored with* $\mathbb{Q}$, *the sequence,* $\mathcal{J}$, *is exact except in the middle, where its homology is* $H_1(\partial\overline{N};\mathbb{Q}) = \coprod_{1\leq i\leq k} H_1(T_i;\mathbb{Q})$.

For a proof, see [11]. We use this to compute the rank of $\mathbf{R}$. However, we will use the original chain with coefficients in $\mathbb{Z}$ to show that the rank of the matrix obtained by concatenating $\mathbf{U}$, as defined on page 14, with $n-k$ linearly independent rows of $\mathbf{R}$, is n.

2.3.1 Rank of R

The matrix of the linear transformation, β, is closely related to $\mathbf{R}^t$, the transpose of $\mathbf{R}$, and they have the same rank. Since $\operatorname{rank}\mathbf{R} = \operatorname{rank}\mathbf{R}^t$, $\operatorname{rank}\mathbf{R} =$ rank of the matrix of β. The edges $E_1, \ldots, E_n$ are a basis of C_1 as a vector space, so the vectors $\beta(E_i)$ for $1 \leq i \leq n$ are the columns of the matrix of β. From the definition of β, we see that in J_{σ_j},

$$\beta(E_i) = \sum_{\substack{1 \leq \tau \leq 6 \\ E_i \text{ is identified with } e_{j\tau}}} e_{j\tau} \qquad \text{modulo relations on } J. \tag{8}$$

Thus, if:

e_{j1} or e_{j4} occur, it means E_i is identified with the z_j parameter

e_{j2} or e_{j5} occur, it means E_i is identified with the $\frac{1}{1-z_j}$ parameter

e_{j3} or e_{j6} occur, it means E_i is identified with the $1 - \frac{1}{z_j}$ parameter.

In J_{σ_j}, $e_{j3} = -e_{j1} - e_{j2}$; also, $1 - \frac{1}{z_j} = -\frac{1-z_j}{z_j}$. Hence, the sum of the coefficients of e_{j1} in $\beta(E_i)$ is r'_{ij}, the sum of the exponents of z_j with respect to the edge E_i in the consistency equations, and the sum of the coefficients of e_{j2} in $\beta(E_i)$ is $-r''_{ij}$, which is -1 times the sum of the exponents of $1 - z_j$ with respect to the edge E_i in the consistency equations, as seen on page 7. Consequently, $\underline{\mathbf{R}}$, the $2n \times n$ matrix of β is

$$\underline{\mathbf{R}} = \begin{pmatrix} r'_{11} & \cdots & r'_{n1} \\ -r''_{11} & \cdots & -r''_{n1} \\ \vdots & \ddots & \vdots \\ r'_{1n} & \cdots & r'_{nn} \\ -r''_{1n} & \cdots & -r''_{nn} \end{pmatrix}$$

We see that $\operatorname{rank} \underline{\mathbf{R}} = \operatorname{rank} \mathbf{R}^t$, so $\operatorname{rank} \mathbf{R} = \operatorname{rank} \underline{\mathbf{R}}$. By definition, the rank of $\underline{\mathbf{R}}$ is equal to the dimension of the image of β. By Lemma 2.2, α is injective, making $\dim \operatorname{im}(\alpha) = \dim C_0 = k$, and $\operatorname{im}(\alpha) = \ker(\beta)$. The matrix of β this way would still be $\underline{\mathbf{R}}$, so

$$\begin{aligned} \operatorname{rank} \underline{\mathbf{R}} &= \dim \operatorname{im}(\beta) \\ &= \dim C_1 - \dim \operatorname{kernel}(\beta) \end{aligned}$$

$$= \dim C_1 - \dim \mathrm{im}(\alpha)$$
$$= n - k$$

Therefore, $\operatorname{rank} \mathbf{R} = n - k$. Let $\mathbf{R}$, $\mathbf{C}$ and $\overline{\mathbf{R}}$ be the matrices associated with the consistency equations, as on page 7. Consider the matrix equation $\mathbf{R} \cdot x = -\mathbf{C}$. In [11] it is proved that there is an $\tilde{x} \in \mathbb{Q}^{2n}$ that is a solution. Then $-\mathbf{C}$ is a linear combination of the columns of $\mathbf{R}$, so $\mathbf{R}$ concatenated with $-\mathbf{C}$ has the same rank as $\mathbf{R}$ since row rank is the same as column rank. That is,

$$\begin{aligned} n - k &= \operatorname{rank} \mathbf{R} = \text{column rank}\, \mathbf{R} = \text{column rank}\, (\mathbf{R}|-\mathbf{C}) \\ &= \text{column rank}\, (\mathbf{R}|\mathbf{C}) = \text{column rank}\, \overline{\mathbf{R}} = \operatorname{rank} \overline{\mathbf{R}}, \end{aligned}$$

so $\operatorname{rank} \overline{\mathbf{R}} = n - k$. Let $\mathbf{R}_\beta$ = matrix consisting of $n - k$ linearly independent rows of $\overline{\mathbf{R}}$.

2.3.2 Rank of $(\mathbf{S}|\mathbf{M_h}|\mathbf{R}_\beta)$

For now we will include all k cusps of N. Let $S_1(\partial\overline{N}) = \mathbb{Z}$ module of simplicial 1-chains, $Z_1(\partial\overline{N}) = \mathbb{Z}$ module of 1-cycles and $B_1(\partial\overline{N}) = \mathbb{Z}$ module of 1-boundaries. Let $e_{j\tau} \in J_{\sigma_j}$ for $\tau = 1, 2$. If the two vertices at the ends of $e_{j\tau}$ in σ_j are $v_{j\tau 1}$ and $v_{j\tau 2}$, let $\zeta_{j\tau 1}$ and $\zeta_{j\tau 2}$ be the respective edges of $L_{\sigma_j}(v_{j\tau 1})$ and $L_{\sigma_j}(v_{j\tau 2})$ that do not intersect $e_{j\tau}$. Do the same for $e_{j(\tau+3)}$, so we have four 1-simplices identified in $\partial\overline{N}$. They are $\zeta_{j\tau 1}$, $\zeta_{j\tau 2}$, $\zeta_{j(\tau+3)1}$ and $\zeta_{j(\tau+3)2}$, with one for each vertex of σ_j. Now define $\hat{\gamma}_0$.

$$\begin{aligned} \hat{\gamma}_0 : J_{\sigma_j} &\rightarrow S_1(\partial\overline{N}) \\ e_{j\tau} &\rightarrow \zeta_{j\tau 1} + \zeta_{j\tau 2} + \zeta_{j(\tau+3)1} + \zeta_{j(\tau+3)2} \end{aligned}$$

We have, by [11],

$$\begin{aligned} \hat{\gamma}_0 :\ \mathrm{im}(\beta) &\ \rightarrow\ B_1(\partial \overline{N}) \\ \hat{\gamma}_0 :\ \ker(\beta^*) &\ \rightarrow\ Z_1(\partial \overline{N}) \end{aligned}$$

so there is the induced map

$$\hat{\gamma} : \ker(\beta^*)/\mathrm{im}(\beta) \rightarrow H_1(\partial \overline{N}) = \coprod_{1 \leq i \leq k} H_1(T_i)$$

Next, let $\hat{\delta}_0 :\ H_1(\partial \overline{N}) \rightarrow J$ be defined as follows. Let Γ be a simple simplicial loop on the torus, T_i, associated with the i^{th} cusp of N. In figure 5, γ is such a loop. Each vertex, w_r, of γ, is the vertex of p_r triangles $L_{\sigma_{r_1}}(v), \ldots, L_{\sigma_{rp_r}}(v)$ where T_i is the link of v, a vertex of K. Define the simple cellular path $\overline{\Gamma}$, by starting at the midpoint of the edge of $L_{\sigma_{11}}(v)$ that ends in w_1 but is not s_1. Continue across the $\{L_{\sigma_{1q}}(v)\}_{2 \leq q \leq p_1 - 1}$ by crossing from one triangle to another at the midpoint of the edges that have w_1 as a vertex, ending at the edge of $L_{\sigma_{1p_1}}(v)$ that is not s_2. Then continue across $L_{\sigma_{1p_1}}(v) = L_{\sigma_{21}}(v)$ to the edge of $L_{\sigma_{21}}(v)$ that has w_2 as a vertex but is not s_2. Repeat the process until the loop is closed by going from the edge of $L_{\sigma_{dp_d}}(v) = L_{\sigma_{11}}(v)$ that contains w_d but is not s_1 to the starting point. When $\overline{\Gamma}$ crosses $L_{\sigma_{rq}}(v)$ for $2 \leq q \leq p_r - 1$, it goes counterclockwise around the vertex w_r, as a vertex of $L_{\sigma_{rq}}(v)$, and when it crosses $L_{\sigma_{rp_r}}(v) = L_{\sigma_{(r+1)1}}(v)$, it goes clockwise around the vertex of this triangle that is opposite to s_{r+1}. When one of these vertices belongs to the triangle $L_{\sigma_{rq}}(v)$, the vertex is associated with an edge, $e_{rq\tau}$, of σ_{rq} for some $1 \leq \tau \leq 6$, as defined at the beginning of Section 2.3, "Matrix Rank", and this edge is an element of $J_{\sigma_{rq}} \subset J$. To each of these edges assign a "+" if $\overline{\Gamma}$ goes around its corresponding vertex counterclockwise, and a "−" if $\overline{\Gamma}$ goes around its corresponding vertex clockwise. Γ is homotopic to $\overline{\Gamma}$, so we can define $\hat{\delta}_0 : Z_1(\partial\overline{N}) \to J$ such that $\hat{\delta}_0(\overline{\Gamma}) = \hat{\delta}_0(\Gamma)$ is the signed sum of these edges in J. That is,

$$\hat{\delta}_0(\overline{\Gamma}) = \sum_{\substack{1\leq r\leq d \\ 2\leq q\leq p_r}} (-1)^t e_{rq\tau} \tag{9}$$

where $t = 0$ when $e_{rq\tau}$ is assigned a "+" and $t = 1$ when $e_{rq\tau}$ is assigned a "−". In $J_{\sigma_{rq}}$, $e_{rq\tau} = e_{rq(\tau+3)}$ for $1 \leq \tau \leq 3$ with the last subscript mod 6, and $e_{rq1}+e_{rq2}+e_{rq3} = 0$, so $-e_{rq\tau} = e_{rq(\tau+1)} + e_{rq(\tau+2)}$ with the last two subscripts mod 6. Therefore, when $e_{rq\tau}$ is assigned a "−", we substitute $e_{rq(\tau+1)} + e_{rq(\tau+2)}$ with both subscripts mod 6. Hence,

$$\hat{\delta}_0(\overline{\Gamma}) = \sum_{\substack{1\leq r\leq d \\ 1\leq q\leq p_r}} e_{rq\tau} \tag{10}$$

where $e_{rq\tau}$ is an edge of σ_{rq} that is associated with w_r, a vertex of $L_{\sigma_{rq}}(v)$ and w_r is a vertex of the simple simplicial loop Γ in T_i. The relations of J also mean that $e_{rq3} = e_{rq6} = -e_{rq1} - e_{rq2}$, so

$$\hat{\delta}_0(\overline{\Gamma}) = \sum_{\substack{1\leq j\leq n \\ \overline{\Gamma} \text{ crosses } L_{\sigma_j}(v)}} g'_{j\overline{\Gamma}} e_{j1} + g''_{j\overline{\Gamma}} e_{j2} \tag{11}$$

where, with respect to σ_j, $g'_{j\overline{\Gamma}}$ is the number of occurrences of the z_j parameter minus the number of occurrences of the $1-\frac{1}{z_j}$ parameter and $g''_{j\overline{\Gamma}}$ is the number of occurrences of the $\frac{1}{1-z_j}$ parameter minus the number of occurrences of the $1 - \frac{1}{z_j}$ parameter in Equation 10.

