# Arbitrage and Duality in Nondominated Discrete-Time Models*

Bruno Bouchard† Marcel Nutz‡

February 9, 2014

**Abstract**

We consider a nondominated model of a discrete-time financial market where stocks are traded dynamically and options are available for static hedging. In a general measure-theoretic setting, we show that absence of arbitrage in a quasi-sure sense is equivalent to the existence of a suitable family of martingale measures. In the arbitrage-free case, we show that optimal superhedging strategies exist for general contingent claims, and that the minimal superhedging price is given by the supremum over the martingale measures. Moreover, we obtain a nondominated version of the Optional Decomposition Theorem.

*Keywords* Knightian uncertainty; Nondominated model; Fundamental Theorem of Asset Pricing; Martingale measure; Superhedging; Optional decomposition

*AMS 2010 Subject Classification* 60G42; 91B28; 93E20; 49L20

*The authors thank Mathias Beiglböck and Jan Obłój for their constructive comments.

†CEREMADE, Université Paris Dauphine and CREST-ENSAE, bouchard@ceremade.dauphine.fr. Research supported by ANR Liquirisk and Investissements d'Avenir (ANR-11-IDEX-0003/Labex Ecodec/ANR-11-LABX-0047).

‡Department of Mathematics, Columbia University, mnutz@math.columbia.edu. Research supported by NSF Grant DMS-1208985.

## 1 Introduction

We consider a discrete-time financial market where stocks and, possibly, options are available as hedging instruments. The market is governed by a set $\mathcal{P}$ of probability measures, not necessarily equivalent, whose role is to determine which events are negligible (polar) and which ones are relevant. We thus unify two approaches: On the one hand, the framework of model uncertainty, where each $P \in \mathcal{P}$ is seen as a possible model for the stocks and a robust analysis over the class $\mathcal{P}$ is performed. On the other hand, the model-free approach, where no attempt is made to model the stocks directly, but one sees the distribution of the stocks as partially described by the current prices of the traded options—an ill-posed inverse problem. Both approaches typically lead to a set $\mathcal{P}$ which is nondominated in the sense that there exists no reference probability measure with respect to which all $P \in \mathcal{P}$ are absolutely continuous.

We answer three fundamental questions of mathematical finance in this context. The first one is how to formulate a condition of market viability and relate it to the existence of consistent pricing mechanisms. The condition $\mathrm{NA}(\mathcal{P})$ stated below postulates that there is no trading strategy which yields a nonnegative gain that is strictly positive on a relevant event; that is, an event that has positive probability for at least one $P \in \mathcal{P}$. We obtain a version of the "First Fundamental Theorem" stating that $\mathrm{NA}(\mathcal{P})$ holds if and only if there exists a family $\mathcal{Q}$ of martingale measures which is equivalent to $\mathcal{P}$ in the sense that $\mathcal{Q}$ and $\mathcal{P}$ have the same polar sets. The next question is the one of superhedging: for a contingent claim $f$, what is the minimal price $\pi(f)$ that allows the seller to offset the risk of $f$ by trading appropriately in the given stocks and options? We show in the "Superhedging Theorem" below that $\pi(f) = \sup_{Q \in \mathcal{Q}} E_Q[f]$, which corresponds to the absence of a duality gap in the sense of convex analysis, and moreover that an optimal superhedging strategy exists. From these two theorems, it will follow that the precise range of arbitrage-free prices of $f$ is given by the open interval $(-\pi(-f), \pi(f))$ if $-\pi(-f) \neq \pi(f)$, and by the singleton $\pi(f)$ otherwise. This latter case occurs if and only if $f$ can be replicated (up to a polar set) by trading in the hedging instruments. Finally, we obtain a version of the "Second Fundamental Theorem," stating that the market is complete (i.e., all claims are replicable) if and only if $\mathcal{Q}$ is a singleton.

In addition to these main financial results, let us mention two probabilistic conclusions that arise from our study. The first one is a nondominated version of the Optional Decomposition Theorem, stating that a process which is a supermartingale under all of the (not necessarily equivalent) martingale measures $Q \in \mathcal{Q}$ can be written as the difference of a martingale transform and an increasing process. Second, our theory can be used to prove an intriguing conjecture of [1]; namely, that every martingale inequality (in finite discrete time) can be reduced to a deterministic inequality in Euclidean space; cf. Remark 1.4.

The main difficulty in our endeavor is that $\mathcal{P}$ can be nondominated, which leads to the failure of various tools of probability theory and functional analysis; in particular, the Dominated Convergence Theorem, Komlós' lemma (in the sense of [18]) and important parts of the theory of $L^p$ spaces. As a consequence, we have not been able to reach general results by using separation arguments in appropriate function spaces, which is the classical approach of mathematical finance in the case where $\mathcal{P}$ is a singleton. Instead, we proceed in a "local" fashion; our basic strategy is to first answer our questions in the case of a one-period market with deterministic initial data and then "glue" the solutions together to obtain the multi-period case. To solve the superhedging problem in the one-period case, we show in a first step that (the nontrivial inequality of) the duality holds for certain approximate martingale measures, and in a second step, we use a strengthened version of the First Fundamental Theorem to show that a suitable perturbation allows one to turn an approximate martingale measure into a true one. To perform the gluing, we tailor our theory such as to be compatible with the classical theory of analytic sets.

In the remainder of the Introduction, we detail the notation, state the main financial results and review the extant literature. In Section 2, we obtain the existence of optimal superhedging strategies; this part is formulated in a more general setting than the main results because we do not resort to a local analysis. In Section 3, we prove the First Fundamental Theorem and the Superhedging Theorem in the one-period case. In Section 4, we obtain the same theorems in the multi-period case, under the hypothesis that only stocks are traded. In Section 5, we add the options to the picture and prove the main results (as well as a slightly more precise form of them). Section 6 discusses the Optional Decomposition Theorem, and Appendix A collects some results of martingale theory that are used in the body of the paper.

## 1.1 Notation

Given a measurable space $(\Omega, \mathcal{A})$, we denote by $\mathfrak{P}(\Omega)$ the set of all probability measures on $\mathcal{A}$. If $\Omega$ is a topological space, $\mathcal{B}(\Omega)$ denotes its Borel $\sigma$-field and we always endow $\mathfrak{P}(\Omega)$ with the topology of weak convergence; in particular, $\mathfrak{P}(\Omega)$ is Polish whenever $\Omega$ is Polish. The *universal completion* of $\mathcal{A}$ is the $\sigma$-field $\cap_{P \in \mathfrak{P}(\Omega)} \mathcal{A}^P$, where $\mathcal{A}^P$ is the $P$-completion of $\mathcal{A}$. If $\Omega$ is Polish, $A \subseteq \Omega$ is *analytic* if it is the image of a Borel subset of another Polish space under a Borel-measurable mapping. A function $f : \Omega \to \overline{\mathbb{R}} := [-\infty, \infty]$ is *upper semianalytic* if the super-level set $\{f > c\}$ is analytic for all $c \in \mathbb{R}$. Any Borel set is analytic, and any analytic set is universally measurable (i.e., measurable for the universal completion of $\mathcal{B}(\Omega)$); similarly, any Borel function is upper semianalytic, and any upper semianalytic function is universally measurable. We refer to [6, Chapter 7] for these facts. Given $P \in \mathfrak{P}(\Omega)$, we define the $P$-expectation for *any* measurable function $f : \Omega \to \overline{\mathbb{R}}$ by

$$E_P[f] := E_P[f^+] - E_P[f^-], \quad \text{with the convention } \infty - \infty := -\infty. \tag{1.1}$$

The term *random variable* is reserved for measurable functions with values in $\mathbb{R}$. We shall often deal with a family $\mathcal{P} \subseteq \mathfrak{P}(\Omega)$ of measures. Then, a subset $A \subseteq \Omega$ is called *$\mathcal{P}$-polar* if $A \subseteq A'$ for some $A' \in \mathcal{A}$ satisfying $P(A') = 0$ for all $P \in \mathcal{P}$, and a property is said to hold *$\mathcal{P}$-quasi surely* or *$\mathcal{P}$-q.s.* if it holds outside a $\mathcal{P}$-polar set. (In accordance with this definition, any set is $\mathcal{P}$-polar in the trivial case $\mathcal{P} = \emptyset$.)

## 1.2 Main Results

Let $T \in \mathbb{N}$ be the time horizon and let $\Omega_1$ be a Polish space. For $t \in \{0, 1, \dots, T\}$, let $\Omega_t := \Omega_1^t$ be the $t$-fold Cartesian product, with the convention that $\Omega_0$ is a singleton. We denote by $\mathcal{F}_t$ the universal completion of $\mathcal{B}(\Omega_t)$ and write $(\Omega, \mathcal{F})$ for $(\Omega_T, \mathcal{F}_T)$. This will be our basic measurable space and we shall often see $(\Omega_t, \mathcal{F}_t)$ as a subspace of $(\Omega, \mathcal{F})$. For each $t \in \{0, 1, \dots, T-1\}$ and $\omega \in \Omega_t$, we are given a nonempty convex set $\mathcal{P}_t(\omega) \subseteq \mathfrak{P}(\Omega_1)$ of probability measures; we think of $\mathcal{P}_t(\omega)$ as the set of possible models for the $t$-th period, given state $\omega$ at time $t$. Intuitively, the role of $\mathcal{P}_t(\omega)$ is to determine which events at the "node" $(t, \omega)$ are negligible (polar) and which ones are relevant (have positive probability under at least one model). We assume that for each $t$,

$$\operatorname{graph}(\mathcal{P}_t) := \{(\omega, P) : \omega \in \Omega_t,\ P \in \mathcal{P}_t(\omega)\} \subseteq \Omega_t \times \mathfrak{P}(\Omega_t) \quad \text{is analytic.}$$

This ensures that $\mathcal{P}_t$ admits a universally measurable selector; that is, a universally measurable kernel $P_t : \Omega_t \to \mathfrak{P}(\Omega_1)$ such that $P_t(\omega) \in \mathcal{P}_t(\omega)$ for all $\omega \in \Omega_t$. If we are given such a kernel $P_t$ for each $t \in \{0, 1, \dots T-1\}$, we can define a probability $P$ on $\Omega$ by Fubini's theorem,

$$P(A) = \int_{\Omega_1} \cdots \int_{\Omega_1} \mathbf{1}_A(\omega_1, \dots, \omega_T) P_{T-1}(\omega_1, \dots, \omega_{T-1}; d\omega_T) \cdots P_0(d\omega_1), \quad A \in \Omega,$$

where we write $\omega = (\omega_1, \dots, \omega_T)$ for a generic element of $\Omega \equiv \Omega_1^T$. The above formula will be abbreviated as $P = P_0 \otimes P_1 \otimes \cdots \otimes P_{T-1}$ in the sequel. We can then introduce the set $\mathcal{P} \subseteq \mathfrak{P}(\Omega)$ of possible models for the multi-period market up to time $T$,

$$\mathcal{P} := \{P_0 \otimes P_1 \otimes \cdots \otimes P_{T-1} : P_t(\cdot) \in \mathcal{P}_t(\cdot),\ t = 0, 1, \dots, T-1\},$$

where, more precisely, each $P_t$ is a universally measurable selector of $\mathcal{P}_t$. Or equivalently, to state the same in reverse, $\mathcal{P}$ is the set of all $P \in \mathfrak{P}(\Omega)$ such that any decomposition $P = P_0 \otimes P_1 \otimes \cdots \otimes P_{T-1}$ into kernels $P_t$ satisfies $P_t(\cdot) \in \mathcal{P}_t(\cdot)$ up to a $(P_0 \otimes P_1 \otimes \cdots \otimes P_{t-1})$-nullset.

Let $d \in \mathbb{N}$ and let $S_t = (S_t^1, \dots, S_t^d) : \Omega_t \to \mathbb{R}^d$ be Borel-measurable for all $t \in \{0, 1, \dots, T\}$. We think of $S_t$ as the (discounted) prices of $d$ traded *stocks* at time $t$. Moreover, let $\mathcal{H}$ be the set of all predictable $\mathbb{R}^d$-valued processes, the *trading strategies*. Given $H \in \mathcal{H}$, the corresponding wealth process (from vanishing initial capital) is the discrete-time integral

$$H \bullet S = (H \bullet S_t)_{t \in \{0,1,\dots,T\}}, \quad H \bullet S_t = \sum_{u=1}^{t} H_u \Delta S_u, \tag{1.2}$$

where $\Delta S_u = S_u - S_{u-1}$ is the price increment and $H_u \Delta S_u$ is a shorthand for the inner product $\sum_{i=1}^d H_u^i \Delta S_u^i$ on $\mathbb{R}^d$.

Moreover, let $e \in \mathbb{N} \cup \{0\}$ and let $g = (g^1, \dots, g^e) : \Omega \to \mathbb{R}^d$ be Borel-measurable. Each $g^i$ is seen as a traded *option* which can be bought or sold at time $t = 0$ at the price $g_0^i$; without loss of generality, $g_0^i = 0$. Following [26], the options can be traded only statically (i.e., at $t = 0$), which accounts for the difference in liquidity and other market parameters compared to stocks. Given a vector $h \in \mathbb{R}^e$, the value of the corresponding option portfolio is then given by $hg = \sum_{i=1}^{e} h^i g^i$ and a pair $(H, h) \in \mathcal{H} \times \mathbb{R}^e$ is called a *semistatic hedging strategy*.

As in any model of mathematical finance, a no-arbitrage condition is needed for the model to be viable. The following formulation seems natural in our setup.

**Definition 1.1.** Condition $\mathrm{NA}(\mathcal{P})$ holds if for all $(H, h) \in \mathcal{H} \times \mathbb{R}^e$,

$$H \bullet S_T + hg \geq 0 \quad \mathcal{P}\text{-q.s.} \qquad \text{implies} \qquad H \bullet S_T + hg = 0 \quad \mathcal{P}\text{-q.s.}$$

We observe that $\mathrm{NA}(\mathcal{P})$ reduces to the classical condition "NA" in the case where $\mathcal{P}$ is a singleton. More generally, if $\mathrm{NA}(\{P\})$ holds for all $P \in \mathcal{P}$, then $\mathrm{NA}(\mathcal{P})$ holds trivially, but the converse is false.

The two extreme cases of our setting occur when $\mathcal{P}$ consists of all measures on $\Omega$ and when $\mathcal{P}$ is a singleton.

**Example 1.2.** (i) Let $\Omega = (\mathbb{R}^d)^T$ and let $S$ be the coordinate-mapping process; more precisely, $S_0 \in \mathbb{R}^d$ is fixed and $S_t(\omega) = \omega_t$ for $\omega = (\omega_1, \dots, \omega_T) \in (\mathbb{R}^d)^T$ and $t > 0$. Moreover, let $\mathcal{P}_t(\omega) = \mathfrak{P}(\mathbb{R}^d)$ for all $(t, \omega)$; then $\mathcal{P}$ is simply the collection of all probability measures on $\Omega$. We emphasize that a $\mathcal{P}$-q.s. inequality is in fact a pointwise inequality in this setting, as $\mathcal{P}$ contains all Dirac measures. Consider the case without options ($e = 0$); we show that $\mathrm{NA}(\mathcal{P})$ holds. Indeed, suppose there exists $H \in \mathcal{H}$ such that $H \bullet S_T(\omega) \geq 0$ for all $\omega \in \Omega$ and $H \bullet S_T(\bar{\omega}) > 0$ for some $\bar{\omega} \in \Omega$. Then, if we consider the smallest $t \in \{1, \dots, T\}$ such that $H \bullet S_t(\bar{\omega}) > 0$ and let $\bar{\omega}'_t := -\bar{\omega}_t + 2\bar{\omega}_{t-1}$, the path $\tilde{\omega} := (\bar{\omega}_1, \dots, \bar{\omega}_{t-1}, \bar{\omega}'_t, \dots, \bar{\omega}'_t)$ satisfies

$$H \bullet S_T(\tilde{\omega}) = H \bullet S_{t-1}(\bar{\omega}) - H_t(\bar{\omega})(\bar{\omega}_t - \bar{\omega}_{t-1}) < 0,$$

a contradiction.

We note that in this canonical setup, a contingent claim is necessarily a functional of $S$. We can generalize this setting by taking $\Omega = (\mathbb{R}^{d'})^T$ for some $d' > d$ and defining $S$ to be the first $d$ coordinates of the coordinate-mapping process. Then, the remaining $d' - d$ components play the role of nontradable assets and we can model claims that depend on additional risk factors.

(ii) Suppose we want to retrieve the classical case where we have a single measure $P$. Given $P \in \mathfrak{P}(\Omega)$, the existence of conditional probability distributions on Polish spaces [55, Theorem 1.1.6, p. 13] implies that there are Borel kernels $P_t : \Omega_t \to \mathfrak{P}(\Omega_1)$ such that $P = P_0 \otimes P_1 \otimes \cdots \otimes P_{T-1}$, where $P_0$ is the restriction $P|_{\Omega_1}$. If we then take $\mathcal{P}_t(\omega) := \{P_t(\omega)\}$, the Borel-measurability of $\omega \mapsto P_t(\omega)$ implies that $\operatorname{graph}(\mathcal{P}_t)$ is Borel for all $t$, and we have $\mathcal{P} = \{P\}$ as desired.

Similarly as in the classical theory, the absence of arbitrage will be related to the existence of linear pricing mechanisms. A probability $Q \in \mathfrak{P}(\Omega)$ is a *martingale measure* if $S$ is a $Q$-martingale in the filtration $(\mathcal{F}_t)$. We are interested in martingale measures $Q$ which are absolutely continuous with respect to $\mathcal{P}$ in the sense that

$Q(N) = 0$ whenever $N$ is $\mathcal{P}$-polar; it seems natural to denote this relation by $Q \ll \mathcal{P}$. Let us also introduce a stronger notion of absolute continuity: we write

$$Q \lll \mathcal{P} \quad \text{if} \quad \text{there exists } P \in \mathcal{P} \text{ such that } Q \ll P.$$

We shall use this stronger notion, mainly because it is better suited to the theory of analytic sets which will be used extensively later on. (However, a posteriori, one can easily check that our main results hold also with the weaker notion $Q \ll \mathcal{P}$.)

If $Q$ is a martingale measure, then $Q$ is consistent with the given prices of the traded options $g^i$ if $E_Q[g^i] = 0$ for $i = 1, \dots, e$. We thus define the set

$$\mathcal{Q} = \big\{Q \lll \mathcal{P} : Q \text{ is a martingale measure and } E_Q[g^i] = 0 \text{ for } i = 1, \dots, e\big\}. \tag{1.3}$$

Our first main result states that $\mathrm{NA}(\mathcal{P})$ is equivalent to $\mathcal{Q}$ being sufficiently rich, in the sense that any $\mathcal{Q}$-polar set is $\mathcal{P}$-polar. In this case, $\mathcal{Q}$ is equivalent to $\mathcal{P}$ in terms of polar sets, which yields an appropriate substitute for the classical notion of an equivalent martingale measure. In the body of this paper, we shall work with the seemingly stronger condition (ii) below; it turns out to be the most convenient formulation for our theory.

**First Fundamental Theorem.** *The following are equivalent:*

*(i)* $\mathrm{NA}(\mathcal{P})$ *holds.*

*(ii) For all* $P \in \mathcal{P}$ *there exists* $Q \in \mathcal{Q}$ *such that* $P \ll Q$.

*(ii') $\mathcal{P}$ and $\mathcal{Q}$ have the same polar sets.*

The following result, which is the main goal of our study, establishes the dual characterization for the superhedging price $\pi(f)$ of a contingent claim $f$ and the existence of an optimal superhedging strategy. We use the standard convention $\inf \emptyset = +\infty$.

**Superhedging Theorem.** *Let* $\mathrm{NA}(\mathcal{P})$ *hold and let* $f : \Omega \to \mathbb{R}$ *be upper semianalytic. Then the minimal superhedging price*

$$\pi(f) := \inf \big\{x \in \mathbb{R} : \exists\, (H, h) \in \mathcal{H} \times \mathbb{R}^e \textit{ such that } x + H \bullet S_T + hg \geq f \ \mathcal{P}\textit{-q.s.}\big\}$$

*satisfies*

$$\pi(f) = \sup_{Q \in \mathcal{Q}} E_Q[f] \in (-\infty, \infty]$$

*and there exist* $(H, h) \in \mathcal{H} \times \mathbb{R}^e$ *such that* $\pi(f) + H \bullet S_T + hg \geq f$ $\mathcal{P}$*-q.s.*

We remark that the supremum over $\mathcal{Q}$ is not attained in general. For instance, if $\mathcal{P} = \mathfrak{P}(\Omega)$ while $S$ is constant and $e = 0$, we have $\sup_{Q \in \mathcal{Q}} E_Q[f] = \sup_{\omega \in \Omega} f(\omega)$, which clearly need not be attained. On the other hand, the supremum is attained if, for instance, $f$ is a bounded continuous function and $\mathcal{Q}$ is weakly compact.

**Remark 1.3.** The Superhedging Theorem also yields the following Lagrange duality: if $\mathcal{Q}'$ is the set of *all* martingale measures $Q \lll \mathcal{P}$ (not necessarily consistent with the prices of the options $g^i$), then

$$\pi(f) = \inf_{h\in\mathbb{R}^e} \sup_{Q\in\mathcal{Q}'} E_Q[f - hg].$$

**Remark 1.4.** The Superhedging Theorem implies that every martingale inequality can be derived from a deterministic inequality, in the following sense. Consider the canonical setting of Example 1.2 and let $f$ be as in the theorem. Then $\mathcal{Q}$ is the set of all martingale laws and hence the inequality $E[f(M_1,\dots,M_T)] \leq a$ holds for every martingale $M$ starting at $M_0 = S_0$ if and only if $\sup_{Q\in\mathcal{Q}} E_Q[f] \leq a$. Thus, the theorem implies that any martingale inequality of the form $E[f(M_1,\dots,M_T)] \leq a$ can be deduced from a completely deterministic ("pathwise") inequality of the form $f \leq a + H \bullet S_T$ by taking expectations. This fact is studied in more detail in [5].

A random variable $f$ is called *replicable* if there exist $x \in \mathbb{R}$ and $(H,h) \in \mathcal{H}\times\mathbb{R}^e$ such that $x + H \bullet S_T + hg = f$ $\mathcal{P}$-q.s. We say that the market is *complete* if all Borel-measurable functions $f : \Omega \to \mathbb{R}$ are replicable. Similarly as in the classical theory, we have the following result.

**Second Fundamental Theorem.** *Let* $\mathrm{NA}(\mathcal{P})$ *hold and let* $f : \Omega \to \mathbb{R}$ *be upper semianalytic. The following are equivalent:*

*(i)* $f$ *is replicable.*

*(ii) The mapping* $Q \mapsto E_Q[f]$ *is constant (and finite) on* $\mathcal{Q}$*.*

*(iii) For all* $P \in \mathcal{P}$ *there exists* $Q \in \mathcal{Q}$ *such that* $P \ll Q$ *and* $E_Q[f] = \pi(f)$*.*

*Moreover, the market is complete if and only if* $\mathcal{Q}$ *is a singleton.*

Regarding (iii), we remark that the existence of *one* $Q \in \mathcal{Q}$ with $E_Q[f] = \pi(f)$ is not sufficient for $f$ to be replicable. Regarding (ii), we observe that if the market is complete, the unique element $Q \in \mathcal{Q}$ satisfies $E_Q[f] \in \mathbb{R}$ for all real-valued Borel functions $f$. This readily implies that $Q$ must be supported by a finite set.

## 1.3 Literature

Our main financial results are, of course, extensions of the classical case where $\mathcal{P}$ is a singleton; see, e.g., [19, 28] and the references therein. Our local way of proving the First Fundamental Theorem is certainly reminiscent of arguments for the Dalang–Morton–Willinger Theorem in [16] and [34]. To the best of our knowledge, our approach to the Superhedging Theorem is new even in the classical case.

In the setting where no options are traded and $\mathcal{P}$ is supposed to consist of martingale measures in the first place, there is a substantial literature on superhedging under "volatility uncertainty," where $S$ is a (continuous-time) process with continuous trajectories; see, among others, [8, 27, 40, 41, 42, 45, 47, 48, 49, 53, 52, 54]. These results were obtained by techniques which do not apply in the presence of jumps. Indeed, an important difference to our setting is that when $S$ is a continuous (hence predictable) process and each $P \in \mathcal{P}$ corresponds to a complete market, the optional decomposition coincides with the Doob–Meyer decomposition, which allows for a constructive proof of the Superhedging Theorem. In discrete time, a duality result (without existence) for a specific topological setup was obtained in [22]; see also [23] for the case of American and game options. A comparable result for the continuous case was given in [21]. The existence of optimal superhedging strategies in the discrete-time case was established in [44] for the one-dimensional case $d = 1$; its result is generalized by our Theorem 2.3 to the multidimensional case, with a much simpler proof using the arguments of [35]. An abstract duality result was also provided in [44], but it remained open whether there is a duality gap with respect to ($\sigma$-additive) martingale measures.

Roughly speaking, the model-free approach corresponds to the case where $\mathcal{P}$ consists of all probability measures on $\Omega$. Starting with [10, 12, 32], one branch of this literature is devoted to finding explicitly a model-independent semi-static hedging strategy for a specific claim, such as a barrier option, when a specific family of options can be traded, such as call options at specific maturities. We refer to the survey [33] which features an extensive list of references. Needless to say, explicit results are not within reach in our general setting.

A more related branch of this literature, starting with [9], studies when a given set (finite or not) of option prices is generated by a martingale measure and, more generally, is consistent with absence of arbitrage; see [1, 7, 11, 13, 14, 15, 17, 32]. The recent paper [1] is the closest to our study. In the setting of Example 1.2 with $d = 1$, it provides a version of the First Fundamental Theorem and the absence of a duality gap, possibly with infinitely many options, under certain continuity and compactness conditions. The arguments hinge on the weak compactness of a certain set of measures; to enforce the latter, it is assumed that options with specific growth properties can be traded.

Adapting the terminology of the previous papers to our context, our condition NA($\mathcal{P}$) excludes both model-independent and model-dependent arbitrage. This is crucial for the validity of the Second Fundamental Theorem in a general context. For instance, it is not hard to see that the theorem does not hold under the no-arbitrage condition of [1]; the reason is that, in a typical case, the superhedging price $\pi(f)$ will be arbitrage-free for some models $P \in \mathcal{P}$ and fail to be so for others. On a related note, let us stress that the equivalence in our version of the First Fundamental Theorem provides not just one martingale measure as in [1], but a family $\mathcal{Q}$ equivalent to $\mathcal{P}$. This concept seems to be new, and is important for our general version of the Superhedging Theorem. Finally, let us remark that in general, optimal superhedging strategies do not exist when infinitely many options are traded.

In the case where all options (or equivalently, all call options) at one or more maturities are available for static hedging, consistency with the option prices is equivalent to a constraint on the marginals of the martingale measures. In this case, the problem dual to superhedging is a so-called martingale optimal transport problem. Following work of [3] in discrete time and [30] in continuous time, this can be used to prove the absence of a duality gap under certain regularity conditions, and in specific cases, to characterize a dual optimizer and a superhedging strategy. See also [4, 24, 25, 31, 56] for recent developments.

The present study is also related to the theory of nonlinear expectations and we make use of ideas developed by [43, 46] in that context. In particular, the dynamic version of the superhedging price used in the proof of the Superhedging Theorem in Section 4 takes the form of a conditional sublinear expectation. In that section, we also make heavy use of the theory of analytic sets and related measurable selections; see [57, 58] for an extensive survey.

# 2 Existence of Optimal Superhedging Strategies

In this section, we obtain the existence of optimal superhedging strategies via an elementary closedness property with respect to pointwise convergence. The technical restrictions imposed on the structure of $\Omega, \mathcal{P}$ and $S$ in the Introduction are not necessary for this, and so we shall work in a more general setting: For the remainder of this section, $\mathcal{P}$ is any nonempty collection of probability measures on a general measurable space $(\Omega, \mathcal{F})$ with filtration $\{\mathcal{F}_t\}_{t\in\{0,1,\dots,T\}}$, and $S = (S_0, S_1, \dots S_T)$ is any collection of $\mathcal{F}$-measurable, $\mathbb{R}^d$-valued random variables $S_t$. To wit, the process $S$ is *not necessarily adapted*. The purpose of this unusual setup is the following.

**Remark 2.1.** Static hedging with options can be incorporated as a special case of a non-adapted stock: Suppose we want to model dynamic trading in (typically adapted) stocks $S^1, \dots, S^d$ as well as static trading in $\mathcal{F}$-measurable random variables (options) $g^1, \dots, g^e$ at initial prices $g_0^i, \dots, g_0^e$ ($\mathcal{F}$-measurable at least, but typically $\mathcal{F}_0$-measurable). Then, we can define the $d+e$ dimensional process $\tilde{S}$ by $\tilde{S}_t^i = S_t^i$ for $i \le d$ and

$$\tilde{S}_0^{i+d} = g_0^i \quad \text{and} \quad \tilde{S}_t^{i+d} = g^i, \quad t = 1, \dots, T$$

for $i = 1, \dots e$. Since $\Delta\tilde{S}_1^{i+d} = g^i - g_0^i$ and $\Delta\tilde{S}_t^{i+d} = 0$ for $t \ge 2$, dynamic trading in $\tilde{S}$ is then equivalent to dynamic trading in $S$ and static ($\mathcal{F}_0$-measurable) trading in $g^1, \dots, g^e$.

In view of the previous remark, we do not have to consider the case with options explicitly; i.e., we take $e = 0$. The only role of the filtration is to determine the set $\mathcal{H}$ of trading strategies; as in the Introduction, this will be the set of all predictable processes. (The arguments in this section could easily be extended to situations such as portfolio constraints.) For $H \in \mathcal{H}$, the wealth process $H \bullet S$ is defined as in (1.2), and the condition NA($\mathcal{P}$) says that $H \bullet S_T \ge 0$ $\mathcal{P}$-q.s. implies $H \bullet S_T = 0$ $\mathcal{P}$-q.s.

We write $\mathcal{L}_+^0$ for the set of all nonnegative random variables. The following result states that the cone $\mathcal{C}$ of all claims which can be superreplicated from initial capital $x = 0$ is closed under pointwise convergence.

**Theorem 2.2.** *Let $\mathcal{C} := \{H \bullet S_T : H \in \mathcal{H}\} - \mathcal{L}^0_+$. If* NA($\mathcal{P}$) *holds, then $\mathcal{C}$ is closed under $\mathcal{P}$-q.s. convergence; i.e., if $\{W^n\}_{n\geq 1} \subseteq \mathcal{C}$ and $W$ is a random variable such that $W^n \to W$ $\mathcal{P}$-q.s., then $W \in \mathcal{C}$.*

Before stating the proof, let us show how this theorem implies the existence of optimal superreplicating strategies.