Now let m_i and l_i for $1 \leq i \leq k$ be the meridianal and longitudinal simple simplicial loops on T_i, as in Section 2.2, "Cusp Conditions". We get corresponding $\overline{m}_i$ and $\overline{l}_i$, constructed as $\overline{\Gamma}$ was, where m_i and l_i are homologous to $\overline{m}_i$ and $\overline{l}_i$, respectively. So $\overline{m}_i$ and $\overline{l}_i$ are the generators of $H_1(T_i)$ and their image under $\hat{\delta}_0$ are two columns of

$\underline{\mathbf{ML}}$, the matrix of $\hat{\delta}_0$. These two columns are of the form

$$\begin{aligned} \vec{g}_{m_i} &= (g'_{1\overline{m}_i}, g''_{1\overline{m}_i}, \ldots, g'_{n\overline{m}_i}, g''_{n\overline{m}_i}) \qquad \text{with } g'_{j\overline{m}_i} = m'_{ij} \text{ and } g''_{j\overline{m}_i} = -m''_{ij} \\ \vec{g}_{l_i} &= (g'_{1\bar{l}_i}, g''_{1\bar{l}_i}, \ldots, g'_{n_i}, g''_{n\bar{l}_i}) \qquad \text{with } g'_{j\bar{l}_i} = l'_{ij} \text{ and } g''_{j\bar{l}_i} = -l''_{ij} \end{aligned}$$

where m'_{ij}, m''_{ij} and l'_{ij}, l''_{ij} are the components of the matrices $\mathbf{M}$ and $\mathbf{L}$ from "Cusp Conditions" on page 13. Let $\mathbf{ML}$ be $\mathbf{M}$ concatenated with $\mathbf{L}$. For each generator of $H_1(\partial\overline{N}) = \coprod_{1\le i\le k} H_1(T_i)$, there is a column in the matrix of $\hat{\delta}_0$, so $\underline{\mathbf{ML}}$ has $2k$ columns and $2n$ rows, where the $(2j-1)^{th}$ row of $\underline{\mathbf{ML}}$ is equal to the j^{th} column of $\mathbf{ML}$ and the $2j^{th}$ row of $\underline{\mathbf{ML}}$ is (-1) times the $(n+j)^{th}$ column of $\mathbf{ML}$. Thus, $\operatorname{rank}\underline{\mathbf{ML}} = \operatorname{rank}\underline{\mathbf{ML}}^t = \operatorname{rank}\mathbf{ML}$. The next step is to show that $\operatorname{rank}\underline{\mathbf{ML}} = 2k$. We have $\operatorname{im}(\hat{\delta}_0) \subset \ker(\beta^*)$, with $\hat{\delta}_0(B_1(\partial\overline{N})) \subset \operatorname{im}(\beta)$, so there is the induced map

$$\hat{\delta} : \; H_1(\partial\overline{N}) \to \ker(\beta^*)/\operatorname{im}(\beta)$$

Now $\hat{\gamma}\hat{\delta} : \; H_1(\partial\overline{N}) \to H_1(\partial\overline{N})$ is multiplication by 2 [11], so $\hat{\delta}_0$ must be injective. Consequently, the matrix of $\hat{\delta}_0$ has maximal rank, which is $2k$, making the $2k$ vectors, $\{\vec{g}_{m_i}, \vec{g}_{l_i}\}_{1\le i\le k}$, linearly independent.

M is derived from N by the Dehn filling of $k-h$ cusps of N with filling coefficients of (p_i, q_i) for $1 \le i \le k-h$. Let $\vec{g}_{s_i} = p_i\vec{g}_{m_i} + q_i\vec{g}_{l_i}$ for $1 \le i \le k-h$.

Lemma 2.3 *The $k+h$ vectors*

$$\{\vec{g}_{s_1}, \ldots, \vec{g}_{s_{k-h}}, \vec{g}_{m_{k-h+1}}, \ldots, \vec{g}_{m_k}, \vec{g}_{l_{k-h+1}}, \ldots, \vec{g}_{l_k}\}$$

are linearly independent.

Proof. Assume otherwise. Then there exists ϕ_{si} for $1 \le i \le k-h$ and ξ_{mi} and φ_{li}

for $k-h+1 \leq i \leq k$ such that

$$\begin{aligned} 0 &= \sum_{1\leq i\leq k-h} \phi_{si}\vec{g}_{s_i} + \sum_{k-h+1\leq i\leq k} (\xi_{mi}\vec{g}_{m_i} + \varphi_{li}\vec{g}_{l_i}) \\ &= \sum_{1\leq i\leq k-h} \phi_{si}(p_i\vec{g}_{m_i} + q_i\vec{g}_{l_i}) + \sum_{k-h+1\leq i\leq k} (\xi_{mi}\vec{g}_{m_i} + \varphi_{li}\vec{g}_{l_i}) \\ &= \sum_{1\leq i\leq k-h} \phi_{si}p_i\vec{g}_{m_i} + \sum_{1\leq i\leq k-h} \phi_{si}q_i\vec{g}_{l_i} + \sum_{k-h+1\leq i\leq k} (\xi_{mi}\vec{g}_{m_i} + \varphi_{li}\vec{g}_{l_i}) \end{aligned}$$

We have just seen that $\{\vec{g}_{m_i}, \vec{g}_{l_i}\}_{1\leq i\leq k}$, is linearly independent, so $\xi_{mi} = \varphi_{li} = 0$ for $k-h+1 \leq i \leq k$ and $\phi_{si}p_i = \phi_{si}q_i = 0$ for $1 \leq i \leq k-h$. But at least one of p_i or q_i is not 0, so $\phi_{si} = 0$ for $1 \leq i \leq k-h$. ■

Since rank $\underline{\mathbf{R}} = n-k$, select $n-k$ linearly independent vectors in $\mathrm{im}(\beta)$ that are columns of the matrix $\underline{\mathbf{R}}$, and denote them by $\vec{g}_{\beta_i}$ for $k+1 \leq i \leq n$. Observe that $\mathrm{im}(\hat{\delta}_0) \cap \mathrm{im}(\beta) = \{0\}$, because otherwise, there is a non-trivial $x \in H_1(\partial\overline{N})$ such that $\hat{\gamma}_0\hat{\delta}_0(x) = \hat{\gamma}_0(\text{element of } \mathrm{im}(\beta)) \in B_1(\partial\overline{N})$. Then $\hat{\gamma}\hat{\delta}(x) = 0 \in H_1(\partial\overline{N})$. But $\hat{\gamma}\hat{\delta}$ is multiplication by 2 on $H_1(\partial\overline{N})$, so $x = 0$, which is a contradiction.

Lemma 2.4 *Let*

1) $\underline{\mathbf{S}} =$ *the* $2n \times (k-h)$ *matrix whose columns are the vectors* $\vec{g}_{s_i}$, *for* $1 \leq i \leq k-h$

2) $\underline{\mathbf{M}}_{\mathbf{h}} =$ *the* $2n \times h$ *matrix whose columns are the linearly independent vectors* $\vec{g}_{m_i}$, *for* $k-h+1 \leq i \leq k$

3) $\underline{\mathbf{R}}_\beta =$ *the* $2n \times (n-k)$ *matrix whose columns are the linearly independent vectors* $\vec{g}_{\beta_i}$, *for* $k+1 \leq i \leq n$

Concatenate these matrices to get the $2n \times n$ *matrix* $\underline{\mathbf{F}} = (\underline{\mathbf{S}}|\underline{\mathbf{M}}_{\mathbf{h}}|\underline{\mathbf{R}}_\beta)$. *Rank* $\underline{\mathbf{F}} = n$.

Proof. Assume otherwise. Then the vectors that are the columns of $\underline{\mathbf{F}}$ are not linearly independent, so there are ξ_{si} for $1 \leq i \leq k-h$, φ_{mi} for $k-h+1 \leq i \leq k$ and

$\phi_{\beta i}$ for $k+1 \leq i \leq n$, where not all are zero, such that

$$0 = \sum_{1 \leq i \leq k-h} \xi_{si} \vec{g}_{s_i} + \sum_{k-h+1 \leq i \leq k} \varphi_{mi} \vec{g}_{m_i} + \sum_{k+1 \leq i \leq n} \phi_{\beta i} \vec{g}_{\beta_i} \tag{12}$$

Therefore,

$$\begin{aligned} 0 &= \hat{\gamma}_0(0) \\ &= \hat{\gamma}_0\Big(\sum_{1 \leq i \leq k-h} \xi_{si} \vec{g}_{s_i} + \sum_{k-h+1 \leq i \leq k} \varphi_{mi} \vec{g}_{m_i} + \sum_{k+1 \leq i \leq n} \phi_{\beta i} \vec{g}_{\beta_i} \Big) \\ &= \hat{\gamma}_0\Big(\sum_{1 \leq i \leq k-h} \xi_{si} \vec{g}_{s_i} + \sum_{k-h+1 \leq i \leq k} \varphi_{mi} \vec{g}_{m_i} \Big) + \hat{\gamma}_0\Big(\sum_{k+1 \leq i \leq n} \phi_{\beta i} \vec{g}_{\beta_i} \Big) \\ &= \hat{\gamma}_0\Big[\hat{\delta}_0\Big(\sum_{1 \leq i \leq k-h} \xi_{si}(p_i \overline{m}_i + q_i \bar{l}_i) + \sum_{k-h+1 \leq i \leq k} \varphi_{mi} \overline{m}_i \Big) \Big] + \hat{\gamma}_0(\text{element in im}(\beta)) \end{aligned}$$