**Theorem 2.3.** *Let* NA($\mathcal{P}$) *hold and let $f$ be a random variable. Then*

$$\pi(f) := \inf\big\{x \in \mathbb{R} : \exists\, H \in \mathcal{H} \text{ such that } x + H \bullet S_T \geq f\ \mathcal{P}\text{-q.s.}\big\} > -\infty$$

*and there exists $H \in \mathcal{H}$ such that $\pi(f) + H \bullet S_T \geq f$ $\mathcal{P}$-q.s.*

*Proof.* The claim is trivial if $\pi(f) = \infty$. Suppose that $\pi(f) = -\infty$. Then, for all $n \geq 1$, there exists $H^n \in \mathcal{H}$ such that $-n + H^n \bullet S_T \geq f$ $\mathcal{P}$-q.s. and hence

$$H^n \bullet S_T \geq f + n \geq (f + n) \wedge 1 \quad \mathcal{P}\text{-q.s.}$$

That is, $W^n := (f + n) \wedge 1 \in \mathcal{C}$ for all $n \geq 1$. Now Theorem 2.2 yields that $1 = \lim W^n$ is in $\mathcal{C}$, which clearly contradicts NA($\mathcal{P}$).

On the other hand, if $\pi(f)$ is finite, then $W^n := f - \pi(f) - 1/n \in \mathcal{C}$ for all $n \geq 1$ and thus $f - \pi(f) = \lim W^n \in \mathcal{C}$ by Theorem 2.2, which yields the existence of $H$. $\square$

*Proof of Theorem 2.2.* We follow quite closely the arguments of [35]; as observed in [36], they can be used even in the case where $S$ is not adapted. Let

$$W^n = H^n \bullet S_T - K^n$$

be a sequence in $\mathcal{C}$ which converges $\mathcal{P}$-q.s. to a random variable $W$; we need to show that $W = H \bullet S_T - K$ for some $H \in \mathcal{H}$ and $K \in \mathcal{L}^0_+$. We shall use an induction over the number of periods in the market. The claim is trivial when there are zero periods. Hence, we show the passage from $T-1$ to $T$ periods; more precisely, we shall assume that the claim is proved for any market with dates $\{1, 2, \dots, T\}$ and we deduce the case with dates $\{0, 1, \dots, T\}$.

For any real matrix $M$, let index($M$) be the number of rows in $M$ which vanish identically. Now let $\mathbb{H}_1$ be the random $(d \times \infty)$-matrix whose columns are given by the vectors $H^1_1, H^2_1, \dots$. Then index($\mathbb{H}_1$) is a random variable with values in $\{0, 1, \dots, d\}$. If index($\mathbb{H}_1$) $= d$ $\mathcal{P}$-q.s., we have $H^n_1 = 0$ for all $n$, so that setting $H_1 = 0$, we conclude immediately by the induction assumption. For the general case, we use another induction over $i = d, d-1, \dots, 0$; namely, we assume that the result is proved whenever index($\mathbb{H}_1$) $\geq i$ $\mathcal{P}$-q.s. and we show how to pass to $i - 1$.

Indeed, assume that index($\mathbb{H}_1$) $\geq i - 1 \in \{0, \dots, d-1\}$; we shall construct $H$ separately on finitely many sets forming a partition of $\Omega$. Consider first the set

$$\Omega_1 := \{\liminf |H^n_1| < \infty\} \in \mathcal{F}_0.$$

By a standard argument (e.g., [35, Lemma 2]), we can find $\mathcal{F}_0$-measurable random indices $n_k$ such that on $\Omega_1$, $H_1^{n_k}$ converges pointwise to a (finite) $\mathcal{F}_0$-measurable random vector $H_1$. As the sequence

$$\tilde{W}^k := W^{n_k} - H_1^{n_k}\Delta S_1 = \sum_{t=2}^{T} H_t^{n_k}\Delta S_t - K^{n_k}$$

converges to $W - H_1\Delta S_1 =: \tilde{W}$ $\mathcal{P}$-q.s. on $\Omega_1$, we can now apply the induction assumption to obtain $H_2, \dots, H_T$ and $K \geq 0$ such that

$$\tilde{W} = \sum_{t=2}^{T} H_t \Delta S_t - K$$

and therefore $W = H \bullet S_T - K$ on $\Omega_1$. It remains to construct $H$ on

$$\Omega_2 := \Omega_1^c = \{\liminf |H_1^n| = +\infty\}.$$

Let

$$G_1^n := \frac{H_1^n}{1 + |H_1^n|}.$$

As $|G_1^n| \leq 1$, there exist $\mathcal{F}_0$-measurable random indices $n_k$ such that $G_1^{n_k}$ converges pointwise to an $\mathcal{F}_0$-measurable random vector $G_1$, and clearly $|G_1| = 1$ on $\Omega_2$. Moreover, on $\Omega_2$, we have $W^{n_k}/(1 + |H_1^{n_k}|) \to 0$ and hence $-G_1\Delta S_1$ is the $\mathcal{P}$-q.s. limit of

$$\sum_{t=2}^{T} \frac{H_t^{n_k}}{1 + |H_1^{n_k}|}\Delta S_t - \frac{K^{n_k}}{1 + |H_1^{n_k}|}.$$

By the induction assumption, it follows that there exist $\tilde{H}_2, \dots, \tilde{H}_T$ such that $\sum_{t=2}^{T} \tilde{H}_t \Delta S_t \geq -G_1\Delta S_1$ on $\Omega_2 \in \mathcal{F}_0$. Therefore,

$$G_1\Delta S_1 + \sum_{t=2}^{T} \tilde{H}_t \Delta S_t = 0 \quad \text{on } \Omega_2, \tag{2.1}$$

since otherwise the trading strategy $(G_1, \tilde{H}_2, \dots, \tilde{H}_T)\mathbf{1}_{\Omega_2}$ would violate NA($\mathcal{P}$). As $|G_1| = 1$ on $\Omega_2$, we have that for every $\omega \in \Omega_2$, at least one component $G_1^j(\omega)$ of $G_1(\omega)$ is nonzero. Therefore,

$$\Lambda_1 := \Omega_2 \cap \{G_1^1 \neq 0\}, \quad \Lambda_j := (\Omega_2 \cap \{G_1^j \neq 0\}) \setminus (\Lambda_1 \cup \dots \cup \Lambda_{j-1}), \quad j = 2, \dots, d$$

defines an $\mathcal{F}_0$-measurable partition of $\Omega_2$. We then consider the vectors

$$\bar{H}_t^n := H_t^n - \sum_{j=1}^{d} \mathbf{1}_{\Lambda_j} \frac{H_1^{n,j}}{G_1^j}\big(G_1 \mathbf{1}_{\{t=1\}} + \tilde{H}_t \mathbf{1}_{\{t \geq 2\}}\big),\ t = 1, \dots, T.$$

Note that $\bar{H}^n \bullet S_T = H^n \bullet S_T$ by (2.1). Hence, we still have $W = \bar{H}^n \bullet S_T - K^n$. However, on $\Omega_2$, the resulting matrix $\bar{\mathbb{H}}_1$ now has index$(\bar{\mathbb{H}}_1) \geq i$ since we have created an additional vanishing row: the $j$-th component of $\bar{H}_1^n$ vanishes on $\Lambda_j$ by construction, while the $j$-th row of $\mathbb{H}_1$ cannot vanish on $\Lambda_j \subseteq \{G_1^j \neq 0\}$ by the definition of $G_1$. We can now apply the induction hypothesis for indices greater or equal to $i$ to obtain $H$ on $\Omega_2$. Recalling that $\Omega = \Omega_1 \cup \Omega_2$, we have shown that there exist $H \in \mathcal{H}$ and $K \geq 0$ such that $W = H \bullet S_T - K$. $\square$

# 3 The One-Period Case

In this section, we prove the First Fundamental Theorem and the Superhedging Theorem in the one-period case; these results will serve as building blocks for the multi-period case. We consider an arbitrary measurable space $(\Omega, \mathcal{F})$ with a filtration $(\mathcal{F}_0, \mathcal{F}_1)$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and a nonempty convex set $\mathcal{P} \subseteq \mathfrak{P}(\Omega)$. The stock price process is given by a deterministic vector $S_0 \in \mathbb{R}^d$ and an $\mathcal{F}_1$-measurable, $\mathbb{R}^d$-valued random vector $S_1$. We write $\Delta S$ for $S_1 - S_0$ and note that $Q$ is a martingale measure simply if $E_Q[\Delta S] = 0$; recall the convention (1.1). Moreover, we have $\mathcal{H} = \mathbb{R}^d$. We do not consider hedging with options explicitly (i.e., we again take $e = 0$) as there is no difference between options and stocks in the one-period case. Thus, we have $\mathcal{Q} = \{Q \in \mathfrak{P}(\Omega) : Q \lll \mathcal{P}, E_Q[\Delta S] = 0\}$.

## 3.1 First Fundamental Theorem

In the following version of the Fundamental Theorem, we omit the equivalent condition (ii') stated in the Introduction; it is not essential and we shall get back to it only in Remark 5.2.

**Theorem 3.1.** *The following are equivalent:*

*(i)* $\mathrm{NA}(\mathcal{P})$ *holds.*

*(ii) For all* $P \in \mathcal{P}$ *there exists* $Q \in \mathcal{Q}$ *such that* $P \ll Q$.

For the proof, we first recall the following well-known fact; cf. [20, Theorem VII.57, p. 246].

**Lemma 3.2.** *Let* $\{g_n\}_{n \geq 1}$ *be a sequence of random variables and let* $P \in \mathfrak{P}(\Omega)$. *There exists a probability measure* $R \sim P$ *with bounded density* $dR/dP$ *such that* $E_R[|g_n|] < \infty$ *for all* $n$.

The following lemma is a strengthening of the nontrivial implication in Theorem 3.1. Rather than just showing the existence of a martingale measure (i.e., $E_R[\Delta S] = 0$ for some $R \in \mathfrak{P}(\Omega)$), we show that the vectors $E_R[\Delta S]$ fill a relative neighborhood of the origin; this property will be of key importance in the proof of the duality. We write $\operatorname{ri} A$ for the relative interior of a set $A$.

**Lemma 3.3** (Fundamental Lemma). *Let* $\mathrm{NA}(\mathcal{P})$ *hold and let* $f$ *be a random variable. Then*

$$0 \in \operatorname{ri} \big\{ E_R[\Delta S] : R \in \mathfrak{P}(\Omega),\, R \lll \mathcal{P},\, E_R[|\Delta S| + |f|] < \infty \big\} \subseteq \mathbb{R}^d.$$

*Similarly, given $P \in \mathcal{P}$, we also have*

$$0 \in \operatorname{ri}\big\{E_R[\Delta S] : R \in \mathfrak{P}(\Omega),\, P \ll R \lll \mathcal{P},\, E_R[|\Delta S| + |f|] < \infty\big\} \subseteq \mathbb{R}^d.$$

*Proof.* We show only the second claim; the first one can be obtained by omitting the lower bound $P$ in the subsequent argument. We fix $P$ and $f$; moreover, we set $\mathcal{I}_k := \{1, \dots, d\}^k$ for $k = 1, \dots, d$ and

$$\Theta := \{R \in \mathfrak{P}(\Omega) : P \ll R \lll \mathcal{P},\, E_R[|\Delta S| + |f|] < \infty\}.$$

Note that $\Theta \neq \emptyset$ as a consequence of Lemma 3.2. Given $I \in \mathcal{I}_k$, we denote

$$\Gamma_I := \{E_R[(\Delta S^i)_{i \in I}] : R \in \Theta\} \subseteq \mathbb{R}^k;$$

then our claim is that $0 \in \operatorname{ri} \Gamma_I$ for $I = (1, \dots, d) \in \mathcal{I}_d$. It is convenient to show more generally that $0 \in \operatorname{ri} \Gamma_I$ for all $I \in \mathcal{I}_k$ and all $k = 1, \dots, d$. We proceed by induction.

Consider first $k = 1$ and $I \in \mathcal{I}_k$; then $I$ consists of a single number $i \in \{1, \dots, d\}$. If $\Gamma_I = \{0\}$, the result holds trivially, so we suppose that there exists $R \in \Theta$ such that $E_R[\Delta S^i] \neq 0$. We focus on the case $E_R[\Delta S^i] > 0$; the reverse case is similar. Then, $\mathrm{NA}(\mathcal{P})$ implies that $A := \{\Delta S^i < 0\}$ satisfies $R_1(A) > 0$ for some $R_1 \in \mathcal{P}$. By replacing $R_1$ with $R_2 := (R_1 + P)/2$ we also have that $R_2 \gg P$, and finally Lemma 3.2 allows to replace $R_2$ with an equivalent probability $R_3$ such that $E_{R_3}[|\Delta S| + |f|] < \infty$; as a result, we have found $R_3 \in \Theta$ satisfying $E_{R_3}[\mathbf{1}_A \Delta S^i] < 0$. But then $R' \sim R_3 \gg P$ defined by

$$\frac{dR'}{dR_3} = \frac{\mathbf{1}_A + \varepsilon}{E_{R_3}[\mathbf{1}_A + \varepsilon]} \tag{3.1}$$

satisfies $R' \in \Theta$ and $E_{R'}[\Delta S^i] < 0$ for $\varepsilon > 0$ chosen small enough. Now set

$$R_\lambda := \lambda R + (1 - \lambda) R' \in \Theta$$

for each $\lambda \in (0, 1)$; then

$$0 \in \{E_{R_\lambda}[\Delta S^i] : \lambda \in (0,1)\} \subseteq \operatorname{ri}\{E_R[\Delta S^i] : R \in \Theta\},$$

which was the claim for $k = 1$.

Let $1 < k \leq d$ be such that $0 \in \operatorname{ri} \Gamma_I$ for all $I \in \mathcal{I}_{k-1}$; we show hat $0 \in \operatorname{ri} \Gamma_I$ for all $I \in \mathcal{I}_k$. Suppose that there exists $I = (i_1, \dots, i_k) \in \mathcal{I}_k$ such that $0 \notin \operatorname{ri} \Gamma_I$. Then, the convex set $\Gamma_I$ can be separated from the origin; that is, we can find $y = (y^1, \dots, y^k) \in \mathbb{R}^k$ such that $|y| = 1$ and

$$0 \leq \inf\bigg\{E_R\bigg[\sum_{j=1}^k y^j \Delta S^{i_j}\bigg] : R \in \Theta\bigg\}.$$

Using a similar argument as before (3.1), this implies that $\sum_{j=1}^k y^j \Delta S^{i_j} \geq 0$ $\mathcal{P}$-q.s., and thus $\sum_{j=1}^k y^j \Delta S^{i_j} = 0$ $\mathcal{P}$-q.s. by NA($\mathcal{P}$). As $|y| = 1$, there exists $1 \leq l \leq k$ such that $y^l \neq 0$, and we obtain that

$$\Delta S^l = -\sum_{j=1}^k \delta_{j \neq l} (y^j/y^l) \Delta S^{i_j} \quad \mathcal{P}\text{-q.s.}$$

Using the definition of the relative interior, the assumption that $0 \notin \operatorname{ri} \Gamma_I$ then implies that $0 \notin \operatorname{ri} \Gamma_{I'}$, where $I' \in \mathcal{I}_{k-1}$ is the vector obtained from $I$ by deleting the $l$-th entry. This contradicts our induction hypothesis. □

*Proof of Theorem 3.1.* (i) implies (ii): This is a special case of the Lemma 3.3, applied with $f \equiv 0$.

(ii) implies (i): Let (ii) hold and let $H \in \mathbb{R}^d$ be such that $H \Delta S \geq 0$ $\mathcal{P}$-q.s. Suppose that there exists $P \in \mathcal{P}$ such that $P\{H\Delta S > 0\} > 0$. By (ii), there exists a martingale measure $Q$ such that $P \ll Q \lll \mathcal{P}$. Thus $Q\{H\Delta S > 0\} > 0$, contradicting that $E_Q[H\Delta S] = 0$. □

## 3.2 Superhedging Theorem

We can now establish the Superhedging Theorem in the one-period case. Recall that the convention (1.1) is in force.

**Theorem 3.4.** *Let* NA($\mathcal{P}$) *hold and let $f$ be a random variable. Then*

$$\sup_{Q \in \mathcal{Q}} E_Q[f] = \pi(f) := \inf \big\{ x \in \mathbb{R} : \exists\, H \in \mathbb{R}^d \text{ such that } x + H\Delta S \geq f \ \mathcal{P}\text{-q.s.} \big\}. \tag{3.2}$$

*Moreover, $\pi(f) > -\infty$ and there exists $H \in \mathbb{R}^d$ such that $\pi(f) + H\Delta S \geq f$ $\mathcal{P}$-q.s.*

The last statement is a consequence of Theorem 2.3. For the proof of (3.2), the inequality "$\geq$" is the nontrivial one; that is, we need to find $Q_n \in \mathcal{Q}$ such that $E_{Q_n}[f] \to \pi(f)$. Our construction proceeds in two steps. In the subsequent lemma, we find "approximate" martingale measures $R_n$ such that $E_{R_n}[f] \to \pi(f)$; in its proof, it is important to relax the martingale property as this allows us to use arbitrary measure changes. In the second step, we replace $R_n$ by true martingale measures, on the strength of the Fundamental Lemma: it implies that if $R$ is any probability with $E_R[\Delta S]$ close to the origin, then there exists a perturbation of $R$ which is a martingale measure.

**Lemma 3.5.** *Let* NA($\mathcal{P}$) *hold and let $f$ be a random variable with $\pi(f) = 0$. There exist probabilities $R_n \lll \mathcal{P}$, $n \geq 1$ such that*

$$E_{R_n}[\Delta S] \to 0 \quad \textit{and} \quad E_{R_n}[f] \to 0.$$

*Proof.* It follows from Lemma 3.2 that the set

$$\Theta := \{R \in \mathfrak{P}(\Omega) : R \lll \mathcal{P},\, E_R[|\Delta S| + |f|] < \infty\}$$

is nonempty. Introduce the set

$$\Gamma := \big\{E_R[(\Delta S, f)] : P \in \Theta\big\} \subseteq \mathbb{R}^{d+1};$$

then our claim is equivalent to $0 \in \overline{\Gamma}$, where $\overline{\Gamma}$ denotes the closure of $\Gamma$ in $\mathbb{R}^{d+1}$. Suppose for contradiction that $0 \notin \overline{\Gamma}$, and note that $\Gamma$ is convex because $\mathcal{P}$ is convex. Thus, $\overline{\Gamma}$ can be separated strictly from the origin; that is, there exist $(y, z) \in \mathbb{R}^d \times \mathbb{R}$ with $|(y, z)| = 1$ and $\alpha > 0$ such that

$$0 < \alpha = \inf_{R \in \Theta} E_R[y\Delta S + zf].$$

Using again a similar argument as before (3.1), this implies that

$$0 < \alpha \leq y\Delta S + zf \quad \mathcal{P}\text{-q.s.} \tag{3.3}$$

Suppose that $z < 0$; then this yields that

$$f \leq |z^{-1}|y\Delta S - |z^{-1}|\alpha \quad \mathcal{P}\text{-q.s.},$$

which implies that $\pi(f) \leq -|z^{-1}|\alpha < 0$ and thus contradicts the assumption that $\pi(f) = 0$. Hence, we must have $0 \leq z \leq 1$. But as $\pi(zf) = z\pi(f) = 0 < \alpha/2$, there exists $H \in \mathbb{R}^d$ such that $\alpha/2 + H\Delta S \geq zf$ $\mathcal{P}$-q.s., and then (3.3) yields

$$0 < \alpha/2 \leq (y + H)\Delta S \quad \mathcal{P}\text{-q.s.},$$

which contradicts $\mathrm{NA}(\mathcal{P})$. This completes the proof. □

**Lemma 3.6.** *Let* $\mathrm{NA}(\mathcal{P})$ *hold, let* $f$ *be a random variable, and let* $R \in \mathfrak{P}(\Omega)$ *be such that* $R \lll \mathcal{P}$ *and* $E_R[|\Delta S| + |f|] < \infty$*. Then, there exists* $Q \in \mathcal{Q}$ *such that* $E_Q[|f|] < \infty$ *and*

$$|E_Q[f] - E_R[f]| \leq c(1 + |E_R[f]|)|E_R[\Delta S]|,$$

*where* $c > 0$ *is a constant independent of* $R$ *and* $Q$*.*

*Proof.* Let $\Theta = \{R' \in \mathfrak{P}(\Omega) : R' \lll \mathcal{P},\, E_{R'}[|\Delta S| + |f|] < \infty\}$ and

$$\Gamma = \{E_{R'}[\Delta S] : R' \in \Theta\}.$$

If $\Gamma = \{0\}$, then $R \in \Theta$ is itself a martingale measure and we are done. So let us assume that the vector space $\operatorname{span}\Gamma \subseteq \mathbb{R}^d$ has dimension $k > 0$ and let $e_1, \dots, e_k$ be an orthonormal basis. Lemma 3.3 shows that $0 \in \operatorname{ri}\Gamma$; hence, we can find $P_i^{\pm} \in \Theta$ and $\alpha_i^{\pm} > 0$ such that

$$\alpha_i^{\pm} E_{P_i^{\pm}}[\Delta S] = \pm e_i, \quad 1 \leq i \leq k.$$

Note also that $P_i^\pm, \alpha_i^\pm$ do not depend on $R$.

Let $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ be such that $-E_R[\Delta S] = \sum_{i=1}^k \lambda_i e_i$. Then, we have $|\lambda| = |E_R[\Delta S]|$ and

$$-E_R[\Delta S] = \int \Delta S \, d\mu \quad \text{for} \quad \mu := \sum_{i=1}^k \lambda_i^+ \alpha_i^+ P_i^+ + \lambda_i^- \alpha_i^- P_i^-,$$

where $\lambda_i^+$ and $\lambda_i^-$ denote the positive and the negative part of $\lambda_i$. Define the probability $Q$ by

$$Q = \frac{R + \mu}{1 + \mu(\Omega)};$$

then $R \ll Q \lll \mathcal{P}$ and $E_Q[\Delta S] = 0$ by construction. Moreover,

$$\begin{aligned} |E_Q[f] - E_R[f]| &= \left| \frac{1}{1 + \mu(\Omega)} \int f \, d\mu - \frac{\mu(\Omega)}{1 + \mu(\Omega)} E_R[f] \right| \\ &\leq \left| \int f \, d\mu \right| + \mu(\Omega) \, |E_R[f]| \\ &\leq c|\lambda|(1 + |E_R[f]|), \end{aligned}$$

where $c$ is a constant depending only on $\alpha_i^\pm$ and $E_{P_i^\pm}[f]$. It remains to recall that $|\lambda| = |E_R[\Delta S]|$. $\square$

*Proof of Theorem 3.4.* The last claim holds by Theorem 2.3, so $\pi(f) > -\infty$. Let us first assume that $f$ is bounded from above; then $\pi(f) < \infty$, and by a translation we may even suppose that $\pi(f) = 0$. By Theorem 3.1, the set $\mathcal{Q}$ of martingale measures is nonempty; moreover, $E_Q[f] \leq \pi(f) = 0$ for all $Q \in \mathcal{Q}$ by Lemma A.2. Thus, we only need to find a sequence $Q_n \in \mathcal{Q}$ such that $E_{Q_n}[f] \to 0$. Indeed, Lemma 3.5 yields a sequence $R_n \lll \mathcal{P}$ such that $E_{R_n}[\Delta S] \to 0$ and $E_{R_n}[f] \to 0$. Applying Lemma 3.6 to each $R_n$, we obtain a sequence $Q_n \in \mathcal{Q}$ such that $E_{Q_n}[|f|] < \infty$ and

$$|E_{Q_n}[f] - E_{R_n}[f]| \leq c(1 + |E_{R_n}[f]|)|E_{R_n}[\Delta S]| \to 0;$$

as a result, we have $E_{Q_n}[f] \to 0$ as desired.

It remains to discuss the case where $f$ is not bounded from above. By the previous argument, we have

$$\sup_{Q \in \mathcal{Q}} E_Q[f \wedge n] = \pi(f \wedge n), \quad n \in \mathbb{N}; \tag{3.4}$$

we pass to the limit on both sides. Indeed, on the one hand, we have

$$\sup_{Q \in \mathcal{Q}} E_Q[f \wedge n] \nearrow \sup_{Q \in \mathcal{Q}} E_Q[f]$$

by the monotone convergence theorem (applied to all $Q$ such that $E_Q[f^-] < \infty$). On the other hand, it also holds that

$$\pi(f \wedge n) \nearrow \pi(f),$$

because if $\alpha := \sup_n \pi(f \wedge n)$, then $(f \wedge n) - \alpha \in \mathcal{C}$ for all $n$ and thus $f - \alpha \in \mathcal{C}$ by Theorem 2.2, and in particular $\pi(f) \leq \alpha$. $\square$

# 4 The Multi-Period Case Without Options

In this section, we establish the First Fundamental Theorem and the Superhedging Theorem in the market with $T$ periods, in the case where only stocks are traded.

## 4.1 Preliminaries on Quasi-Sure Supports

We first fix some terminology. Let $(\Omega, \mathcal{F})$ be any measurable space and let $Y$ be a topological space. A mapping $\Psi$ from $\Omega$ into the power set of $Y$ will be denoted by $\Psi : \Omega \twoheadrightarrow Y$ and called a random set or a set-valued mapping. We say that $\Psi$ is measurable (in the sense of set-valued mappings) if

$$\{\omega \in \Omega : \Psi(\omega) \cap A \neq \emptyset\} \in \mathcal{F} \quad \text{for all closed } A \subseteq Y. \tag{4.1}$$

It is called closed-valued if $\Psi(\omega) \subseteq Y$ is closed for all $\omega \in \Omega$.

**Remark 4.1.** The mapping $\Psi$ is called weakly measurable if

$$\{\omega \in \Omega : \Psi(\omega) \cap O \neq \emptyset\} \in \mathcal{F} \quad \text{for all open } O \subseteq Y. \tag{4.2}$$

This condition is indeed weaker than (4.1), whenever $Y$ is metrizable [2, Lemma 18.2, p. 593]. If $Y = \mathbb{R}^d$, then (4.2) is equivalent to (4.1); cf. [50, Proposition 1A].

Another useful notion of measurability refers to the graph of $\Psi$, defined as

$$\operatorname{graph}(\Psi) = \{(\omega, y) : \omega \in \Omega,\ y \in \Psi(\omega)\} \subseteq \Omega \times Y.$$

In particular, if $\Omega$ and $Y$ are Polish spaces and $\operatorname{graph}(\Psi) \subseteq \Omega \times Y$ is analytic, then $\Psi$ admits a universally measurable selector $\psi$ on the (universally measurable) set $\{\Psi \neq \emptyset\} \subseteq \Omega$; that is, $\psi : \{\Psi \neq \emptyset\} \to Y$ satisfies $\psi(\omega) \in \Psi(\omega)$ for all $\omega$ such that $\Psi(\omega) \neq \emptyset$.

**Lemma 4.2.** *Given a nonempty family $\mathcal{R}$ of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, let*

$$\operatorname{supp}(\mathcal{R}) := \bigcap \big\{A \subseteq \mathbb{R}^d \text{ closed: } R(A) = 1 \text{ for all } R \in \mathcal{R}\big\} \subseteq \mathbb{R}^d.$$

*Then $\operatorname{supp}(\mathcal{R})$ is the smallest closed set $A \subseteq \mathbb{R}^d$ such that $R(A) = 1$ for all $R \in \mathcal{R}$. In particular, if $\mathcal{P} \subseteq \mathfrak{P}(\Omega)$, $X : \Omega \to \mathbb{R}^d$ is measurable, and $\mathcal{R} = \{P \circ X^{-1} : P \in \mathcal{P}\}$ is the associated family of laws, then*

$$\operatorname{supp}_{\mathcal{P}}(X) := \operatorname{supp}(\mathcal{R})$$

*is the smallest closed set $A \subseteq \mathbb{R}^d$ such that $P\{X \in A\} = 1$ for all $P \in \mathcal{P}$.*

*Proof.* Let $\{B_n\}_{n \geq 1}$ be a countable basis of the topology of $\mathbb{R}^d$ and let $A_n = B_n^c$. Since the intersection

$$\operatorname{supp}(\mathcal{R}) = \bigcap \{A_n : R(A_n) = 1 \text{ for all } R \in \mathcal{R}\}$$

is countable, we see that $R(\operatorname{supp}(\mathcal{R})) = 1$ for all $R \in \mathcal{R}$. The rest is clear. $\square$

**Lemma 4.3.** *Let $\Omega, \Omega_1$ be Polish spaces, let $\mathcal{P} : \Omega \twoheadrightarrow \mathfrak{P}(\Omega_1)$ have nonempty values and analytic graph, and let $X : \Omega \times \Omega_1 \to \mathbb{R}^d$ be Borel. Then the random set $\Lambda : \Omega \twoheadrightarrow \mathbb{R}^d$,*

$$\Lambda(\omega) := \operatorname{supp}_{\mathcal{P}(\omega)}(X(\omega, \cdot)) \subseteq \mathbb{R}^d, \quad \omega \in \Omega$$

*is closed-valued and universally measurable. Moreover, its polar cone $\Lambda^\circ : \Omega \twoheadrightarrow \mathbb{R}^d$,*

$$\Lambda^\circ(\omega) := \{y \in \mathbb{R}^d : \, yv \geq 0 \textit{ for all } v \in \Lambda(\omega)\}, \quad \omega \in \Omega$$

*is nonempty-closed-valued and universally measurable, and it satisfies*

$$\Lambda^\circ(\omega) = \{y \in \mathbb{R}^d : \, yX(\omega, \cdot) \geq 0 \; \mathcal{P}(\omega)\text{-}q.s.\}, \quad \omega \in \Omega. \tag{4.3}$$

*Proof.* Consider the mapping $\ell$ which associates to $\omega \in \Omega$ and $P \in \mathfrak{P}(\Omega_1)$ the law of $X(\omega, \cdot)$ under $P$,

$$\ell : \Omega \times \mathfrak{P}(\Omega_1) \to \mathfrak{P}(\mathbb{R}^d), \quad \ell(\omega, P) := P \circ X(\omega, \cdot)^{-1}.$$

If $X$ is continuous and bounded, it is elementary to check that $\ell$ is separately continuous (using the weak convergence on $\mathfrak{P}(\Omega_1)$, as always) and thus Borel. A monotone class argument then shows that $\ell$ is Borel whenever $X$ is Borel. Consider also the random set consisting of the laws of $X(\omega, \cdot)$,

$$\mathcal{R} : \Omega \twoheadrightarrow \mathfrak{P}(\mathbb{R}^d), \quad \mathcal{R}(\omega) := \ell(\omega, \mathcal{P}(\omega)) \equiv \{P \circ X(\omega, \cdot)^{-1} : \, P \in \mathcal{P}(\omega)\}.$$

Its graph is the image of graph($\mathcal{P}$) under the Borel mapping $(\omega, P) \mapsto (\omega, \ell(\omega, P))$; in particular, graph($\mathcal{R}$) is again analytic by [6, Proposition 7.40, p. 165]. Now let $O \subseteq \mathbb{R}^d$ be an open set; then

$$\begin{aligned} \{\omega \in \Omega : \, \Lambda(\omega) \cap O \neq \emptyset\} &= \{\omega \in \Omega : \, R(O) > 0 \text{ for some } R \in \mathcal{R}(\omega)\} \\ &= \operatorname{proj}_\Omega\{(\omega, R) \in \operatorname{graph}(\mathcal{R}) : \, R(O) > 0\}, \end{aligned}$$

where $\operatorname{proj}_\Omega$ denotes the canonical projection $\Omega \times \mathfrak{P}(\mathbb{R}^d) \to \Omega$. Since $R \mapsto R(O)$ is semicontinuous and in particular Borel, this shows that $\{\omega \in \Omega : \, \Lambda(\omega) \cap O \neq \emptyset\}$ is the continuous image of an analytic set, thus analytic and in particular universally measurable. In view of Remark 4.1, it follows that $\Lambda$ is universally measurable as claimed. Moreover, $\Lambda$ is closed-valued by its definition. Finally, the polar cone $\Lambda^\circ$ is then also universally measurable [50, Proposition 1H, Corollary 2T], and it is clear that $\Lambda^\circ$ is closed-valued and contains the origin.