But $\hat{\gamma}_0(\text{im}(\beta)) \subset B_1(\partial\overline{N})$, so $\hat{\gamma}\hat{\delta}\Big(\sum_{1 \leq i \leq k-h} \xi_{si}(p_i \overline{m}_i + q_i \bar{l}_i) + \sum_{k-h+1 \leq i \leq k} \varphi_{mi} \overline{m}_i \Big) = 0$. Therefore, $\sum_{1 \leq i \leq k-h} \xi_{si}(p_i \overline{m}_i + q_i \bar{l}_i) + \sum_{k-h+1 \leq i \leq k} \varphi_{mi} \overline{m}_i = 0$ since $\hat{\gamma}\hat{\delta}$ is injective. Hence,

$$\begin{aligned} 0 &= \hat{\delta}_0(0) \\ &= \hat{\delta}_0\Big(\sum_{1 \leq i \leq k-h} \xi_{si}(p_i \overline{m}_i + q_i \bar{l}_i) + \sum_{k-h+1 \leq i \leq k} \varphi_{mi} \overline{m}_i \Big) \\ &= \sum_{1 \leq i \leq k-h} \xi_{si} \hat{\delta}_0(p_i \overline{m}_i + q_i \bar{l}_i) + \sum_{k-h+1 \leq i \leq k} \varphi_{mi} \hat{\delta}_0(\overline{m}_i) \\ &= \sum_{1 \leq i \leq k-h} \xi_{si} \vec{g}_{s_i} + \sum_{k-h+1 \leq i \leq k} \varphi_{mi} \vec{g}_{m_i} \end{aligned}$$

By Lemma 2.3, ξ_{si} for $1 \leq i \leq k-h$ and φ_{mi} for $k-h+1 \leq i \leq k$ are all zero. Then, Equation 12 becomes $0 = \sum_{k+1 \leq i \leq n} \phi_{\beta i} \vec{g}_{\beta_i}$. However, the $\vec{g}_{\beta_i}$ for $k+1 \leq i \leq n$ were selected to be linearly independent, so $\phi_{\beta i} = 0$ for $k+1 \leq i \leq n$. This is a contradiction. ■

Corollary 2.5 *Each column of $\underline{\mathbf{R}}_\beta$ has a corresponding row in $\mathbf{R}$, the matrix associated with the consistency equations. Let $\mathbf{R}_\beta$ be the matrix comprised of only these*

$n - k$ rows of $\mathbf{R}$ *and let*

$$\mathbf{F} = \begin{pmatrix} \mathbf{S} \\ \mathbf{M_h} \\ \mathbf{R}_\beta \end{pmatrix}.$$

Then rank $\mathbf{F} = n$.

Proof. As before, every $(2j-1)^{th}$ row of $\underline{\mathbf{F}}$ is equal to the j^{th} column of $\mathbf{F}$ and every $2j^{th}$ row of $\underline{\mathbf{F}}$ is (-1) times the $(n+j)^{th}$ column of $\mathbf{F}$. Thus,

$$\operatorname{rank} \mathbf{F} = \operatorname{rank} \underline{\mathbf{F}}^t = \operatorname{rank} \underline{\mathbf{F}} = n$$

That is, rank $\mathbf{F} = n$. ■

3 How to Test for a Solution

Let

$$\begin{aligned}
f_i(z_1,\ldots,z_n) &= \sum_{j=1}^{n}\Big((p_i m'_{ij}+q_i l'_{ij})\log(z_j)+(p_i m''_{ij}+q_i l''_{ij})\log(1-z_j)\Big) \\
&\qquad -c_{si}\pi i \qquad (i=1,\ldots,k-h) \\
f_i(z_1,\ldots,z_n) &= \sum_{j=1}^{n}(m'_{ij}\log(z_j)+m''_{ij}\log(1-z_j))-c_{mi}\pi i \\
&\qquad (i=k-h+1,\ldots,k) \\
f_i(z_1,\ldots,z_n) &= \sum_{j=1}^{n}(r'_{ij}\log(z_j)+r''_{ij}\log(1-z_j))-c_i\pi i \\
&\qquad (i=k+1,\ldots,n)
\end{aligned}$$

and let

$$\begin{aligned}
f:\mathbb{C}^n &\to \mathbb{C}^n \qquad \text{such that} \\
z=(z_1,\ldots,z_n) &\to f(z)=(f_1(z),\ldots,f_n(z)).
\end{aligned}$$

Then let

$$\begin{array}{llll}
t'_{ij}=p_i m'_{ij}+q_i l'_{ij} & t''_{ij}=p_i m''_{ij}+q_i l''_{ij} & t'''_i=c_{si} & (i=1,\ldots,k-h) \\
t'_{ij}=m'_{ij} & t''_{ij}=m''_{ij} & t'''_i=c_{mi} & (i=k-h+1,\ldots,k) \\
t'_{ij}=r'_{ij} & t''_{ij}=r''_{ij} & t'''_i=c_i & (i=k+1,\ldots,n).
\end{array}$$

The resulting components of f are

$$f_i(z_1,\ldots,z_n)=\sum_{j=1}^{n}(t'_{ij}\log(z_j)+t''_{ij}\log(1-z_j))-t'''_i\pi i \qquad (i=1,\ldots,n). \tag{13}$$

Then $\frac{\partial f_i(z)}{\partial z_j}=\frac{t'_{ij}}{z_j}-\frac{t''_{ij}}{1-z_j}$ for $1\le i\le n$, so

$$\frac{\partial f(z)}{\partial z_j}=\Big(\frac{t'_{1j}}{z_j}-\frac{t''_{1j}}{1-z_j},\ldots,\frac{t'_{nj}}{z_j}-\frac{t''_{nj}}{1-z_j}\Big). \tag{14}$$

Since we are only working with manifolds where SnapPea finds an approximate solution to f in $\mathbb{C}^n_+$, the upper half plane in $\mathbb{C}^n$, there is an $a \in \mathbb{C}^n_+$ such that $f(a) = b$ and b is extremely close to $0 \in \mathbb{C}^n$. We begin with the method that uses many of the constructs in the proof of the Inverse Function Theorem, as presented in [21], and find that these quantities are used again by the Kantorovich method. Each of these methods provides a sufficient condition for a manifold to have a complete hyperbolic structure. Consequently, a manifold may not satisfy either condition and still be complete hyperbolic. However, if it satisfies at least one we know it has a complete hyperbolic structure.

3.1 Inverse Function Theorem

The Inverse Function Theorem states that there exists U', a neighborhood of a, that is mapped homeomorphically by f onto V, a neighborhood of b if the determinant of $f'(a)$, the derivative of f at a, is not zero. It is our aim to describe V because if $0 \in V$ and $U' \subset \mathbb{C}^n_+$, there is a solution to $f(z) = 0$ in $\mathbb{C}^n_+$ that gives a complete hyperbolic structure on the manifold. Let $H = \mathbb{C}^n_+$. H is open in $\mathbb{C}^n$. Each f_i is holomorphic on H, so f is holomorphic on H [14]. Thus f is smooth on H, with the *differential of f at z, $df(z)$*, being the linear map

$$\begin{aligned} df(z) : \mathbb{C}^n &\to \mathbb{C}^n \\ v &\to f'(z) \cdot v, \end{aligned}$$

where $f'(z) = \left(\frac{\partial f_i(z)}{\partial z_j}\right)_{1 \le i,j \le n}$ [21]. We know that $\det f'(a) \neq 0$ [4], so rank $f'(a) = n$ and f is regular at a. Since $|df(z)(v)|$, as a function of v, is a continuous function on $\mathbb{C}^n$, it will attain a maximum and minimum on the compact set $\{v \in \mathbb{C}^n : |v| = 1\}$.

Thus, for any point $z \in H$, we can define

$$|df(z)|_{\inf} = \inf_{|v|=1} |df(z)(v)|.$$

We will be interested in calculating the value of $|df(a)|_{\inf}$, however now we introduce $\eta = \frac{|df(a)|_{\inf}}{2}$. The open ball of radius r about z in $\mathbb{C}^n$, $\{w \in \mathbb{C}^n : |w - z| < r\}$, will be denoted by $B_r(z)$. Whitney [21] proves that:

- $|df(a)|_{\inf} > 0$.
- There exists a $\delta > 0$ such that if $z', z'' \in B_\delta(a)$ then $|f(z') - f(z'')| \geq \eta |z' - z''|$, so that f is 1-1 on $B_\delta(a)$. This is done by letting $\epsilon = \frac{|df(a)|_{\inf} - \eta}{n}$ and selecting $\delta > 0$ such that $|\frac{\partial f(z)}{\partial z_j} - \frac{\partial f(a)}{\partial z_j}| < \epsilon$ for $1 \leq j \leq n$ whenever $|z - a| < \delta$. This can be done since the partial derivatives, $\frac{\partial f(z)}{\partial z_j}$ for $1 \leq j \leq n$, are continuous functions of z on H.
- When we take δ small enough, $B_\delta(a) \subset H$, and the rank of $f'(z)$ for $z \in B_\delta(a)$ is n. If $U = B_\delta(a)$ and $V = B_{\frac{\eta\delta}{4}}(b)$, then $V \subset f(U)$.

Therefore, if $0 \in V$ and $U' = U \cap f^{-1}(V)$, there is a unique $\tilde{z} \in U'$ such that $f(\tilde{z}) = 0$. This $\tilde{z}$ is a solution of the surgery, completeness and consistency equations. Consequently, we need $|b| < \frac{\eta\delta}{4}$, so it remains to calculate $|df(a)|_{\inf}$ and δ.

3.1.1 Calculate $|df(a)|_{\inf}$

We know that $\det f'(a) \neq 0$, so $f'(a)^{-1}$ exists. Let

$$B = \{f'(a) \cdot v : |v| = 1\} = \{w \in \mathbb{C}^n : |f'(a)^{-1} \cdot w| = 1\}.$$

We look at the continuous real valued function μ on the compact set B such that

$$\begin{aligned} \mu : B &\to \mathbb{R} \\ w &\to |w|^2. \end{aligned}$$

Let $S = \{v \in \mathbb{C}^n : |v| = 1\}$. Then μ attains a minimum at some $\tilde{w} \in B$ and the function $|df(a)|$ will attain a minimum at some $\tilde{v} \in S$ where $\tilde{w} = df(a)(\tilde{v}) = f'(a) \cdot \tilde{v}$. Now let $A = f'(a)^{-1}$. This is a complex matrix, so

$$\begin{aligned} |Aw|^2 &= (Aw)^t(\overline{Aw}) \qquad \overline{A} = \text{conjugate of } A \text{ and } t = \text{transpose of } A \\ &= (w^t A^t)(\overline{Aw}) \\ &= w^t(A^t\overline{A})\overline{w}. \end{aligned}$$

Let $D = (A^t\overline{A})$. This is a self adjoint matrix so it has real eigenvalues [6]. Then,

$$\begin{aligned} B &= \{w : |Aw| = 1\} \\ &= \{w : |Aw|^2 = 1\} \\ &= \{w : w^t D\overline{w} = 1\}. \end{aligned}$$

Using the Lagrange multiplier method to minimize $g = |w|^2$ on B [6], let

$$\begin{aligned} H(w_1, \ldots, w_n, \lambda) &= |w|^2 - \lambda(w^t D\overline{w} - 1) \\ &= \sum_{i=1}^{n} w_i\overline{w_i} - \lambda\Big(\sum_{i=1}^{n} w_i(\sum_{j=1}^{n} d_{ij}\overline{w_j}) - 1\Big). \end{aligned}$$