It remains to prove (4.3). Let $\omega \in \Omega$. Clearly, $y \in \Lambda^\circ(\omega)$ implies that

$$P\{yX(\omega, \cdot) \geq 0\} = P\{X(\omega, \cdot) \in \Lambda(\omega)\} = 1$$

for all $P \in \mathcal{P}(\omega)$, so $y$ is contained in the right-hand side of (4.3). Conversely, let $y \notin \Lambda^\circ(\omega)$; then there exists $v \in \Lambda(\omega)$ such that $yv < 0$, and thus $yv' < 0$ for all $v'$ in an open neighborhood $B(v)$ of $v$. By the minimality property of the support $\Lambda(\omega)$, it follows that there exists $P \in \mathcal{P}(\omega)$ such that $P\{X(\omega, \cdot) \in B(v)\} > 0$. Therefore, $P\{yX(\omega, \cdot) < 0\} > 0$ and $y$ is not contained in the right-hand side of (4.3). $\square$

**Remark 4.4.** The assertion of Lemma 4.3 cannot be generalized to the case where $X$ is universally measurable instead of Borel (which would be quite handy in Section 4.3 below). Indeed, if $\Omega = \Omega_1 = [0,1]$, $\mathcal{P} \equiv \mathfrak{P}(\Omega)$ and $X = \mathbf{1}_A$ for some $A \subseteq \Omega \times \Omega_1$, then $\{\Lambda \cap \{1\} \neq \emptyset\} = \operatorname{proj}_\Omega(A)$. But there exists a universally measurable subset $A$ of $[0,1] \times [0,1]$ whose projection is not universally measurable; cf. the proof of [29, 439G].

## 4.2 First Fundamental Theorem

We now consider the setup as detailed in Section 1.2, for the case when there are no options ($e = 0$). In brief, $(\Omega, \mathcal{F}) = (\Omega_T, \mathcal{F}_T)$, where $\Omega_t = \Omega_1^t$ is the $t$-fold product of a Polish space $\Omega_1$ (and often identified with a subset of $\Omega$) and $\mathcal{F}_T$ is the universal completion of $\mathcal{B}(\Omega_T)$. The set $\mathcal{P}$ is determined by the random sets $\mathcal{P}_t(\cdot)$, which have analytic graphs, and $S_t$ is $\mathcal{B}(\Omega_t)$-measurable. It will be convenient to write $S_{t+1}$ as a function on $\Omega_t \times \Omega_1$,

$$S_{t+1}(\omega, \omega') = S_{t+1}(\omega_1, \dots, \omega_t, \omega'), \quad (\omega, \omega') = ((\omega_1, \dots, \omega_t), \omega') \in \Omega_t \times \Omega_1,$$

since the random variable $\Delta S_{t+1}(\omega, \cdot) = S_{t+1}(\omega, \cdot) - S_{t+1}(\omega)$ on $\Omega_1$ will be of particular interest. Indeed, $\Delta S_{t+1}(\omega, \cdot)$ determines a one-period market on $(\Omega_1, \mathcal{B}(\Omega_1))$ under the set $\mathcal{P}_t(\omega) \subseteq \mathfrak{P}(\Omega_1)$. Endowed with deterministic initial data, this market is of the type considered in Section 3, and we shall write $\mathrm{NA}(\mathcal{P}_t(\omega))$ for the no-arbitrage condition in that market. The following result contains the First Fundamental Theorem for the multi-period market, and also shows that absence of arbitrage in the multi-period market is equivalent to absence of arbitrage in all the one-period markets, up to a polar set.

**Theorem 4.5.** *The following are equivalent:*

*(i)* $\mathrm{NA}(\mathcal{P})$ *holds.*

*(ii) The set* $N_t = \{\omega \in \Omega_t : \mathrm{NA}(\mathcal{P}_t(\omega)) \text{ fails}\}$ *is* $\mathcal{P}$*-polar for all* $t \in \{0, \dots, T-1\}$.

*(iii) For all* $P \in \mathcal{P}$ *there exists* $Q \in \mathcal{Q}$ *such that* $P \ll Q$.

Before stating the proof, let us isolate some preliminary steps.

**Lemma 4.6.** *Let* $t \in \{0, \dots, T-1\}$. *The set*

$$N_t = \{\omega \in \Omega_t : \mathrm{NA}(\mathcal{P}_t(\omega)) \text{ fails}\}$$

*is universally measurable, and if* $\mathrm{NA}(\mathcal{P})$ *holds, then* $N_t$ *is* $\mathcal{P}$*-polar.*

*Proof.* We fix $t$ and set $X^\omega(\cdot) := S_{t+1}(\omega, \cdot) - S_t(\omega)$ for $\omega \in \Omega_t$. Let $\Lambda(\omega) = \operatorname{supp}_{\mathcal{P}_t(\omega)}(X^\omega)$ and let $\Lambda^\circ(\omega)$ be its polar cone. Our first claim is that

$$N_t^c \equiv \{\omega : \mathrm{NA}(\mathcal{P}_t(\omega)) \text{ holds}\} = \{\omega : \Lambda^\circ(\omega) = -\Lambda^\circ(\omega)\}. \tag{4.4}$$

Indeed, suppose that $\Lambda^\circ(\omega) = -\Lambda^\circ(\omega)$; then (4.3) shows that $yX^\omega \geq 0$ $\mathcal{P}_t(\omega)$-q.s. implies $-yX^\omega \geq 0$ $\mathcal{P}_t(\omega)$-q.s., and hence $\mathrm{NA}(\mathcal{P}_t(\omega))$ holds. On the other hand, if there exists $y \in \Lambda^\circ(\omega)$ such that $-y \notin \Lambda^\circ(\omega)$, then we have $yX^\omega \geq 0$ $\mathcal{P}_t(\omega)$-q.s. while $\{yX^\omega > 0\}$ is not $\mathcal{P}_t(\omega)$-polar, meaning that $\mathrm{NA}(\mathcal{P}_t(\omega))$ is violated. Therefore, (4.4) holds.

Since $\Lambda^\circ(\cdot)$ is universally measurable by Lemma 4.3, it follows from the representation (4.4) (and, e.g., [50, Proposition 1A]) that $N_t$ is universally measurable.

It remains to show that $N_t$ is $\mathcal{P}$-polar under $\mathrm{NA}(\mathcal{P})$. Suppose for contradiction that there exists $P_* \in \mathcal{P}$ such that $P_*(N_t) > 0$; then we need to construct (measurable) arbitrage strategies $y(\omega) \in \mathbb{R}^d$ and measures $P(\omega) \in \mathcal{P}_t(\omega)$ under which $y(\omega)$ makes a riskless profit with positive probability, for $\omega \in N_t$. In other words, we need to select $P_*$-a.s. from

$$\{(y, P) \in \Lambda^\circ(\omega) \times \mathcal{P}_t(\omega) : \, E_P[yX^\omega] > 0\}, \quad \omega \in N_t.$$

To this end, note that by modifying as in [6, Lemma 7.27, p. 173] each member of a universally measurable Castaing representation [2, Corollary 18.14, p. 601] of $\Lambda^\circ$, we can find a Borel-measurable mapping $\Lambda^\circ_* : \Omega_t \twoheadrightarrow \mathbb{R}^d$ with nonempty closed values such that

$$\Lambda^\circ_* = \Lambda^\circ \quad P_*\text{-a.s.} \tag{4.5}$$

This implies that $\mathrm{graph}(\Lambda^\circ_*) \subseteq \Omega_t \times \mathbb{R}^d$ is Borel [2, Theorem 18.6, p. 596]. Let

$$\Phi(\omega) := \{(y, P) \in \Lambda^\circ_*(\omega) \times \mathcal{P}_t(\omega) : \, E_P[yX^\omega] > 0\}, \quad \omega \in \Omega_t.$$

After using a monotone class argument to see that the function

$$\psi : \Omega \times \mathfrak{P}(\Omega_1) \times \mathbb{R}^d \to \mathbb{R}, \quad \psi(\omega, P, y) := E_P[yX^\omega]$$

is Borel, we deduce that (with a minor abuse of notation)

$$\mathrm{graph}(\Phi) = [\mathrm{graph}(\mathcal{P}_t) \times \mathbb{R}^d] \cap [\mathfrak{P}(\Omega_1) \times \mathrm{graph}(\Lambda^\circ_*)] \cap \{\psi > 0\}$$

is an analytic set. Now, we can apply the Jankov–von Neumann Theorem [6, Proposition 7.49, p. 182] to obtain a universally measurable function $\omega \mapsto (y(\omega), P(\omega))$ such that $(y(\cdot), P(\cdot)) \in \Phi(\cdot)$ on $\{\Phi \neq \emptyset\}$. By the definition of $N_t$ and (4.5), we have

$$N_t = \{\Phi \neq \emptyset\} \quad P_*\text{-a.s.},$$

so that $y$ is $P_*$-a.s. an arbitrage on $N_t$. On the universally measurable $P_*$-nullset $\{y \notin \Lambda^\circ\}$, we may redefine $y := 0$ to ensure that $y$ takes values in $\Lambda^\circ$. Similarly, on $\{\Phi = \emptyset\}$, we may define $P$ to be any universally measurable selector of $\mathcal{P}_t$. Setting $H_{t+1} := y$ and $H_s := 0$ for $s \neq t+1$, we have thus defined $H \in \mathcal{H}$ and $P(\cdot) \in \mathcal{P}_t(\cdot)$ such that

$$P(\omega)\{H_{t+1}(\omega)X^\omega > 0\} > 0 \quad \text{for } P_*\text{-a.e. } \omega \in N_t \tag{4.6}$$

and $H_{t+1}(\omega)X^\omega \geq 0$ $\mathcal{P}_t(\omega)$-q.s. for all $\omega \in \Omega$; cf. (4.3). As any $P' \in \mathcal{P}$ satisfies a decomposition of the form $P'|_{\Omega_{t+1}} = P'|_{\Omega_t} \otimes P'_t$ for some selector $P'_t$ of $\mathcal{P}_t$, Fubini's theorem yields that $H \bullet S_T \geq 0$ $\mathcal{P}$-q.s. On the other hand, let

$$P^* := P_*|_{\Omega_t} \otimes P \otimes \tilde{P}_{t+1} \otimes \cdots \otimes \tilde{P}_{T-1},$$

where $\tilde{P}_s$ is any universally measurable selector of $\mathcal{P}_s$, for $s = t+1, \dots, T-1$. Then $P^* \in \mathcal{P}$, while $P_*(N_t) > 0$ and (4.6) imply that $P^*\{H \bullet S_T > 0\} > 0$. This contradicts NA($\mathcal{P}$) and completes the proof that $N_t$ is $\mathcal{P}$-polar. $\square$

We recall the following result of Doob; it based on the fact that the Borel $\sigma$-field of $\Omega_1$ is countably generated (see [20, Theorem V.58, p. 52] and the subsequent remark).

**Lemma 4.7.** *Let $\mu, \mu' : \Omega \times \mathfrak{P}(\Omega_1) \to \mathfrak{P}(\Omega_1)$ be Borel. There exists a Borel function $D : \Omega \times \Omega \times \mathfrak{P}(\Omega_1) \times \mathfrak{P}(\Omega_1) \times \Omega_1 \to \overline{\mathbb{R}}$ such that*

$$D(\omega, \omega', P, P'; \tilde{\omega}) = \frac{d\mu(\omega, P)}{d\mu'(\omega', P')}\bigg|_{\mathcal{B}(\Omega_1)} (\tilde{\omega})$$

*for all $\omega, \omega' \in \Omega$ and $P, P' \in \mathfrak{P}(\Omega_1)$; i.e., $D(\omega, \omega', P, P'; \cdot)$ is a version of the Radon–Nikodym derivative of the absolutely continuous part of $\mu(\omega, P)$ with respect to $\mu'(\omega', P')$, on the Borel $\sigma$-field of $\Omega_1$.*

**Lemma 4.8.** *Let $t \in \{0, \dots, T-1\}$, let $P(\cdot) : \Omega_t \to \mathfrak{P}(\Omega_1)$ be Borel, and let*

$$\mathcal{Q}_t(\omega) := \{Q \in \mathfrak{P}(\Omega_1) : \ Q \lll \mathcal{P}_t(\omega), \ E_Q[\Delta S_{t+1}(\omega, \cdot)] = 0\}, \quad \omega \in \Omega_t.$$

*Then $\mathcal{Q}_t$ has an analytic graph and there exist universally measurable mappings $Q(\cdot), \hat{P}(\cdot) : \Omega_t \to \mathfrak{P}(\Omega_1)$ such that*

$$P(\omega) \ll Q(\omega) \ll \hat{P}(\omega) \quad \textit{for all } \omega \in \Omega_t,$$
$$\hat{P}(\omega) \in \mathcal{P}_t(\omega) \quad \textit{if } P(\omega) \in \mathcal{P}_t(\omega),$$
$$Q(\omega) \in \mathcal{Q}_t(\omega) \quad \textit{if } \mathrm{NA}(\mathcal{P}_t(\omega)) \textit{ holds and } P(\omega) \in \mathcal{P}_t(\omega).$$

*Proof.* Given $\omega \in \Omega_t$, we write $X^\omega$ for $\Delta S_{t+1}(\omega, \cdot)$. Let $P(\cdot) : \Omega_t \to \mathfrak{P}(\Omega_1)$ be Borel. As a first step, we show that the random set

$$\Xi(\omega) := \{(Q, \hat{P}) \in \mathfrak{P}(\Omega_1) \times \mathfrak{P}(\Omega_1) : \ E_Q[X^\omega] = 0, \ \hat{P} \in \mathcal{P}_t(\omega), \ P(\omega) \ll Q \ll \hat{P}\} \tag{4.7}$$

has an analytic graph. To this end, let

$$\Psi(\omega) := \{Q \in \mathfrak{P}(\Omega_1) : \ E_Q[X^\omega] = 0\}, \quad \omega \in \Omega_t.$$

Since the function

$$\psi : \Omega_t \times \mathfrak{P}(\Omega_1) \to \mathbb{R}, \quad \psi(\omega, R) := E_R[X^\omega]$$

is Borel, we see that

$$\mathrm{graph}(\Psi) = \{\psi = 0\} \subseteq \Omega_t \times \mathfrak{P}(\Omega_1) \quad \text{is Borel.}$$

Next, consider the random set

$$\Phi(\omega) := \{(R, \hat{R}) \in \mathfrak{P}(\Omega_1) \times \mathfrak{P}(\Omega_1) : \ P(\omega) \ll R \ll \hat{R}\}, \quad \omega \in \Omega_t.$$

Moreover, let

$$\phi : \Omega_t \times \mathfrak{P}(\Omega_1) \times \mathfrak{P}(\Omega_1) \to \mathbb{R}, \quad \phi(\omega, R, \hat{R}) := E_{P(\omega)}[dR/dP(\omega)] + E_R[d\hat{R}/dR],$$

where $dR/dP(\omega)$ and $d\hat{R}/dR$ are the jointly Borel-measurable Radon–Nikodym derivatives as in Lemma 4.7. Then, $P(\omega) \ll R \ll \hat{R}$ if and only if $\phi(\omega, R, \hat{R}) = 2$. Since Lemma 4.7 and [6, Proposition 7.26, p. 134] imply that $\phi$ is Borel, we deduce that

$$\operatorname{graph}(\Phi) = \{\phi = 2\} \subseteq \Omega_t \times \mathfrak{P}(\Omega_1) \quad \text{is Borel.}$$

As a result, we obtain that

$$\Xi(\omega) = [\Psi(\omega) \times \mathcal{P}_t(\omega)] \cap \Phi(\omega), \quad \omega \in \Omega_t$$

has an analytic graph. Thus, the Jankov–von Neumann Theorem [6, Proposition 7.49, p. 182] allows us to find universally measurable selectors $Q(\cdot), \hat{P}(\cdot)$ for $\Xi$ on the universally measurable set $\{\Xi \neq \emptyset\}$. Setting $Q(\cdot) := P(\cdot)$ and $\hat{P}(\cdot) := P(\cdot)$ on $\{\Xi \neq \emptyset\}$ completes the construction, as by Theorem 3.1, $\Xi(\omega) = \emptyset$ is possible only if $\mathrm{NA}(\mathcal{P}_t(\omega))$ does not hold or $P(\omega) \notin \mathcal{P}_t(\omega)$.