In order to find a critical point for H, all partials with respect to $w_1, \ldots, w_n$ and λ must be 0. We set

$$0 = \frac{\partial H}{\partial w_i} = \overline{w_i} - \lambda(\sum_{j=1}^{n} d_{ij}\overline{w_j}) \qquad (i = 1, \ldots, n),$$

so,

$$\begin{aligned} 0 &= \overline{w} - \lambda D\overline{w} \\ &= (I - \lambda D)\overline{w} \\ &= (\frac{1}{\lambda}I - D)\overline{w}. \end{aligned}$$

Then $D\overline{w} = \frac{\overline{w}}{\lambda}$, making $\frac{1}{\lambda}$ an eigenvalue of D. Also,

$$\begin{aligned} 0 &= \frac{\partial H}{\partial \lambda} \\ &= w^t D\overline{w} - 1. \end{aligned}$$

Thus, $w^t D\overline{w} = 1$, and substituting $\frac{\overline{w}}{\lambda}$ for $D\overline{w}$ from above, we have $w^t \frac{\overline{w}}{\lambda} = 1$. That is, $w^t\overline{w} = \lambda$. But $w^t\overline{w} = |w|^2$, so $\min|w|^2 = \min\lambda$ such that $\frac{1}{\lambda}$ is an eigenvalue of D. From this we see that on B,

$$\begin{aligned} \min|w|^2 &= \frac{1}{\text{maximum eigenvalue of } D} \\ &= \text{smallest eigenvalue of } D^{-1} \text{ [5]} \\ &= \text{smallest eigenvalue of } (A^t\overline{A})^{-1} \\ &= \text{smallest eigenvalue of } \overline{f'(a)}f'(a)^t. \end{aligned}$$

By definition, $|df(a)|_{\inf} = \min_{\text{on}B}|w|$, so

$$|df(a)|_{\inf} = \sqrt{\text{smallest eigenvalue of } \overline{f'(a)}f'(a)^t}. \tag{15}$$

The result is that we calculate the eigenvalues of $\overline{f'(a)}f'(a)^t$ using its characteristic polynomial and then take the square root of the smallest one to get $|df(a)|_{\inf}$. We can now set $\eta = \frac{|df(a)|_{\inf}}{2}$.

3.1.2 Calculate δ

Since the partial derivatives $\frac{\partial f(z)}{\partial z_j}$ for $1 \leq j \leq n$ are continuous functions of z on H, there exists a $\delta_j > 0$ such that $|\frac{\partial f(z)}{\partial z_j} - \frac{\partial f(a)}{\partial z_j}| < \epsilon$ when $|z - a| < \delta_j$ for each j. Let $\delta = \min_j(\delta_j)$, so that for all j, $|\frac{\partial f(z)}{\partial z_j} - \frac{\partial f(a)}{\partial z_j}| < \epsilon$ whenever $|z - a| < \delta$. Thus, we must identify these δ_j for $1 \leq j \leq n$.

$$\begin{aligned} \Big|\frac{\partial f(z)}{\partial z_j} - \frac{\partial f(a)}{\partial z_j}\Big| &= \Big|\Big(\frac{\partial f_1(z)}{\partial z_j} - \frac{\partial f_1(a)}{\partial z_j}, \ldots, \frac{\partial f_n(z)}{\partial z_j} - \frac{\partial f_n(a)}{\partial z_j}\Big)\Big| \qquad (16) \\ &\leq \sum_{i=1}^{n} \Big|\frac{\partial f_i(z)}{\partial z_j} - \frac{\partial f_i(a)}{\partial z_j}\Big|. \qquad (17) \end{aligned}$$

If there exists a δ_j such that when $|z - a| < \delta_j$ we have $|\frac{\partial f_i(z)}{\partial z_j} - \frac{\partial f_i(a)}{\partial z_j}| < \frac{\epsilon}{n}$ for all $1 \leq i \leq n$, we can set $\delta = \min_j \delta_j$ and our conditions are satisfied. From page 25,

$$\begin{aligned} \frac{\partial f_i(z)}{\partial z_j} - \frac{\partial f_i(a)}{\partial z_j} &= \frac{t'_{ij}}{z_j} - \frac{t''_{ij}}{1 - z_j} - \Big(\frac{t'_{ij}}{a_j} - \frac{t''_{ij}}{1 - a_j}\Big) \\ &= \Big(\frac{t'_{ij}}{z_j} - \frac{t'_{ij}}{a_j}\Big) - \Big(\frac{t''_{ij}}{1 - z_j} - \frac{t''_{ij}}{1 - a_j}\Big) \\ &= \Big(\frac{a_j t'_{ij} - z_j t'_{ij}}{a_j z_j}\Big) - \Big(\frac{(1 - a_j)t''_{ij} - (1 - z_j)t''_{ij}}{(1 - a_j)(1 - z_j)}\Big) \\ &= \Big(\frac{a_j t'_{ij} - z_j t'_{ij}}{a_j z_j}\Big) - \Big(\frac{t''_{ij} - a_j t''_{ij} - t''_{ij} + z_j t''_{ij}}{(1 - a_j)(1 - z_j)}\Big) \\ &= t'_{ij}\frac{(a_j - z_j)}{a_j z_j} + t''_{ij}\frac{(a_j - z_j)}{(1 - a_j)(1 - z_j)} \\ &= (a_j - z_j)\Big(\frac{t'_{ij}}{a_j z_j} + \frac{t''_{ij}}{(1 - a_j)(1 - z_j)}\Big). \qquad (18) \end{aligned}$$

From this we see that we need only concern ourselves when both t'_{ij} and t''_{ij} are not zero, because otherwise $\frac{\partial f_i(z)}{\partial z_j} - \frac{\partial f_i(a)}{\partial z_j} = 0$.

Let $\hat{M} = \min(|a_j|, |1 - a_j|)$ and $\delta_j \leq \min(\frac{\hat{M}}{2}, \text{Im}(a_j))$. We note that $a \in H$ so $\text{Im}(a_j) > 0$ and $1 - a_j \neq 0$, so $\delta_j > 0$. By restricting δ_j to be less than or equal to

$\mathrm{Im}(a_j)$, we guarantee solutions only in H. Then $\delta_j \leq \frac{|a_j|}{2}$, so

$$|a_j| - \delta_j \geq \frac{|a_j|}{2} > 0. \tag{19}$$

Thus,

$$\frac{1}{|a_j| - \delta_j} \leq \frac{1}{\frac{|a_j|}{2}}. \tag{20}$$

Similarly, $\delta_j \leq \frac{|1-a_j|}{2}$, so

$$|1 - a_j| - \delta_j \geq \frac{|1 - a_j|}{2} > 0, \tag{21}$$

giving

$$\frac{1}{|1 - a_j| - \delta_j} \leq \frac{1}{\frac{|1-a_j|}{2}}. \tag{22}$$

Let $z \in B_{\delta_j}(a)$, so that $|z_j - a_j| < \delta_j$. In Equation 18 there are two quantities to consider: $\frac{t'_{ij}}{a_j z_j}$ and $\frac{t''_{ij}}{(1-a_j)(1-z_j)}$.

1. $\frac{t'_{ij}}{a_j z_j}$

   $|a_j| - |z_j| \leq |z_j - a_j| < \delta_j$, so that $|z_j| > |a_j| - \delta_j$. However, $|a_j| - \delta_j > 0$, whereby

   $$\frac{1}{|z_j|} < \frac{1}{|a_j| - \delta_j}. \tag{23}$$

   Hence, by Equations 23 and 20, we have

   $$\left|\frac{t'_{ij}}{a_j z_j}\right| < \frac{|t'_{ij}|}{|a_j|(|a_j| - \delta_j)} \leq \frac{|t'_{ij}|}{|a_j|\frac{|a_j|}{2}}.$$

   That is,

   $$\left|\frac{t'_{ij}}{a_j z_j}\right| < \frac{|t'_{ij}|}{|a_j|\frac{|a_j|}{2}}. \tag{24}$$

2. $\frac{t''_{ij}}{(1-a_j)(1-z_j)}$

$$\begin{aligned} |1-a_j| - |1-z_j| &\leq |(1-z_j)-(1-a_j)| \\ &= |a_j - z_j| \\ &< \delta_j \end{aligned}$$

so that

$$|1-a_j| - \delta_j < |1-z_j|. \tag{25}$$

But $|1-a_j| - \delta_j > 0$, so that

$$\Big|\frac{1}{1-z_j}\Big| < \frac{1}{|1-a_j| - \delta_j}.$$

Combining with Equation 22, we get

$$\Big|\frac{1}{1-z_j}\Big| < \frac{1}{\frac{|1-a_j|}{2}}. \tag{26}$$

Hence,

$$\Big|\frac{t''_{ij}}{(1-a_j)(1-z_j)}\Big| < \frac{|t''_{ij}|}{|1-a_j|\frac{|1-a_j|}{2}}. \tag{27}$$

Going back to Equations 18, 24 and 27, we conclude:

$$\Big|\frac{\partial f_i(z)}{\partial z_j} - \frac{\partial f_i(a)}{\partial z_j}\Big| < 2\delta_j \frac{|t'_{ij}|}{|a_j|^2} + 2\delta_j \frac{|t''_{ij}|}{|1-a_j|^2} = 2\delta_j \left[\frac{|t'_{ij}|}{|a_j|^2} + \frac{|t''_{ij}|}{|1-a_j|^2}\right]. \tag{28}$$

Let

$$\delta_j^i = \min\left(\frac{\epsilon}{2n}\left[\frac{1}{\frac{|t'_{ij}|}{|a_j|^2} + \frac{|t''_{ij}|}{|1-a_j|^2}}\right], \frac{\hat{M}}{2}, \mathrm{Im}(a_j)\right) \text{ and } \delta_j = \min_{1 \leq i \leq n} \delta_j^i.$$

Then, for $|z-a| < \delta_j$,

$$\Big|\frac{\partial f_i(z)}{\partial z_j} - \frac{\partial f_i(a)}{\partial z_j}\Big| < 2\frac{\epsilon}{2n}\left(\frac{1}{\frac{|t'_{ij}|}{|a_j|^2} + \frac{|t''_{ij}|}{|1-a_j|^2}}\right)\left(\frac{|t'_{ij}|}{|a_j|^2} + \frac{|t''_{ij}|}{|1-a_j|^2}\right). \tag{29}$$

That is, for $|z-a| < \delta_j$, we have

$$\left|\frac{\partial f_i(z)}{\partial z_j} - \frac{\partial f_i(a)}{\partial z_j}\right| < \frac{\epsilon}{n} \qquad (i = 1, \ldots, n).$$

In conclusion, let $\delta = \min_{1 \le j \le n}(\delta_j)$. Using the values of η and δ just calculated, the test for a solution to the equations, as presented on page 27, means checking if

$$|b| < \frac{\eta\delta}{4}.$$

This completes the first part of the proof of Theorem 1.2.