It remains to show that $\mathcal{Q}_t$ has an analytic graph. Using the same arguments as for the measurability of $\Xi$, but omitting the lower bound $P(\cdot)$, we see that the random set

$$\tilde{\Xi}(\omega) := \{(Q, \hat{P}) \in \mathfrak{P}(\Omega_1) \times \mathfrak{P}(\Omega_1) : \; E_Q[\Delta S_{t+1}(\omega, \cdot)] = 0, \, \hat{P} \in \mathcal{P}_t(\omega), \, Q \ll \hat{P}\} \tag{4.8}$$

has an analytic graph. Moreover, we observe that $\operatorname{graph}(\mathcal{Q}_t)$ is the image of $\operatorname{graph}(\tilde{\Xi})$ under the canonical projection $\Omega_t \times \mathfrak{P}(\Omega_1) \times \mathfrak{P}(\Omega_1) \to \Omega_t \times \mathfrak{P}(\Omega_1)$, so that $\operatorname{graph}(\mathcal{Q}_t)$ is indeed analytic. □

*Proof of Theorem 4.5.* Lemma 4.6 shows that (i) implies (ii). Let (iii) hold, then a set is $\mathcal{P}$-polar if and only if it is $\mathcal{Q}$-polar. Let $H \in \mathcal{H}$ be such that $H \bullet S_T \geq 0$ $\mathcal{P}$-q.s. For all $Q \in \mathcal{Q}$, we have $H \bullet S_T \geq 0$ $Q$-a.s., and as Lemma A.1 shows that the local $Q$-martingale $H \bullet S$ is a true one, it follows that $H \bullet S_T = 0$ $Q$-a.s. Hence, (i) holds.

It remains to show that (ii) implies (iii). Let $P \in \mathcal{P}$; then $P = P_0 \otimes \cdots \otimes P_{T-1}$ for some universally measurable selectors $P_t(\cdot)$ of $\mathcal{P}_t(\cdot)$, $t = 0, \dots, T-1$. We first focus on $t = 0$. Using Theorem 3.1, we can find $\hat{P}_0 \in \mathcal{P}_0$ and $Q_0 \in \mathcal{Q}_0$ such that $P_0 \ll Q_0 \ll \hat{P}_0$. Next, consider $t = 1$. By changing $P_1(\cdot)$ on a $\hat{P}_0$-nullset (hence a $P_0$-nullset), we find a Borel kernel which takes values in $\mathcal{P}_1(\cdot)$ $P_0$-a.e.; we again denote this kernel by $P_1(\cdot)$. As any $\hat{P}_0$-nullset is a $P_0$-nullset, this change does not affect the identity $P = P_0 \otimes \cdots \otimes P_{T-1}$. In view of (ii), we can apply Lemma 4.8 to find universally measurable kernels $Q_1(\cdot)$ and $\hat{P}_1(\cdot)$ such that

$$P_1(\cdot) \ll Q_1(\cdot) \ll \hat{P}_1(\cdot), \tag{4.9}$$

$Q_1(\cdot)$ takes values in $\mathcal{Q}_1(\cdot)$ $\hat{P}_0$-a.s. (and hence $Q_0$-a.s.), and $\hat{P}_1(\cdot)$ takes values in $\mathcal{P}_1(\cdot)$ $\hat{P}_0$-a.s. Let

$$P^1 := P_0 \otimes P_1, \quad \hat{P}^1 := \hat{P}_0 \otimes \hat{P}_1, \quad Q^1 := Q_0 \otimes Q_1.$$

Then (4.9) and Fubini's theorem show that $P^1 \ll Q^1 \ll \hat{P}^1$ and so we can proceed with $t = 2$ as above, using $\hat{P}^1$ instead of $\hat{P}_0$ as a reference measure, and continue up to $t = T - 1$. We thus find kernels

$$P_t(\cdot) \ll Q_t(\cdot) \ll \hat{P}_t(\cdot), \quad t = 0, \dots, T-1,$$

and we define

$$Q := Q_0 \otimes \cdots \otimes Q_{T-1}, \quad \hat{P} := \hat{P}_0 \otimes \cdots \otimes \hat{P}_{T-1}.$$

By construction, we have $P \ll Q \ll \hat{P}$ and $\hat{P} \in \mathcal{P}$; in particular, $P \ll Q \lll \mathcal{P}$. Moreover, the fact that $Q_t(\cdot) \in \mathcal{Q}_t(\cdot)$ holds $(Q_0 \otimes \cdots \otimes Q_{t-1})$-a.s. and Fubini's theorem yield that $S$ is a generalized martingale under $Q$ in the sense stated before Lemma A.1. (We do not have that $S_t \in L^1(Q)$, which is the missing part of the martingale property.) However, Lemma A.1 and Lemma A.3 imply that there exists $Q' \sim Q$ under which $S$ is a true martingale, which completes the proof. $\square$

## 4.3 Superhedging Theorem

We continue with the same setting as in the preceding subsection; that is, the setup detailed in Section 1.2 for the case when there are no options ($e = 0$). Our next aim is to prove the Superhedging Theorem in the multi-period market where only stocks are traded. To avoid some integrability problems in the subsequent section, we use a slightly smaller set of martingale measures: given a random variable ("weight function") $\varphi \geq 1$, we define $\mathcal{Q}_\varphi := \{Q \in \mathcal{Q} : E_Q[\varphi] < \infty\}$.

**Theorem 4.9.** *Let* $\mathrm{NA}(\mathcal{P})$ *hold, let* $\varphi \geq 1$ *be a random variable and let* $f : \Omega \to \mathbb{R}$ *be an upper semianalytic function such that* $|f| \leq \varphi$. *Then*

$$\sup_{Q \in \mathcal{Q}_\varphi} E_Q[f] = \pi(f) := \inf\big\{x \in \mathbb{R} : \exists\, H \in \mathcal{H} \text{ such that } x + H \bullet S_T \geq f\ \mathcal{P}\text{-}q.s.\big\}.$$

We shall give the proof through two key lemmas. First, we provide a measurable version of the one-step duality (Theorem 3.4). Then, we apply this result in a recursive fashion, where the claim is replaced by a dynamic version of the superhedging price. The relevance of upper semianalytic functions in this context is that semianalyticity is preserved through the recursion, whereas Borel-measurability is not. For the next statement, recall the random set $\mathcal{Q}_t$ introduced in Lemma 4.8.

**Lemma 4.10.** *Let* $\mathrm{NA}(\mathcal{P})$ *hold, let* $t \in \{0, \dots, T-1\}$ *and let* $f : \Omega_t \times \Omega_1 \to \overline{\mathbb{R}}$ *be upper semianalytic. Then*

$$\mathcal{E}_t(f) : \Omega_t \to \overline{\mathbb{R}}, \quad \mathcal{E}_t(f)(\omega) := \sup_{Q \in \mathcal{Q}_t(\omega)} E_Q[f(\omega, \cdot)]$$

*is upper semianalytic. Moreover, there exists a universally measurable function* $y(\cdot) : \Omega_t \to \mathbb{R}^d$ *such that*

$$\mathcal{E}_t(f)(\omega) + y(\omega)\Delta S_{t+1}(\omega, \cdot) \geq f(\omega, \cdot) \quad \mathcal{P}_t(\omega)\text{-}q.s. \tag{4.10}$$

*for all* $\omega \in \Omega_t$ *such that* $\mathrm{NA}(\mathcal{P}_t(\omega))$ *holds and* $f(\omega, \cdot) > -\infty$ $\mathcal{P}_t(\omega)$*-q.s.*

*Proof.* It is easy to see that the mapping $(\omega, Q) \mapsto E_Q[f(\omega, \cdot)]$ is continuous when $f$ is bounded and uniformly continuous. By a monotone class argument, the same mapping is therefore Borel when $f$ is bounded and Borel. Using [6, Proposition 7.26, p. 134] and [6, Proposition 7.48, p. 180], it follows that $(\omega, Q) \mapsto E_Q[f(\omega, \cdot)]$ is upper semianalytic when $f$ is. After recalling from Lemma 4.8 that the graph of $\mathcal{Q}_t$ is analytic, it follows from the Projection Theorem in the form of [6, Proposition 7.47, p. 179] that $\mathcal{E}_t(f)$ is upper semianalytic.

To avoid dealing with too many infinities in what follows, set

$$\mathcal{E}_t(f)' := \mathcal{E}_t(f)\mathbf{1}_{\mathbb{R}}(\mathcal{E}_t(f));$$

then $\mathcal{E}_t(f)'$ is universally measurable and we have $\mathcal{E}_t(f)'(\omega) = \mathcal{E}_t(f)(\omega)$ unless $\mathcal{E}_t(f)(\omega) = \pm\infty$. Consider the random set

$$\Psi(\omega) := \{y \in \mathbb{R}^d : \mathcal{E}_t(f)'(\omega) + y\Delta S_{t+1}(\omega, \cdot) \geq f(\omega, \cdot)\ \mathcal{P}_t(\omega)\text{-q.s.}\}, \quad \omega \in \Omega_t.$$

We show below that $\{\Psi \neq \emptyset\}$ is universally measurable and that $\Psi$ admits a universally measurable selector $y(\cdot)$ on that set. Before stating the proof, let us check that this fact implies the lemma. Indeed, define also $y(\cdot) = 0$ on $\{\Psi = \emptyset\}$. For $\omega$ such that $\Psi(\omega) = \emptyset$, there are two possible cases. Either $\mathcal{E}_t(f)'(\omega) < \mathcal{E}_t(f)(\omega)$; then necessarily $\mathcal{E}_t(f)(\omega) = +\infty$, and in this case (4.10) is trivially satisfied for our choice $y(\omega) = 0$. Or, there exists no $y'(\omega) \in \mathbb{R}^d$ solving (4.10) with $y(\omega)$ replaced by $y'(\omega)$; but then it follows from Theorem 3.4 that we must be in one of three sub-cases: either $\omega \in N_t$ (that is, $\mathrm{NA}(\mathcal{P}_t(\omega))$ fails), or $\{f(\omega, \cdot) = -\infty\}$ is not $\mathcal{P}_t(\omega)$-polar, or $\{f(\omega, \cdot) = +\infty\}$ is not $\mathcal{P}_t(\omega)$-polar. In the first two sub-cases, the lemma claims nothing. If we are not in the first two sub-cases and $\{f(\omega, \cdot) = +\infty\}$ is not $\mathcal{P}_t(\omega)$-polar, it follows via Theorem 3.1 and Lemma A.3 that $\mathcal{E}_t(f)(\omega) = +\infty$, so (4.10) is again trivially satisfied for $y(\omega) = 0$. On the other hand, if $\Psi(\omega) \neq \emptyset$, then our selector $y(\omega)$ solves (4.10) unless $\mathcal{E}_t(f)'(\omega) > \mathcal{E}_t(f)(\omega)$; that is, $\mathcal{E}_t(f)(\omega) = -\infty$. However, it follows from Theorem 3.4 that then again either $\omega \in N_t$ or $\{f(\omega, \cdot) = -\infty\}$ is not $\mathcal{P}_t(\omega)$-polar.

As a result, it remains to construct a universally measurable selector for $\Psi$. To this end, it will be necessary to consider the *difference* $f - \mathcal{E}_t(f)'$ which, in general, fails to be upper semianalytic. For that reason, we shall consider a larger class of functions. Given a Polish space $\Omega'$, recall that we are calling a function $g$ on $\Omega_t \times \Omega'$ upper semianalytic if for any $c \in \mathbb{R}$, the set $\{g > c\}$ is analytic, or equivalently, the nucleus of a Suslin scheme on $\mathcal{B}(\Omega_t \otimes \Omega') = \mathcal{B}(\Omega_t) \otimes \mathcal{B}(\Omega')$. Let us denote more generally by $\mathbf{A}[\mathcal{A}]$ the set of all nuclei of Suslin schemes on a paving $\mathcal{A}$; the mapping $\mathbf{A}$ is called the Suslin operation. Moreover, let $\mathrm{USA}[\mathcal{A}]$ be the set of all $\overline{\mathbb{R}}$-valued functions $g$ such that $\{g > c\} \in \mathbf{A}[\mathcal{A}]$ for all $c \in \mathbb{R}$. Hence, if $\mathcal{A}$ is the Borel $\sigma$-field of a Polish space, $\mathrm{USA}[\mathcal{A}]$ is the set of upper semianalytic functions in the classical sense. However, we shall extend consideration to the class $\mathrm{USA}[\mathcal{U}(\Omega_t) \otimes \mathcal{B}(\Omega')]$, where $\mathcal{U}(\Omega_t)$ is the universal $\sigma$-field on $\Omega_t$. It is a convex cone containing both $\mathrm{USA}[\mathcal{B}(\Omega_t) \otimes \mathcal{B}(\Omega')]$ and the linear space of $\mathcal{U}(\Omega_t) \otimes \mathcal{B}(\Omega')$-measurable function; in particular, is contains the function $f - \mathcal{E}_t(f)'$.

On the other hand, the class $\mathrm{USA}[\mathcal{U}(\Omega_t) \otimes \mathcal{B}(\Omega')]$ shares the main benefits of the classical upper semianalytic functions. This is due to the following fact;

it is a special case of [39, Theorem 5.5] (by the argument in the proof of that theorem's corollary and the subsequent scholium), in conjunction with the fact that $\mathbf{A}[\mathcal{U}(\Omega_t)] = \mathcal{U}(\Omega_t)$ since $\mathcal{U}(\Omega_t)$ is universally complete [6, Proposition 7.42, p. 167].

**Lemma 4.11.** *Let* $\Gamma \in \mathbf{A}[\mathcal{U}(\Omega_t)\otimes\mathcal{B}(\Omega')]$. *Then* $\mathrm{proj}_{\Omega_t}(\Gamma) \in \mathcal{U}(\Omega_t)$ *and there exists a* $\mathcal{U}(\Omega_t)$*-measurable mapping* $\gamma : \mathrm{proj}_{\Omega_t}(\Gamma) \to \Omega'$ *such that* $\mathrm{graph}(\gamma) \subseteq \Gamma$.

We can now complete the proof of Lemma 4.10 by showing that $\{\Psi \neq \emptyset\}$ is universally measurable and $\Psi$ admits a universally measurable selector $y(\cdot)$ on that set. Fix $y \in \mathbb{R}^d$. Given $\omega \in \Omega_t$, we have $y \in \Psi(\omega)$ if and only if

$$\theta_y(\omega,\cdot) := f(\omega,\cdot) - \mathcal{E}_t(f)'(\omega) - y\Delta S_{t+1}(\omega,\cdot) \leq 0 \quad \mathcal{P}_t(\omega)\text{-q.s.}$$

Moreover, we have $\theta_y \in \mathrm{USA}[\mathcal{U}(\Omega_t)\otimes\mathcal{B}(\Omega_1)]$ as explained above. Note that for any $P \in \mathcal{P}_t(\omega)$, the condition $\theta_y(\omega,\cdot) \leq 0$ $P$-a.s. holds if and only if

$$E_{\tilde{P}}[\theta_y(\omega,\cdot)] \leq 0 \quad \text{for all} \quad \tilde{P} \ll P,$$

and by an application of Lemma 3.2, it is further equivalent to have this only for $\tilde{P} \ll P$ satisfying $E_{\tilde{P}}[|\Delta S_{t+1}(\omega,\cdot)|] < \infty$. Therefore, we introduce the random set

$$\widetilde{\mathcal{P}}_t(\omega) := \{\tilde{P} \in \mathfrak{P}(\Omega_1) : \tilde{P} \lll \mathcal{P}_t(\omega),\, E_{\tilde{P}}[|\Delta S_{t+1}(\omega,\cdot)|] < \infty\}.$$

By the same arguments as at the end of the proof of Lemma 4.8, we can show that $\widetilde{\mathcal{P}}_t$ has an analytic graph (in the classical sense). Define also

$$\Theta_y(\omega) := \sup_{\tilde{P}\in\tilde{\mathcal{P}}_t(\omega)} E_{\tilde{P}}[\theta_y(\omega,\cdot)];$$

the next step is to show that $\omega \mapsto \Theta_y(\omega)$ is universally measurable. Indeed, following the arguments in the very beginning of this proof, the first term in the difference

$$E_{\tilde{P}}[\theta_y(\omega,\cdot)] = E_{\tilde{P}}[f(\omega,\cdot)] - \mathcal{E}_t(f)'(\omega) - yE_{\tilde{P}}[\Delta S_{t+1}(\omega,\cdot)]$$

is an upper semianalytic function of $(\omega,\tilde{P})$; that is, in $\mathrm{USA}[\mathcal{B}(\Omega_t)\otimes\mathcal{B}(\mathfrak{P}(\Omega_1))]$. Moreover, the second term can be seen as a $\mathcal{U}(\Omega_t)\otimes\mathcal{B}(\mathfrak{P}(\Omega_1))$-measurable function of $(\omega,\tilde{P})$, and the third is even Borel. As a result, $(\omega,P) \mapsto E_{\tilde{P}}[\theta_y(\omega,\cdot)]$ is in $\mathrm{USA}[\mathcal{U}(\Omega_t)\otimes\mathcal{B}(\mathfrak{P}(\Omega_1))]$. Thus, by the Projection Theorem in the form of Lemma 4.11,

$$\{\Theta_y > c\} = \mathrm{proj}_{\Omega_t}\big\{(\omega,\tilde{P}) \in \mathrm{graph}\,(\widetilde{\mathcal{P}}_t) : E_{\tilde{P}}[\theta_y(\omega,\cdot)] > c\big\} \in \mathcal{U}(\Omega_t)$$

for all $c \in \mathbb{R}$. That is, $\omega \mapsto \Theta_y(\omega)$ is universally measurable for any fixed $y$.

On the other hand, fix $\omega \in \Omega_t$ and $m \geq 1$. Then, the function $y \mapsto \Theta_y(\omega)\wedge m$ is lower semicontinuous because it is the supremum of the continuous functions

$$y \mapsto E_{\tilde{P}}[\theta_y(\omega,\cdot)]\wedge m = \big(E_{\tilde{P}}[f(\omega,\cdot)] - \mathcal{E}_t(f)'(\omega) - yE_{\tilde{P}}[\Delta S_{t+1}(\omega,\cdot)]\big)\wedge m$$

over $\widetilde{\mathcal{P}}_t(\omega)$. (To be precise, $\Theta_y(\omega) \wedge m$ may be infinite for some $\omega$. For such $\omega$, the function $y \mapsto \Theta_y(\omega) \wedge m$ is semicontinuous in that it is constant.) By Lemma 4.12 below, it follows that $(y, \omega) \mapsto \Theta_y(\omega) \wedge m$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}(\Omega_t)$-measurable. Passing to the supremum over $m \geq 1$, we obtain that $(y, \omega) \mapsto \Theta_y(\omega)$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}(\Omega_t)$-measurable as well. As a result, we have that

$$\operatorname{graph}(\Psi) = \{(y, \omega) : \Theta_y(\omega) \leq 0\} \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}(\Omega_t).$$

Now, Lemma 4.11 yields that $\{\Psi \neq \emptyset\} \in \mathcal{U}(\Omega_t)$ and that $\Psi$ admits a $\mathcal{U}(\Omega_t)$-measurable selector on that set, which completes the proof of Lemma 4.10. $\square$

The following statement was used in the preceding proof; it is a slight generalization of the fact that Carathéodory functions are jointly measurable.

**Lemma 4.12.** *Let $(A, \mathcal{A})$ be a measurable space and let $\theta : \mathbb{R}^d \times A \to \overline{\mathbb{R}}$ be a function such that $\omega \mapsto \theta(y, \omega)$ is $\mathcal{A}$-measurable for all $y \in \mathbb{R}^d$ and $y \mapsto \theta(y, \omega)$ is lower semicontinuous for all $\omega \in A$. Then $\theta$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{A}$-measurable.*

*Proof.* Let $\{y_k\}_{k \geq 1}$ be a dense sequence in $\mathbb{R}^d$. For all $c \in \mathbb{R}$, we have

$$\{\theta \geq c\} = \bigcap_{n \geq 1} \bigcup_{k \geq 1} B_{1/n}(y_k) \times \{\theta(y_k, \cdot) > c - 1/n\},$$

where $B_{1/n}(y_k)$ is the open ball of radius $1/n$ around $y_k$. The right-hand side is clearly in $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{A}$. $\square$

We now turn to the second key lemma for the proof of Theorem 4.9. Recall that $\mathcal{Q}_\varphi = \{Q \in \mathcal{Q} : E_Q[\varphi] < \infty\}$.

**Lemma 4.13.** *Let $\mathrm{NA}(\mathcal{P})$ hold, let $f : \Omega \to \mathbb{R}$ be upper semianalytic and bounded from above, and let $\varphi \geq 1$ be a random variable such that $|f| \leq \varphi$. Then there exists $H \in \mathcal{H}$ such that*

$$\sup_{Q \in \mathcal{Q}_\varphi} E_Q[f] + H \bullet S_T \geq f \quad \mathcal{P}\text{-q.s.}$$

*Proof.* Let $\mathcal{E}(f) := (\mathcal{E}_0 \circ \cdots \circ \mathcal{E}_{T-1})(f)$; note that the composition is well defined by Lemma 4.10. We first show that there exists $H \in \mathcal{H}$ such that

$$\mathcal{E}(f) + H \bullet S_T \geq f \quad \mathcal{P}\text{-q.s.} \tag{4.11}$$

Define $\mathcal{E}^t(f)(\omega) := (\mathcal{E}_t \circ \cdots \circ \mathcal{E}_{T-1})(f)(\omega)$, and let $N$ be the set of all $\omega \in \Omega$ such that $\mathrm{NA}(\mathcal{P}_t(\omega))$ fails for some $t \in \{0, \ldots, T-1\}$. Then $N$ is a universally measurable, $\mathcal{P}$-polar set by Lemma 4.6. Using Lemma 4.10, there exist universally measurable functions $y_t : \Omega_t \to \mathbb{R}^d$ such that

$$y_t(\omega) \Delta S_{t+1}(\omega, \cdot) \geq \mathcal{E}^{t+1}(f)(\omega, \cdot) - \mathcal{E}^t(f)(\omega) \quad \mathcal{P}_t(\omega)\text{-q.s.}$$

for all $\omega \in N^c$. Recalling that any $P \in \mathcal{P}$ is of the form $P = P_0 \otimes \cdots \otimes P_{T-1}$ for some selectors $P_t$ of $\mathcal{P}_t$, it follows by Fubini's theorem that

$$\sum_{t=0}^{T-1} y_t \Delta S_{t+1} \geq \sum_{t=0}^{T-1} \mathcal{E}^{t+1}(f) - \mathcal{E}^t(f) = f - \mathcal{E}(f) \quad \mathcal{P}\text{-q.s.};$$

that is, (4.11) holds for $H \in \mathcal{H}$ defined by $H_{t+1} := y_t$. Next, we show that

$$\mathcal{E}(f) \leq \sup_{Q \in \mathcal{Q}_\varphi} E_Q[f]; \tag{4.12}$$

together with (4.11), this will imply the result. (The reverse inequality of this dynamic programming principle is also true, but not needed here.) Recall from below (4.8) that the graph of the random set

$$\tilde{\Xi}_t(\omega) = \{(Q, \hat{P}) \in \mathfrak{P}(\Omega_1) \times \mathfrak{P}(\Omega_1) : \, E_Q[\Delta S_{t+1}(\omega, \cdot)] = 0, \, \hat{P} \in \mathcal{P}_t(\omega), \, Q \ll \hat{P}\}$$

is analytic. We may see its section $\tilde{\Xi}_t(\omega)$ as the set of controls for the control problem $\sup_{Q \in \mathcal{Q}_t(\omega)} E_Q[\mathcal{E}^{t+1}(f)(\omega, \cdot)]$; indeed, $\mathcal{Q}_t(\omega)$ is precisely the projection of $\tilde{\Xi}_t(\omega)$ onto the first component. As in the proof of Lemma 4.10, we obtain that the reward function $(\omega, Q) \mapsto E_Q[\mathcal{E}^{t+1}(f)(\omega, \cdot)]$ is upper semianalytic. Given $\varepsilon > 0$, it then follows from the Jankov-von Neumann Theorem in the form of [6, Proposition 7.50, p. 184] that there exists a universally measurable selector $(Q_t^\varepsilon(\cdot), \hat{P}_t^\varepsilon(\cdot))$ of $\tilde{\Xi}_t(\cdot)$ such that $Q_t^\varepsilon(\omega)$ is an $\varepsilon$-optimal control; that is,

$$E_{Q_t^\varepsilon(\omega)}[\mathcal{E}^{t+1}(f)(\omega, \cdot)] \geq \sup_{Q \in \mathcal{Q}_t(\omega)} E_Q[\mathcal{E}^{t+1}(f)(\omega, \cdot)] - \varepsilon \quad \text{for all } \omega \text{ such that } \tilde{\Xi}_t(\omega) \neq \emptyset. \tag{4.13}$$

Define also $Q_t^\varepsilon$ and $\hat{P}_t^\varepsilon$ as an arbitrary (universally measurable) selector of $\mathcal{P}_t$ on $\{\tilde{\Xi}_t = \emptyset\}$; note that the latter set is contained in the $\mathcal{P}$-polar set $N$ by Theorem 3.1. Now let

$$Q^\varepsilon := Q_0^\varepsilon \otimes \cdots \otimes Q_{T-1}^\varepsilon \quad \text{and} \quad \hat{P}^\varepsilon := \hat{P}_0^\varepsilon \otimes \cdots \otimes \hat{P}_{T-1}^\varepsilon.$$

As in the proof of Theorem 4.5, we then have $Q^\varepsilon \ll \hat{P} \in \mathcal{P}$ (thus $Q^\varepsilon \lll \mathcal{P}$) and $S$ is a generalized martingale under $Q^\varepsilon$. Moreover, recalling that $f$ is bounded from above, we may apply Fubini's theorem $T$ times to obtain that

$$\mathcal{E}(f) = (\mathcal{E}_0 \circ \cdots \circ \mathcal{E}_{T-1})(f) \leq \varepsilon + E_{Q_0^\varepsilon}[\mathcal{E}^1(f)] \leq \cdots \leq T\varepsilon + E_{Q^\varepsilon}[f].$$

By Lemma A.1 and Lemma A.3, there exists $Q_*^\varepsilon \in \mathcal{Q}_\varphi$ such that $E_{Q_*^\varepsilon}[f] \geq E_{Q^\varepsilon}[f]$. Thus, we have

$$\mathcal{E}(f) \leq T\varepsilon + \sup_{Q \in \mathcal{Q}_\varphi} E_Q[f].$$

As $\varepsilon > 0$ was arbitrary, we conclude that (4.12) holds. In view of (4.11), the proof is complete. □

*Proof of Theorem 4.9.* By the same argument as below (3.4), we may assume that $f$ is bounded from above. The inequality "$\geq$" then follows from Lemma 4.13, whereas the inequality "$\leq$" follows from Lemma A.2. □

## 5 The Multi-Period Case With Options

In this section, we prove the main results stated in the Introduction. The setting is as introduced in Section 1.2; in particular, there are $d$ stocks available for dynamic trading and $e$ options available for static trading. Our argument is based on the results of the preceding section: if $e = 1$, we can apply the already proved versions of the Fundamental Theorem and the Superhedging Theorem to study the option $g^1$; similarly, we shall add the options $g^2, \dots, g^e$ one after the other in an inductive fashion. To avoid some technical problems in the proof, we state the main results with an additional weight function $\varphi \geq 1$ that controls the integrability; this is by no means a restriction and actually yields a more precise conclusion (Remark 5.2). We recall the set $\mathcal{Q}$ from (1.3) as well as the notation $\mathcal{Q}_\varphi = \{Q \in \mathcal{Q} : \, E_Q[\varphi] < \infty\}$.

**Theorem 5.1.** *Let $\varphi \geq 1$ be a random variable and suppose that $|g^i| \leq \varphi$ for $i = 1, \dots, e$.*

*(a) The following are equivalent:*

- *(i) $\mathrm{NA}(\mathcal{P})$ holds.*
- *(ii) For all $P \in \mathcal{P}$ there exists $Q \in \mathcal{Q}_\varphi$ such that $P \ll Q$.*

*(b) Let $\mathrm{NA}(\mathcal{P})$ hold and let $f : \Omega \to \mathbb{R}$ be an upper semianalytic function such that $|f| \leq \varphi$. Then*

$$\pi(f) := \inf\big\{x \in \mathbb{R} : \, \exists \, (H, h) \in \mathcal{H} \times \mathbb{R}^e \text{ such that } x + H \bullet S_T + hg \geq f \ \mathcal{P}\text{-q.s.}\big\}$$

*satisfies*

$$\pi(f) = \sup_{Q \in \mathcal{Q}_\varphi} E_Q[f] \in (-\infty, \infty]$$

*and there exist $(H, h) \in \mathcal{H} \times \mathbb{R}^e$ such that $\pi(f) + H \bullet S_T + hg \geq f$ $\mathcal{P}$-q.s.*

*(c) Let $\mathrm{NA}(\mathcal{P})$ hold and let $f : \Omega \to \mathbb{R}$ be an upper semianalytic function such that $|f| \leq \varphi$. The following are equivalent:*

- *(i) $f$ is replicable.*
- *(ii) The mapping $Q \mapsto E_Q[f] \in \mathbb{R}$ is constant on $\mathcal{Q}_\varphi$.*
- *(iii) For all $P \in \mathcal{P}$ there exists $Q \in \mathcal{Q}_\varphi$ such that $P \ll Q$ and $E_Q[f] = \pi(f)$.*

*Proof.* We proceed by induction; note that if $\mathrm{NA}(\mathcal{P})$ holds for the market with options $g^1, \dots, g^e$, then it also holds for the market with options $g^1, \dots, g^{e'}$ for any $e' \leq e$.