3.2 Kantorovich

The Kantorovich Theorem [8] provides a test for the solution of f. The relevance of this theorem to the solution of f was brought to our attention by Joan Birman after the Inverse Function Theorem test of Section 3.1 had been developed. We thank her for telling us about it. The Kantorovich Theorem is usable in our situation because we can identify the quantities used. This is not the case for all functions.

Theorem 3.1 (Kantorovich) *Let U be an open neighborhood of a point, a, in $\mathbb{C}^n$ and $f : U \to \mathbb{C}^n$ a holomorphic mapping with invertible derivative $f'(a)$ at a. Let $hh = -f'(a)^{-1}f(a)$, $\tilde{a} = a + hh$ and $U_0 = B_{|hh|}(\tilde{a})$. If $U_0 \subset U$ and*

1. *The derivative $f'(z)$ satisfies the Lipschitz Condition on U_0, with Lipschitz Ratio, L*

2. *$|f(a)||f'(a)^{-1}|^2 L \le \frac{1}{2}$,*

then $f(z) = 0$ *has a unique solution in* U_0.

The Kantorovich Theorem applied to our function, f, works as follows. Let $U = H$. Given a, an approximate solution to $f(z) = 0$, apply Newton's method to f at a to get an even better approximate solution, $\tilde{a}$. That is, let $hh = -f'(a)^{-1} \cdot f(a)$ and $\tilde{a} = a + hh = (a_1 + hh_1, \ldots, a_n + hh_n)$ so $\tilde{a}_j = a_j + hh_j$. Then see if a Lipschitz Ratio, denoted by L, can be identified for $z \in B_{|hh|}(\tilde{a})$ so that $f'(z)$ satisfies the Lipschitz condition on U_0 with L. One way to do this is to find an upper bound, c_{ijk}, on the second partials, $|\partial_i \partial_j f_k(z)|$ for $1 \leq i, j, k \leq n$ for $z \in B_{|hh|}(\tilde{a})$, and let $L = \sqrt{\sum_{1 \leq i,j,k \leq n} (c_{ijk})^2}$ [8]. This works for us, but in general, the major stumbling block to using this theorem is the difficulty in finding this L. Here, $|f'(a)^{-1}|$, the norm of $f'(a)^{-1}$, can be either the supremum norm, which we will denote by $|f'(a)^{-1}|_{\text{sup}}$, or the length norm, referred to as $|f'(a)^{-1}|_{\text{len}}$, where

$$|f'(a)^{-1}|_{\text{sup}} = \sup_{|v|=1} |f'(a)^{-1} \cdot v|$$

and if a component of $f'(a)^{-1}$ is denoted by h_{ij},

$$|f'(a)^{-1}|_{\text{len}} = \sqrt{\sum_{1 \leq i,j \leq n} |h_{ij}|^2}.$$

Now substitute values in the inequality found in the second part of the Kantorovich Theorem and see if they pass the test. If so, there is a solution in $B_{|hh|}(\tilde{a})$. The c_{ijk} are found by methods similar to those used in the calculation of the δ^i_j, and the calculation of the supremum norm is almost identical to that of $|df(a)|_{\text{inf}}$. In fact, the supremum norm can be expressed in terms of $|df(a)|_{\text{inf}}$.

3.2.1 Calculate $|f'(a)^{-1}|$

Supremum Norm:$|f'(a)^{-1}|_{\text{sup}}$ Follow the work in Section 3.1.1, only this time we are interested in $f'(a)^{-1}$ instead of $f'(a)$. Thus, let A_K, D_K and B_K be defined as follows:

$$A_K = (f'(a)^{-1})^{-1} = f'(a),\ \ D_K = A_K^t \overline{A_K} = f'(a)^t \overline{f'(a)} \text{ and}$$

$$B_K = \{f'(a)^{-1} \cdot v : |v| = 1\} = \{w \in \mathbb{C}^n : |f'(a) \cdot w| = 1\}.$$

Then,

$$\begin{aligned} \max_{\text{on} B_K} |w|^2 &= \frac{1}{\text{smallest eigenvalue of } D_K} \\ &= \frac{1}{\text{smallest eigenvalue of } A_K^t \overline{A_K}} \\ &= \frac{1}{\text{smallest eigenvalue of } f'(a)^t \overline{f'(a)}}. \end{aligned}$$

By definition, $|f'(a)^{-1}|_{\text{sup}} = \max_{\text{on} B_K} |w|$, so

$$|f'(a)^{-1}|_{\text{sup}} = \frac{1}{\sqrt{\text{smallest eigenvalue of } f'(a)^t \overline{f'(a)}}}. \tag{30}$$

Two matrices that are conjugate to each other have the same eigenvalues, so $f'(a)^t \overline{f'(a)}$ and $\overline{f'(a)} f'(a)^t$ have the same eigenvalues. The relation between the Inverse Function Theorem and Kantorovich methods is seen now, since the definition of η on page 29 means that

$$|f'(a)^{-1}|_{\text{sup}} = \frac{1}{2\eta}. \tag{31}$$

Length Norm:$|f'(a)^{-1}|_{\text{len}}$ Let the components of $f'(a)^{-1}$ be $(h_{ij})_{1 \le i,j \le n}$. Then

$$|f'(a)^{-1}|_{\text{len}} = \sqrt{\sum_{1 \le i,j \le n} |h_{ij}|^2}. \tag{32}$$

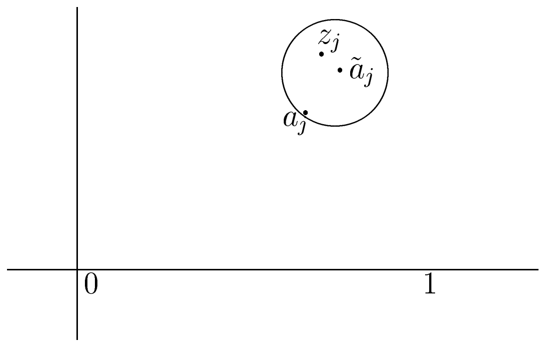

Figure 7: Disc of radius $|hh_j|$ about $\tilde{a}_j$

3.2.2 Calculate c_{ijk}

Let $z \in B_{|hh|}(\tilde{a})$. Then $|z-\tilde{a}| < |hh|$, so $|z_j - \tilde{a}_j| < |hh|$, where $z_j - \tilde{a}_j = z_j - (a_j + hh_j)$ since $\tilde{a}_j = a_j + hh_j$. Figure 7 shows the situation for each j. There are three tests that need to be performed before we test for the inequality in the Kantorovich Theorem. The entire process stops and Kantorovich tells us nothing about a manifold when any of these tests fail.

Test 1 We want a solution in H, so **we require that** $\mathrm{Im}(\tilde{\mathbf{a}}_\mathbf{j}) > |\mathbf{hh}|$. Otherwise, there are $z \in B_{|hh|}(\tilde{a})$ that have $\mathrm{Im}(z_j) \leq 0$, and the solution could be one of these z.

Test 2

$$|(z_j - a_j) - hh_j| = |z_j - (a_j + hh_j)| = |z_j - \tilde{a}_j| < |hh|.$$

Using triangle inequalities,

$$|z_j - a_j| - |hh_j| \leq |(z_j - a_j) - hh_j|.$$

Therefore,$|z_j - a_j| - |hh_j| < |hh|$, giving $|z_j - a_j| < |hh_j| + |hh|$. But $|hh_j| \leq |hh|$, so $|z_j - a_j| < 2|hh|$. Now

$$|z_j| = |a_j + (z_j - a_j)| \geq |a_j| - |(z_j - a_j)|.$$

Thus,

$$|z_j| > |a_j| - 2|hh|. \tag{33}$$

We need $|a_j| - 2|hh| > 0$ in order to define L, so **the second test is to see if** $|\mathbf{hh}| < \frac{1}{2}|\mathbf{a_j}|$. Then,

$$\frac{1}{z_j} < \frac{1}{|a_j| - 2|hh|}. \tag{34}$$

Test 3 We do a similar process as in the previous test. We already know that $|z_j - a_j| < 2|hh|$. Only now, we use $1 - z_j$ instead of z_j, so

$$|1 - z_j| = |1 - a_j - (z_j - a_j)| \geq |1 - a_j| - |(z_j - a_j)|.$$

Hence,

$$|1 - z_j| > |1 - a_j| - 2|hh|. \tag{35}$$

We need $|1 - a_j| - 2|hh| > 0$; **a third test is to check that** $|\mathbf{hh}| < \frac{1}{2}|\mathbf{1 - a_j}|$. Then,

$$\frac{1}{1 - z_j} < \frac{1}{|1 - a_j| - 2|hh|}. \tag{36}$$

Remainder of Calculation

We are now ready to look at the second partials. By page 25 we see that for $z \in B_{|hh|}(\tilde{a})$,

$$\partial_j f_i(z) = \frac{\partial f_i(z)}{\partial z_j} = \frac{t'_{ij}}{z_j} - \frac{t''_{ij}}{1 - z_j}.$$

Therefore,

$$\partial_k \partial_j f_i(z) = 0 \qquad \text{for } k \neq j \tag{37}$$

$$= -\frac{t'_{ij}}{z_j^2} - \frac{t''_{ij}}{(1-z_j)^2} \qquad \text{for } k = j. \tag{38}$$

Consequently,

$$|\partial_j \partial_j f_i(z)| \leq \frac{|t'_{ij}|}{|z_j|^2} + \frac{|t''_{ij}|}{|1-z_j|^2}.$$

Combining this with Equations 34 and 36 yields

$$|\partial_j \partial_j f_i(z)| \leq \frac{|t'_{ij}|}{(|a_j| - 2|hh|)^2} + \frac{|t''_{ij}|}{(|1-a_j| - 2|hh|)^2}. \tag{39}$$

Using this, c_{ijk} for $1 \leq i, j, k \leq n$ is defined as

$$c_{ijk} = 0 \qquad \text{for } j \neq k \tag{40}$$

$$c_{ijj} = \frac{|t'_{ij}|}{(|a_j| - 2|hh|)^2} + \frac{|t''_{ij}|}{(|1-a_j| - 2|hh|)^2}. \tag{41}$$

The Lipschitz Ratio, L, can now be identified as

$$L = \sqrt{\sum_{1 \leq i,j \leq n} (c_{ijj})^2}$$

$$L = \sqrt{\sum_{1 \leq i,j \leq n} \left(\frac{|t'_{ij}|}{(|a_j| - 2|hh|)^2} + \frac{|t''_{ij}|}{(|1-a_j| - 2|hh|)^2} \right)^2}. \tag{42}$$

The theorem can finally be applied, testing to see if $|f(a)||f'(a)^{-1}|^2 L \leq \frac{1}{2}$. Since $b = f(a)$, this can be rewritten as

$$|b| \leq \frac{1}{2|f'(a)^{-1}|^2 L}.$$

We really have two tests, one using the supremum norm and the other using the length norm. This completes the last part of the proof of Theorem 1.2.

4 Examples

The methods presented are implemented by the use of two programs: SNAP to get information about the manifold and Pari-Gp [2] to do calculations. We use Pari-Gp instead of Mathematica because of its high level of precision. Page 41 begins the template for a program that is written in an edit file and then copied into Pari-Gp for execution. The template needs to be adjusted for information gotten from SNAP. Assume we have a manifold file in SNAP for the manifold, M. Once SNAP is open, read in the file for processing. The "pr sol" command will print the type of solution SNAP has found. A geometric solution means that the solution is in H. Any other response is useless here, so there is no need to go any further. Assuming it is geometric, proceed with setting up the template. Issue the "pr sh" command. SNAP will return the transpose of a vector representing an approximate solution to our set of n equations for M. The number of components of the vector will be equal to n, the number of tetrahedra in the triangulation. Then copy this vector from SNAP to the template, replacing $[a_1, \ldots, a_n]$, so that a now has the value of our approximate solution. The tilde at the end of the Snap response must be eliminated so that a appears as a $1 \times n$ matrix. Next comes the "pr fill" command. SNAP will display a $(n + k) \times (2n + 1)$ matrix where the components of each row are the coefficients of a cusp or consistency equation. Assuming M is the result of Dehn filling on h out of k cusps, the first $k - h$ rows represent the cusp surgery equations, the next h rows are the meridian completeness equations for the unsurgered cusps, and the last n rows are all the consistency equations before any have been eliminated. If all cusps are unsurgered, $h = k$, so the first k rows are all meridianal completeness equations. This command to print filling equations is closely related to the "pr gl" command which prints the gluing equations. This latter display presents the k meridianal followed by the k longitudinal completeness equations for the original k cusps before any surgery, and then all of the n consistency equations. But it is simpler to use the filling equations, even for manifolds where no surgery has been done. Copy this matrix from Snap to the template, initializing the matrix FG. The script will then create the matrices F and G, where F consists of the first k rows of FG and G consists of the last n rows of FG. The rows of F are linearly independent and the program selects $n-k$ rows from G so that when added to F, the resulting matrix has rank n.

The only further adjustments may be Pari-Gp punctuation to reflect line continuation. In order to tell Pari-Gp to ignore an end of line from the text editor, a "\" followed immediately by Return must end that text line. This is needed with a large vector or matrix, so it will probably be needed once the values for a, F and G are copied into the template.

4.1 Template

/* set precision to 60 (or higher for very large manifolds) from the default of */

\p 60

/* read the file FILENAME into SNAP */

/* see that there are h unsurgered cusps */

/* (,) (,) (,) (,) */

/* print shapes - the triangulation has n tetrahedra */

/* enter the shapes as a vector, so it is regarded as a $1 \times n$ matrix */

a =

$[a_1, \ldots, a_n]$

/* find n, the number of tetrahedra */

n = matsize(a)[2]

/* print filling equations and use this to initialize the matrix FG. The first $k - h$ \

equations are cusp surgery equations, followed by the h meridianal completeness \

equations, and finally all of the n consistency equations. */

FG =

$[x_{11}, \ldots, x_{1(2n+1)}; \ldots; x_{(n+k)1}, \ldots, x_{(n+k)(2n+1)}]$

/* find $n + k$, the number of equations derived from the Snap command "pr fill" */

numalleq = matsize(FG)[1]

/* find total number of cusps, k */

k = numalleq - n

/* initialize F, the cusp equations matrix, using the first k equations from FG */

F = matrix(k,2*(n) +1,ii,jj,FG[ii,jj])

/* initialize G, the matrix of all consistency equations, using the last n equations\

```
from FG */
G = matrix(n,2*(n) +1,ii,jj,FG[k+ii,jj])
/* define matrix H by eliminating the last column of F representing the πi \
 coefficient */
H = matrix (k,2*n,i,j,F[i,j])
/* define matrix K by eliminating the last column of G representing the πi \
coefficient */
K = matrix (n,2*n,i,j,G[i,j])
/* redefine F and H by adding rows to them from G and K respectively \
  until the rank of F and H are both n */
r = 1
v(r) = ( (2*n)+1,l,G[r,l] )
t(r) = vector( 2*n,l,K[r,l] )
while( n − matrank(H) && (n+1−r), if( (matrank(concat(F,v(r)))  \
                    − matrank(F)) && (matrank(concat(H,t(r))) − matrank(H)), \
                     (F = concat(F,v(r))) &&  (H = concat(H,t(r))), r=r+1))
eval(F)
eval(H)
/* set up the filling equations as log functions evaluated at a */
f(i) = sum( j = 1, n, F[i,j]*log(a[j]) ) + ( j = 1, n, F[i,n+j]*log(1-a[j]) ) \
                                 + F[i,(2*n)+1]*Pi*I
/* define the vector b in ℂ^n */
b = vector( n, i, f(i) )
/* identify the norm of b */
```

```
normb = sqrt( norml2(b) )
/* identify A, the derivative matrix for f at a */
g(i,j) = ( F[i,j]/a[j] ) - ( F[i,n+j]/(1-a[j]) )
A = matrix( n, n, i, j, g(i,j) )
/* check that determinant of A is not zero */
matdet(A)

    /* INVERSE FUNCTION THEOREM PROCESSING */
/* find eigenvalues for D = (conjugate of A) * (transpose of A) */
D = conj(A)*mattranspose(A)
wapprox = polroots( charpoly(D,x) )
w = real( wapprox )
/* define Eta = (square root of smallest eigenvalue of D)/2 */
Eta = sqrt ( vecmin( w ) )/2
/* define Eps = ((square root of smallest eigenvalue of D) - Eta)/n */
Eps = ( sqrt( vecmin( w ) ) - Eta )/n
/* Identify DELTA */
/* If not both F[i,j] and F[i,j+n] are equal to zero, select delta(i,j) according\
  to instructions. If both equal zero, then we get zero as the denominator; however\
  any arbitrarily large delta will work for this i and j, so select \
  delta(i,j)=((4*normb)/Eta) + 1 */
delta(i,j) = if( abs(F[i,j])+abs(F[i,j+n]),( Eps/(2n) ) \
          * ( 1/( (abs(F[i,j])/(abs(a[j]))^2) + (abs(F[i,n+j])/(abs(1-a[j]))^2) ) ),\
                              ((4*normb)/Eta)+1 )
```

```
vecdelta(j) = vector( n, i, delta(i,j) )
mindelta(j) = vecmin( vecdelta(j) )
vecmind elta(j) = [mindelta(j),imag(a[j]),(norml2(a[j]))/2,(norml2(1−a[j]))/2]
mmindelta(j) = vecmin(vecmindelta(j) )
VECDELTA = vector( n, j, mmindelta(j) )
DELTA = vecmin( VECDELTA )
/* find the value that the norm of b must be less than */
(Eta*DELTA)/4
/* compare norm of b to above */
normb < ( (Eta*DELTA)/4 )
/* KANTOROVICH PROCESSING */
/* change b into a matrix to do matrix multiplication */
B = matrix(n,1,j,i,b[j])
/* define the vector hh and find its length, normhh */
hhh = −(A)^(−1)*(B)
hh = vector(n, j, hhh[j,1])
normhh = sqrt(norml2(hh))
/* perform the first three tests to see if this method is applicable */
atilde = a + hh
/* test 1 to see if fat solution; if j > n */
for (j = 1, n, if(normhh < imag(atilde[j]), , \
                    error("failure at atilde[", j, "]")))
/* test 2 to see if c_ijj can be defined */
for(j = 1, n, if(normhh < (1/2)*abs(a[j]), , \
```

```
                                        error("failure at atilde[", j, "]")))
/* test 3; other test to see if c_ijj can be defined */
for(j = 1, n, if(normhh < (1/2)*abs(1 − a[j]), , \
                                        error("failure at atilde[", j, "]")))
/* identify the Lipschitz ratio, Lips */
c(i,j) = (abs(F[i,j])/(abs(a[j]) − 2*normhh)^2) \
                                       + (abs(F[i,j+n])/(abs(1−a[j]) − 2*normhh)^2)
Lips = sqrt( sum( j = 1, n, sum(i = 1, n, c(i,j)^2) ) )
/* identify normAinv, the norm of A^(−1), using the definition of matrix \
 norm as the supremum of A^(−1)v for v on the n-sphere */
normAinv = 1/( 2 * Eta )
/* do the Kantorovich tests */
/* find the value that the norm of b must be less than or equal to with respect \
 to the supremum norm */
1/(2 * (normAinv)^2 * Lips)
normb <= 1/(2 * (normAinv)^2 * Lips)
/* find the length norm and the value that the norm of b must be less than\
  or equal to with respect to the length norm */
sqrt(norml2(A^(−1)))
1/(2 * norml2(A^(−1)) * Lips)
normb <= 1/(2 * norml2(A^(−1)) * Lips)
```

■

4.2 Using the Template

The template is ready to be used. If you want a copy of what has happened, first turn the log on in Pari-Gp by typing "\l logfilename". Then copy the adjusted template to Pari-Gp, wait for the run to complete, and open the log file to see the results. Make sure that there are no error messages from the qualification tests described. If there are, any further results are of no value. If there are no error messages, a response of "1" to at least the Inverse Function Theorem or either of the Kantorovich inequalities will indicate the manifold is complete hyperbolic. A copy of the template can be found at [10]. We now look at six examples. Each example will have two sets of data. The first comes from SNAP and the second is the result of calculations in Pari-Gp. The vectors and matrices are printed as they appear in SNAP. When one of them extends beyond one line, it is edited once copied into the template to add the line continuation character, "\," after each line before its end. The Pari-Gp data has been shortened to 40 decimal places from the calculated precision of 60 decimal places so as to fit on one line since in these examples, it has no effect on understanding the results.

1. *FIGURE* 8 *KNOT COMPLEMENT*

   The simplest is the figure 8 knot complement. We know [18] that this is complete hyperbolic already. However, only sufficiency conditions have been presented here, so it is nice to see that a manifold we know to be complete hyperbolic does not fail the test.

   QUANTITIES FROM SNAP

   $n = 2$

$h = k = 1$

$a =$

[0.50000000000000000000000000000+0.86602540378443864676372317075*I,

0.50000000000000000000000000000+0.86602540378443864676372317075*I]

$F =$[1, 0, 0, 1, 0]

$G =$[2, -1, -1, 2, 0; -2, 1, 1, -2, 0]

Pari-Gp CALCULATIONS

$|b| = 1.296666384352891444530724934775173278518E - 28$

Inverse Function Theorem data:

$\eta = 0.3140257650798779229370147391728000202907$

$\epsilon = 0.1570128825399389614685073695864000101453$

$\delta = 0.01308440687832824678904228079766698232949$

$\frac{\eta\delta}{4} = 0.001027210220145861215835361282642896699237$

Kantorovich data:

$L = 4.472135954999579392818347339211785668123$

$|f'(a)^{-1}|_{\text{sup}} = 1.592226038754547070932399593119376104348$

$|f'(a)^{-1}|_{\text{len}} = 1.632993161855452065464856049716587347937$

$\frac{1}{2|f'(a)^{-1}|^2_{\text{sup}}L} = 0.04410070808503045666350407221846082500302$

$\frac{1}{2|f'(a)^{-1}|^2_{\text{len}}L} = 0.04192627457812105680767200627679720162466$

2. $(3,2)$ *DEHN SURGERY: FIGURE* 8 *KNOT COMPLEMENT*

Dehn surgery on the figure 8 knot complement with coefficients outside the rectangle with vertices $(-4,-1)$, $(-4,\, 1)$, $(4,1)$ and $(4,-1)$ are known to be hyperbolic. Our test confirms this for $(3,2)$ Dehn surgery.

QUANTITIES FROM SNAP

$n = 2$

$h = 0$ and $k = 1$

$a =$

[0.4954656064075546882700742435+0.3184344025224255962108800064*I,

0.2384629932649871770466704016+0.3145299052214213676739562796*I]

$F =$[3, -4, 0, 11, 2]

$G =$[2, -1, -1, 2, 0; -2, 1, 1, -2, 0]

Pari-Gp CALCULATIONS

$|b| = 6.465667870212635404201807594225101849482E - 28$

Inverse Function Theorem data:

$\eta = 1.618205277801913438468680573543738992211$

$\epsilon = 0.8091026389009567192343402867718694961057$

$\delta = 0.004830066958223720619553909664170838579078$

$\frac{\eta\delta}{4} = 0.001954009960983564713909747682988867681415$

Kantorovich data:

$L = 44.60747497092832814628093161972439417944$

$|f'(a)^{-1}|_{\text{sup}} = 0.3089842845397057433721243439808921852965$

$|f'(a)^{-1}|_{\text{len}} = 0.3130058070195557125862651767333350941155$

$\frac{1}{2|f'(a)^{-1}|^2_{\text{sup}}L} = 0.1174058080091984317596820702542925455028$

$\frac{1}{2|f'(a)^{-1}|^2_{\text{len}}L} = 0.1144083109952044299550670223077376358121$

3. ***<u>WHITEHEAD LINK COMPLEMENT</u>***

The Whitehead link complement is known to be complete hyperbolic [12] also.

It is presented here.

QUANTITIES FROM SNAP

$n = 4$

$h = k = 2$

$a =$

[1.0000000000000000000000000000+1.0000000000000000000000000000*I,

0.50000000000000000000000000000+0.50000000000000000000000000000*I,

0.50000000000000000000000000000+0.50000000000000000000000000000*I,

0.50000000000000000000000000000+0.50000000000000000000000000000*I]

$F =$[1, 0, -1, 0, -1, 0, 1, 1, 0; 0, 0, 0, 1, 1, -1, 0, 0, 0]

$G =$

[1, 1, 1, 1, 1, -2, 0, 0, -1; 0, -1, -1, -1, -1, 1, 1, 1, 1;

-1, 1, 1, 1, 1, 0, -2, -2, -1; 0, -1, -1, -1, -1, 1, 1, 1, 1]

Pari-Gp CALCULATIONS

$|b| = 0.E - 67$

Inverse Function Theorem data:

$\eta = 0.34925247275491749971859609154542697014487$

$\epsilon = 0.087313118188729374929649022886356674253719$

$\delta = 0.0018190232955985286443676879767990988028 58$

$\frac{\eta\delta}{4} = 0.00015882459599664634171283647143871597551 76$

Kantorovich data:

$L = 11.247221879201992519034539756955659907 71$

$|f'(a)^{-1}|_{\sup} = 1.4316290907147484594274107284331672948 82$

$|f'(a)^{-1}|_{\text{len}} = 1.77218085984472815036945231783857224O764$

$\frac{1}{2|f'(a)^{-1}|_{\text{sup}}^2 L} = 0.02169020777494946413727384227646169243289$

$\frac{1}{2|f'(a)^{-1}|_{\text{len}}^2 L} = 0.01415496038127401571624506999479209159919$

4. <u>$(9872, 11111)$ *DEHN SURGERY: WHITEHEAD LINK COMPLEMENT*</u>

This example considers Dehn surgery on only one of the two cusps of the Whitehead link complement.

QUANTITIES FROM SNAP

$n = 4$

$h = 1$ and $k = 2$

$a =$

[0.99993437000738276495709924 30+1.00017053625772981772763 0077*I,

0.49991474365975085404436930 49+0.49996718440669707775832117 69*I,

0.50008526752982106519582439 37+0.50003280320702125426589811 40*I,

0.49991474365975085404436930 49+0.49996718440669707775832117 69*I]

$F =$

[20983, 0, -9872, 0, -9872, 11111, -1239, 20983, -2;

0, 0, 0, 1, 1, -1, 0, 0, 0]

$G =$

[1, 1, 1, 1, 1, -2, 0, 0, -1; 0, -1, -1, -1, -1, 1, 1, 1, 1;

-1, 1, 1, 1, 1, 0, -2, -2, -1; 0, -1, -1, -1, -1, 1, 1, 1, 1]

Pari-Gp CALCULATIONS

$|b| = 6.2905460436226495098540673660635089512 85E - 24$

Inverse Function Theorem data:

$\eta = 0.46996460925298636179567532106643085 22515$

$\epsilon = 0.11749115231324659044891883026660771 30628$

$\delta = 0.00000034999605907718972010363855631 92345713977$

$\frac{\eta\delta}{4} = 0.00000004112144028607414931402210141793751163981$

Kantorovich data:

$L = 56237.01131396100111291495604741250466464$

$|f'(a)^{-1}|_{\text{sup}} = 1.06390989907677347115761852905147130 8315$

$|f'(a)^{-1}|_{\text{len}} = 1.23541566132487349717522223681282373 5348$

$\frac{1}{2|f'(a)^{-1}|_{\text{sup}}^2 L} = 0.00000785485319329127816522549498105 3686965848$

$\frac{1}{2|f'(a)^{-1}|_{\text{len}}^2 L} = 0.00000582534387077831797653292041727 8552662252$

5. *SMALLLINK COMPLEMENT*

This is the smaller of two extremely large link complements. See figure 8. It has 32 tetrahedra and 4 cusps. These two links are used by Leininger [9] to construct other knots and links by cut and paste methods, and then looking at their covers. For any even integer $g > 0$, we eventually get from Smalllink a two component link whose complement in S^3 contains an embedded totally geodesic surface of genus g. The importance of Smalllink is that prior to this, such embedded surfaces could only be found in the complement of links with more than two components.

QUANTITIES FROM SNAP

$n = 32$

$h = k = 4$

$a =$

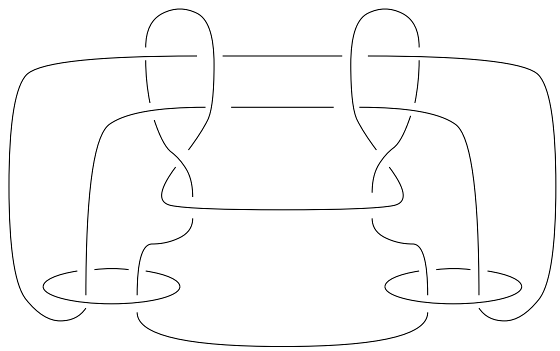

Figure 8: The Link Smalllink

[5.431680776271168985E-77+1.0431901497858949733789949 44*I,

0.4788708557877967957032308372+0.4995533597773714501527030266*I,

-4.471822153042346518E-77+0.9585980084313877504633692171*I,

0.5211291442122032042967691627+0.4995533597773714501527030266*I,

0.2929970420861826752219808548+1.473911044296957810855392169*I,

-0.4509782171525463654321193064+1.200765444220459728291241593*I,

1.000000000000000000000000000+0.9300613056344272435239940348*I,

0.4638110047891777790136229363+0.4986886369525889712130902195*I,

0.2371128008078259554449875702+0.6313317290357266968810549581*I,

0.3060049499572359024927254903+0.4485055715132523515850447879*I,

0.8585412796265143611585133046+1.027932770073775455116265474*I,

0.5000000000000000000000000000+0.4792990042156938752316846085*I,

0.4375240155821198504166057790+0.8813536566549109733830907053*I,

1.042258288424406408593538325+0.9991067195547429003054060533*I,

0.6696776343174901312972923995+0.7426519144895069642064793083*I,

0.4518888703362351094400929330+0.9102903934144040876554144906*I,

0.03927009472823897821842546946+1.571359648665194056162768058*I,

-8.46343996E-78+0.9585980084313877504633692171*I,

1.00000000000000000000000000+1.043190149785894973378994944*I,

0.02451089142372681728394034675+0.5982980953722294364585245396*I,

0.9621628947892310086730291057+0.3083453854492406606721071067*I,

0.7354295168083648566686302069+0.5515583107626382072967381105*I,

0.6213864977872760582396031709+0.4161571993484503024065288682*I,

0.6756917822944407062548472825+0.1978399260627268524593119332*I,

0.5213536432299720005859050458+1.346701507985612627940863123*I,

0.2659365860052524158474189000+0.5690611275237113909012011313*I,

0.8916797222785394793793396465+0.5330292860478110834980601119*I,

0.4489838724616496515202858898+0.4713823217067450930880172825*I,

0.5364433482241135276673629307+0.6234802797569418514001720639*I,

-0.4489884234808609710528328543+0.3884305318039174460267193001*I,

1.000000000000000000000000000+0.9585980084313877504633692171*I,

-8.49494342E-77+1.043190149785894973378994944*I]

$F =$

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1; 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, -1, -1, 1, -1, -1, 0, 1, 1, 1, 2, 0, 1, 0, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, -2, 0, 0, -1, 0, 0, 0, 0, 1, 1, -1, 1]

$G =$

[0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, -1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0; -1, 0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, -1, 0, 0, 0; 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1; 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0,

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0, 0, 2; 0, 0, 1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1,
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0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0,
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0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2; 0, 0, 0, 0, 0, -1, -1, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 1, 0, 0, 0, 0, 0,
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0,
0, 0, 0, 0, 0, 0, 0, -1; 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0,
0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1,
1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1; 0, 0, 0, 0, 0, 0, 0, 0,
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0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, -1, 0,
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0, 0, 0, 1, 0, 0, 0, 0, -1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 1, 0, 1, 0, 0, 0, -1, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, 0, 1, -1, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 1, 1, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1; 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, -1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0,
0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 1, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, -1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, 0, 1, 0, 0,
0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1; 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, 0, 0, 0, 0, 0, -1, 0, 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
1, 1, 0, 0, 0, -1, 1, 0, 0, 0, 1, 0, 0, 2; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 1, -1, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 0, -1]

Pari-Gp CALCULATIONS

$|b| = 2.890741236697218507543429035402903716418E-27$

Inverse Function Theorem data:

$\eta = 0.060880233627393671442074741143869224996 83$

$\epsilon = 0.0019025073008560522325648356607459132811 50$

$\delta = 0.0000026210833933198758532535434514252597974 29$

$\frac{\eta\delta}{4} = 0.000000039893042335548954691989521213442600682 23$

Kantorovich data:

$L = 38.469609270367684652002921675811783438 87$

$|f'(a)^{-1}|_{\text{sup}} = 8.21284627552775992508552565634205331691 5$

$|f'(a)^{-1}|_{\text{len}} = 10.32145710779244812406937753131330598443$

$\frac{1}{2|f'(a)^{-1}|^2_{\text{sup}}L} = 0.00019269251322399044236648498715666824282 36$

$\frac{1}{2|f'(a)^{-1}|^2_{\text{len}}L} = 0.00012200291428418181728451377112277231072 18$

6. *BIGLINK COMPLEMENT*

This example deals with the larger of the two links mentioned in the previous example. See figure 9. It has 57 tetrahedra and 11 cusps. For any $g > 2$, the complement of Biglink contains a closed totally geodesic surface of genus g, and there is an infinite sequence of Dehn filling coefficients $\{(p_{ij}, q_{ij})_{1\leq i\leq 10}\}_{j=1}^{\infty}$ such that for each j, the surface is also included in the Dehn filling of 10 out of the 11 cusps in the complement of Biglink in S^3. By cut and paste on Biglink, there is an infinite sequence of knots, $\{K_j\}_{j=1}^{\infty}$, where for each j, the complement of K_j in S^3 is homeomorphic to the cover of the corresponding Dehn filling described above. The principal curvature of the embedding of the surface into the complement of K_j converges to zero as $j \to \infty$. This gives evidence that

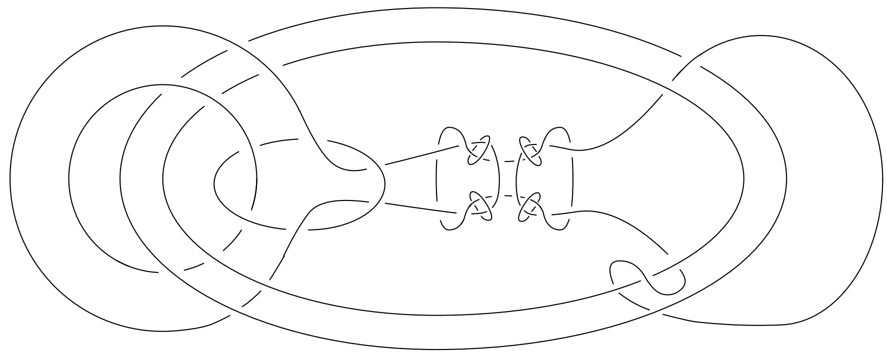

Figure 9: The Link Biglink

there is a counterexample to the conjecture of Menasco and Reid that there are no hyperbolic knots in S^3 which contain closed embedded totally geodesic surfaces in their complement. The calculations in Pari-Gp have been set to 125 significant digits, but as above, they are only displayed to 40 significant digits.

QUANTITIES FROM SNAP

$n = 57$

$h = k = 11$

$a =$

[1.158782270647143247050917636+0.9196847760114970597472726274*I,

0.3692744572717491962139324887+1.541520107870206798762555869*I,

0.4651382994027756058550693063+0.3345287563803869057830781538*I,

0.7418144117219689011103202648+0.6706052330258012594446576994*I,

0.2242167001072306964797384009+1.298688353401913664176971985*I,

0.3023974615050923737812872753+0.5536626255098309764278713932*I,

0.5830321410748256540594026280+1.083022604258112203184180379*I,

0.2519097114045880666087096754+0.5449470633051167964915168767*I,

0.5259845582416346104061851974+0.8805165307787365494680454513*I,

1.000000000000000000000000000+0.6260298578271997951529634803*I,

0.6395741954951861796362729751+0.4801239881029394577432337424*I,

1.000000000000000000000000000+0.9233417622756045666495433329*I,

-2.72517217E-137+1.083022604258112203184180379*I,

0.4602064199688765219662062469+0.4984139554509951427519747932*I,

0.2762905482350840528817033509+0.6901047959738613482364244899*I,

0.5000000000000000000000000000+0.4616708811378022833247716664*I,

0.3743007095348759849629913911+0.7800643549806902744851580640*I,

0.9602254248324689231620227991+0.7762843619303948855002015497*I,

0.5000000000000000000000000000+0.7952032238523349385477646774*I,

0.03977457516753107683797720087+0.7762843619303948855002015497*I,

-0.2791483909903723592725459503+0.9602479762058789154864674849*I,

1.000000000000000000000000000+0.7506931822526181950548573042*I,

0.7184355092002201289002039952+0.4497620796826256977049533749*I,

-6.117559714432124475E-135+0.9233417622756045666495433329*I,

0.5397935800311234780337937530+0.4984139554509951427519747932*I,

0.7914318917644015561752442055+0.6112573604450261180032331368*I,

0.6469534030923889064938529645+1.083022604258112203184180379*I,

0.7370077208791826947969074012+0.4402582653893682747340227256*I,

0.4547246175170458323155054225+1.332676274355746974479468977*I,

1.000000000000000000000000000+1.083022604258112203184180379*I,

0.4602064199688765219662062469+0.4984139554509951427519747932*I,

1.0000000000000000000000000+0.5973590953215259340130126106*I,

0.4065088426210201073063291103+0.6805100078073114320639594485*I,

0.5000000000000000000000000000+1.2220213099363675336560496380*I,

0.5259845582416346104061851974+0.8805165307787365494680454513*I,

0.2293345248982259555296819493+0.6721181753724961795982236043*I,

0.7390728466303382839959029562+0.8004325990940453144914948035*I,

0.6226807304139232560571957507+0.6743773062742306578386953216*I,

0.1266813969858828307645942983+0.6361697448790930107855080342*I,

0.5000000000000000000000000000+1.6320442768933665281294748570*I,

0.4795012707624852590078929240+0.8538624437590239378622942580*I,

0.4740154417583653895938148025+0.8805165307787365494680454513*I,

0.5000000000000000000000000000+0.8370174722641113808522003186*I,

0.5000000000000000000000000000+1.2220213099363675336560496380*I,

0.6904010911733571045007793186+0.8041449942382041126934846798*I,

0.1290927941390155494448398675+0.7477199877452115734618515298*I,

1.0000000000000000000000000+0.9233417622756045666495433329*I,

1.0795871600622469560675875060+0.9968279109019902855039495864*I,

0.5000000000000000000000000000+1.2488751431820246124227323430*I,

0.0795871600622469560675875060+0.9968279109019902855039495864*I,

-0.6050428150975738666770697550+0.7961929363532453752743663404*I,

1.0000000000000000000000000+1.0830226042581122031841803790*I,

0.5397935800311234780337937530+0.4984139554509951427519747932*I,

0.8270620386659764830999618379+2.1660452085162244063683607590*I,

0.9633734779922633804202076464+0.4587466042012591685781199239*I,

1.000000000000000000000000000+0.923341762275604566649543329*I,

1.000000000000000000000000000+1.083022604258112203184180379*I]

$F =$

[0, 0, 0, 0, -1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1; 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 2, 0, 0, 0, 0, -1, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0,
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0,
0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, -1, 0, 0, -1, 0, -1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, -1, 1, 0, 0, 0, 3; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, -1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0,
0, 0, 0, 0, -2; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, -1, 0, -1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, 1, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1; 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0,
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0, 0, 0, 0, 0, 1, 0, 0, -1, 0, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, -1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0,
1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, -1, 0, 0, 0, -1, 0, 0, -1;
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, -1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 1, -1, 0, 0, 0,

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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, -1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0,
0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, -2; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, -1, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, -1, -1; 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0,
0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, -1, 0, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, -1, 0, -1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1; 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0,
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1; 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1,
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 1, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, -1, 1, 0, 0, 0, 0, 0]

Pari-Gp CALCULATIONS

$|b| = 4.561638089524197282780803329134126087047E - 27$

Inverse Function Theorem data:

$\eta = 0.04411048735566768512251112007367997834516$

$\epsilon = 0.0007738681992222400898686161416435083920204$

$\delta = 0.000000785985350252520626292212285928054398293 4$

$\frac{\eta\delta}{4} = 0.000000008667549213513461966953652607177743018047$

Kantorovich data:

$L = 38.69990090618816909906768917402744007679$

$|f'(a)^{-1}|_{\text{sup}} = 11.33517288005560449348826624292354127942$

$|f'(a)^{-1}|_{\text{len}} = 14.58230300417961960848424155929725548467$

$\frac{1}{2|f'(a)^{-1}|_{\text{sup}}^2 L} = 0.0001005550427362150131100157033771382963935$

$\frac{1}{2|f'(a)^{-1}|_{\text{len}}^2 L} = 0.00006075862193098799511624683864295535271091$

4.3 Cusped Census

We can apply the tests of Theorem 1.2 to every manifold in the SnapPea cusped census. The results are found in the following theorem.

Theorem 4.1 *Every manifold in the SnapPea cusped census has a complete hyperbolic structure.*

A program was written in Perl [1] that issues commands to Snap to send tetrahedron shapes and filling equations for each manifold in the cusped census to an output file.

Then a Pari-Gp program reads the file, getting the needed data per manifold, and applies the template using this input. The program then prints out the results. The first run of this process determined that all but four manifolds, 5 168, 6 297, 7 1431 and 7 1927, have a complete hyperbolic structure. The program rejected these four because each one, upon triangulation by Snap, had one tetrahedron shape parameter with an imaginary component that was effectively zero. This was remedied by revising the original Perl program to process only these four manifolds, and including the "randomize" command to get a different, acceptable triangulation. The Pari-Gp program, also revised to process only these four manifolds, was then run using the second Perl output file. The result was a determination that they also have a complete hyperbolic structure.

These programs can be adapted to give other information, such as the maximum value that $normb$, the norm of b, assumes over all the manifolds in the cusped census. Call this $maxnormb$. Similarly, for each manifold, we can ascertain the largest of the three values that $normb$ is compared to, and then the minimum of these maximum comparison values over all the manifolds in this census. We do this because as long as $normb$ of a manifold is less than the largest of the three comparison values for that manifold, the manifold will have a complete hyperbolic structure. Then if $normb$ of a manifold in the census is less than the smallest of these maximum comparison values over the whole census, that manifold is guaranteed to have a complete hyperbolic structure. Call this minimum of maximum comparison values $minmaxvalue$. It tells us the precision needed to evaluate a manifold in the census. We have

$maxnormb = 1.71784409302201522318388858908732142516487589977 8$ E-26

$minmaxvalue = 0.000001478316776918140633809077361402607225498377777 47014$.

Thus, the approximate solution given by SnapPea, which is given to 10 digits but is computed to an internal precision of at least 15 significant digits, is sufficient for use as our $a_1, \ldots, a_n$. It is interesting to see that the largest $normb$ is considerably smaller than the smallest comparison value over the entire cusped census. The Perl programs and output files, as well as the Pari-Gp programs and log files, can be found at [10].

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