In the case $e = 0$, there are no traded options. Then, (a) follows from Theorem 4.5 and Lemma A.3, and (b) holds by Theorem 4.9. To see (c), suppose that (i) holds; i.e., $f = x + H \bullet S_T$ $\mathcal{P}$-q.s. for some $x \in \mathbb{R}$ and $H \in \mathcal{H}$. Using that $|f| \leq \varphi$, we deduce via Lemma A.1 that $E_Q[f] = x$ for all $Q \in \mathcal{Q}_\varphi$, which is (ii). In view of (a), (ii) implies (iii). Let (iii) hold, let $H \in \mathcal{H}$ be an optimal superhedging strategy as in (b), and let $K := \pi(f) + H \bullet S_T - f$. Then $K \geq 0$ $\mathcal{P}$-q.s., but in view of (iii) and (a), we have $K = 0$ $\mathcal{P}$-q.s., showing that $H$ is a replicating strategy.

Next, we assume that the result holds as stated when there are $e \geq 0$ traded options $g^1, \dots, g^e$ and we introduce an additional (Borel-measurable) option $f \equiv g^{e+1}$ with $|f| \leq \varphi$ in the market at price $f_0 = 0$.

We first prove (a). Let NA($\mathcal{P}$) hold. If $f$ is replicable, we can reduce to the case with at most $e$ options, so we may assume that $f$ is not replicable. Let $\pi(f)$ be the superhedging price when the stocks and $g^1, \dots, g^e$ are available for trading (as stated in the theorem); we have $\pi(f) > -\infty$ by (b) of the induction hypothesis. If $f_0 \geq \pi(f)$, then as $f$ is not replicable, we obtain an arbitrage by shortselling one unit of $f$ and using an optimal superhedging strategy for $f$, which exists by (b) of the induction hypothesis. Therefore, $f_0 < \pi(f)$. Together with (b) of the induction hypothesis, we have

$$f_0 < \pi(f) = \sup_{Q \in \mathcal{Q}_\varphi} E_Q[f].$$

As $Q \mapsto E_Q[f]$ is not constant by (c) of the induction hypothesis and $E_Q[|f|] < \infty$ for all $Q \in \mathcal{Q}_\varphi$, it follows that there exists $Q_+ \in \mathcal{Q}_\varphi$ such that

$$f_0 < E_{Q_+}[f] < \pi(f).$$

A similar argument applied to $-f$ (which is Borel-measurable and not replicable like $f$) yields $Q_- \in \mathcal{Q}_\varphi$ such that

$$-\pi(-f) < E_{Q_-}[f] < f_0 < E_{Q_+}[f] < \pi(f). \tag{5.1}$$

Now let $P \in \mathcal{P}$. By (a) of the induction hypothesis, there exists $Q_0 \in \mathcal{Q}_\varphi$ such that $P \ll Q_0 \lll \mathcal{P}$. By choosing suitable weights $\lambda_-, \lambda_0, \lambda_+ \in (0,1)$ such that $\lambda_- + \lambda_0 + \lambda_+ = 1$, we have that

$$P \ll Q := \lambda_- Q_- + \lambda_0 Q_0 + \lambda_+ Q_+ \in \mathcal{Q} \quad \text{and} \quad E_Q[f] = f_0.$$

This completes the proof that (i) implies (ii). The reverse implication can be shown as in the proof of Theorem 4.5, so (a) is established. Moreover, the argument for (c) is essentially the same as in the case $e = 0$.

Next, we prove (b). The argument is in the spirit of Theorem 3.4; indeed, (5.1) states that the price $f_0$ is in the interior of the set $\{E_Q[f] : Q \in \mathcal{Q}_\varphi\}$ and will thus play the role that the Fundamental Lemma (Lemma 3.3) had in the proof of Theorem 3.4. We continue to write $f$ for $g^{e+1}$ and let $f'$ be an upper semianalytic function such that $|f'| \leq \varphi$. Let $\pi'(f')$ be the superhedging price of $f'$ when the stocks and $g^1, \dots, g^e, f \equiv g^{e+1}$ are available for trading. Recall that $\pi'(f') > -\infty$ by Theorem 2.3 which also yields the existence of an optimal strategy. Let $\mathcal{Q}'_\varphi$ be the set of all martingale measures $Q'$ which satisfy $E_{Q'}[g^i] = 0$ for $i = 1, \dots, e+1$ (whereas we continue to write $\mathcal{Q}_\varphi$ for those martingale measures which have that property for $i = 1, \dots, e$). Then, we need to show that

$$\pi'(f') = \sup_{Q' \in \mathcal{Q}'_\varphi} E_{Q'}[f'].$$

As above, we may assume that $f$ is not replicable given the stocks and $g^1, \dots, g^e$, as otherwise we can reduce to the case with $e$ options. The inequality

$$\pi'(f') \geq \sup_{Q' \in \mathcal{Q}'_\varphi} E_{Q'}[f'] \tag{5.2}$$

follows from Lemma A.2; we focus on the reverse inequality

$$\pi'(f') \leq \sup_{Q' \in \mathcal{Q}'_\varphi} E_{Q'}[f']. \tag{5.3}$$

Suppose first that $\pi'(f') < \infty$; then we have $\pi'(f') \in \mathbb{R}$. In this situation, we claim that

$$\text{there exist } Q_n \in \mathcal{Q}_\varphi \text{ such that } \quad E_{Q_n}[f] \to f_0, \quad E_{Q_n}[f'] \to \pi'(f'). \tag{5.4}$$

Before proving this claim, let us see how it implies (5.3). Indeed, as $f$ is not replicable, there exist measures $Q_\pm \in \mathcal{Q}_\varphi$ as in (5.1). Given $Q_n$ from (5.4), we can then find weights $\lambda^n_-, \lambda^n, \lambda^n_+ \in [0,1]$ such that $\lambda^n_- + \lambda^n + \lambda^n_+ = 1$ and

$$Q'_n := \lambda^n_- Q_- + \lambda^n Q_n + \lambda^n_+ Q_+ \in \mathcal{Q}_\varphi \quad \text{satisfies} \quad E_{Q'_n}[f] = f_0 = 0;$$

that is, $Q'_n \in \mathcal{Q}'_\varphi$. Moreover, since $E_{Q_n}[f] \to f_0$, the weights can be chosen such that $\lambda^n_\pm \to 0$. Using that $E_{Q_\pm}[|f'|] < \infty$ as $|f'| \leq \varphi$, we conclude that

$$E_{Q'_n}[f'] \to \pi'(f'),$$

which indeed implies (5.3).

We turn to the proof of (5.4). By a translation we may assume that $\pi'(f') = 0$; thus, if (5.4) fails, we have

$$0 \notin \overline{\{E_Q[(f, f')] : Q \in \mathcal{Q}_\varphi\}} \subseteq \mathbb{R}^2.$$

Then, there exists a separating vector $(y, z) \in \mathbb{R}^2$ with $|(y, z)| = 1$ such that

$$0 > \sup_{Q \in \mathcal{Q}_\varphi} E_Q[yf + zf'].$$

But by (b) of the induction hypothesis, we know that

$$\sup_{Q \in \mathcal{Q}_\varphi} E_Q[yf + zf'] = \pi(yf + zf').$$

Moreover, by the definitions of $\pi$ and $\pi'$, we clearly have $\pi(\psi) \geq \pi'(\psi)$ for any random variable $\psi$. Finally, the definition of $\pi'$ shows that $\pi'(yf + \psi) = \pi'(\psi)$, because $f$ is available at price $f_0 = 0$ for hedging. Hence, we have

$$0 > \sup_{Q \in \mathcal{Q}_\varphi} E_Q[yf + zf'] \geq \pi'(zf'), \tag{5.5}$$

which clearly implies $z \neq 0$. If $z > 0$, the positive homogeneity of $\pi'$ yields that $\pi'(f') < 0$, contradicting our assumption that $\pi'(f') = 0$. Thus, we have $z < 0$. To find a contradiction, recall that we have already shown that there exists some $Q' \in \mathcal{Q}'_\varphi \subseteq \mathcal{Q}_\varphi$. Then, (5.5) yields that $0 > E_{Q'}[yf + zf'] = E_{Q'}[zf']$ and hence $E_{Q'}[f'] > 0 = \pi'(f')$ as $z < 0$, which contradicts (5.2). This completes the proof of (5.3) in the case $\pi'(f') < \infty$.

Finally, to obtain (5.3) also in the case $\pi'(f') = \infty$, we apply the above to $f' \wedge n$ and let $n \to \infty$, exactly as below (3.4). This completes the proof of (b). $\square$

**Remark 5.2.** Given the options $g^1, \dots, g^e$ and $f$, we can always choose the weight function $\varphi := 1 + |g^1| + \dots + |g^e| + |f|$; that is, the presence of $\varphi$ in Theorem 5.1 is not restrictive. The theorem implies the main results as stated in the Introduction. Indeed, the implication from (i) to (ii) in the First Fundamental Theorem as stated in Section 1.2 follows immediately from (a), and it is trivial that (ii) implies (ii'). The fact that (ii') implies (i) is seen as in the proof of Theorem 4.5. The Superhedging Theorem of Section 1.2 is a direct consequence of (b) and Lemma A.2. The equivalence of (i)–(iii) in the Second Fundamental Theorem then follows from (c); in particular, if $\mathcal{Q}$ is a singleton, the market is complete. Conversely, if the market is complete, then using $f = \mathbf{1}_A$, we obtain from (ii) that $Q \mapsto Q(A)$ is constant on $\mathcal{Q}$ for every $A \in \mathcal{B}(\Omega)$. As any probability measure on the universal $\sigma$-field $\mathcal{F}$ is determined by its values on $\mathcal{B}(\Omega)$, this implies that $\mathcal{Q}$ is a singleton.

# 6 Nondominated Optional Decomposition

In this section, we derive a nondominated version of the Optional Decomposition Theorem of [37] and [38], for the discrete-time case. As in Section 4, we consider the setting introduced in Section 1.2 in the case without options ($e = 0$).

**Theorem 6.1.** *Let* $\mathrm{NA}(\mathcal{P})$ *hold and let* $V$ *be an adapted process such that* $V_t$ *is upper semianalytic and in* $L^1(Q)$ *for all* $Q \in \mathcal{Q}$ *and* $t \in \{1, 2, \dots, T\}$*. The following are equivalent:*

*(i)* $V$ *is a supermartingale under each* $Q \in \mathcal{Q}$*.*

*(ii) There exist* $H \in \mathcal{H}$ *and an adapted increasing process* $K$ *with* $K_0 = 0$ *such that*

$$V_t = V_0 + H \bullet S_t - K_t \quad \mathcal{P}\text{-q.s.}, \quad t \in \{0, 1, \dots, T\}.$$

*Proof.* It follows from Lemma A.1 that (ii) implies (i). We show that (i) implies (ii); this proof is similar to the one of Theorem 4.9, so we shall be brief. Recall the operator $\mathcal{E}_t(\cdot)$ from Lemma 4.10. Our first aim is to show that for every $t \in \{0, 1, \dots, T-1\}$, we have

$$\mathcal{E}_t(V_{t+1}) \leq V_t \quad \mathcal{P}\text{-q.s.} \tag{6.1}$$

Let $Q \in \mathcal{Q}$ and $\varepsilon > 0$. Following the same arguments as in (4.13), we can construct a universally measurable $\varepsilon$-optimizer $Q_t^\varepsilon(\cdot) \in \mathcal{Q}_t(\cdot)$; that is,

$$E_{Q_t^\varepsilon(\omega)}[V_{t+1}(\omega, \cdot)] + \varepsilon \geq \sup_{Q \in \mathcal{Q}_t(\omega)} E_Q[V_{t+1}(\omega, \cdot)] \equiv \mathcal{E}_t(V_{t+1})(\omega) \quad \text{for all } \omega \in \Omega_t \setminus N_t,$$

where $N_t = \{\omega \in \Omega_t : \mathrm{NA}(\mathcal{P}_t(\omega)) \text{ fails}\}$ is universally measurable and $\mathcal{P}$-polar (hence $\mathcal{Q}$-polar) by Lemma 4.6. Let

$$Q' := Q|_{\Omega_t} \otimes Q_t^{\varepsilon} \otimes Q_{t+1} \otimes \cdots \otimes Q_{T-1},$$

where $Q_s$ is an arbitrary selector of $\mathcal{Q}_s$ for $s = t+1, \dots, T-1$. Then, $S$ is a generalized martingale under $Q'$ and

$$E_{Q_t^{\varepsilon}(\omega)}[V_{t+1}(\omega, \cdot)] = E_{Q'}[V_{t+1}|\mathcal{F}_t](\omega) \quad \text{for } Q'\text{-a.e. } \omega$$

by Fubini's theorem, where the conditional expectation is understood in the generalized sense (cf. the Appendix). As both sides of this identity are $\mathcal{F}_t$-measurable and $Q = Q'$ on $\mathcal{F}_t$, we also have that

$$E_{Q_t^{\varepsilon}(\omega)}[V_{t+1}(\omega, \cdot)] = E_{Q'}[V_{t+1}|\mathcal{F}_t](\omega) \quad \text{for } Q\text{-a.e. } \omega.$$

Finally, by a conditional version of Lemma A.3, there exists a martingale measure $Q'' \in \mathcal{Q}$ with $Q'' = Q'$ on $\mathcal{F}_t$ such that

$$E_{Q''}[V_{t+1}|\mathcal{F}_t] \geq E_{Q'}[V_{t+1}|\mathcal{F}_t] \quad Q\text{-a.s.},$$

and since $V$ is a $Q''$-supermartingale by (i), we also have

$$V_t \geq E_{Q''}[V_{t+1}|\mathcal{F}_t] \quad Q\text{-a.s.}$$

Collecting the above (in)equalities, we arrive at

$$\mathcal{E}_t(V_{t+1}) - \varepsilon \leq V_t \quad Q\text{-a.s.}$$

Since $\varepsilon > 0$ and $Q \in \mathcal{Q}$ were arbitrary, we deduce that $\mathcal{E}_t(V_{t+1}) \leq V_t$ $\mathcal{Q}$-q.s., and as Theorem 4.5 shows that $\mathcal{Q}$ and $\mathcal{P}$ have the same polar sets, we have established (6.1).

For each $t \in \{0, 1, \dots, T-1\}$, Lemma 4.10 yields a universally measurable function $y_t(\cdot) : \Omega_t \to \mathbb{R}^d$ such that

$$\mathcal{E}_t(f)(\omega) + y_t(\omega)\Delta S_{t+1}(\omega, \cdot) \geq V_{t+1}(\omega, \cdot) \quad \mathcal{P}_t(\omega)\text{-q.s.}$$

for all $\omega \in \Omega_t \setminus N_t$. By Fubini's theorem and (6.1), this implies that

$$V_t + y_t \Delta S_{t+1} \geq V_{t+1} \quad \mathcal{P}\text{-q.s.}$$

Define $H \in \mathcal{H}$ by $H_{t+1} := y_t$; then it follows that

$$V_t \leq V_0 + H \bullet S_t \quad \mathcal{P}\text{-q.s.}, \quad t = 0, 1, \dots, T;$$

that is, (ii) holds with $K_t := V_0 + H \bullet S_t - V_t$. □

# A Appendix

For ease of reference, we collect here some known facts about the classical (dominated) case that are used in the body of the paper. Let $(\Omega, \mathcal{F}, P)$ be a probability space, equipped with a filtration $(\mathcal{F}_t)_{t\in\{0,1,\dots,T\}}$. An adapted process $M$ is a *generalized martingale* if

$$E_P[M_{t+1}|\mathcal{F}_t] = M_t \quad P\text{-a.s.}, \quad t = 0, \dots, T-1$$

holds in the sense of generalized conditional expectations; that is, with the definition

$$E_P[M_{t+1}|\mathcal{F}_t] := \lim_{n\to\infty} E_P[M_{t+1}^+ \wedge n|\mathcal{F}_t] - \lim_{n\to\infty} E[M_{t+1}^- \wedge n|\mathcal{F}_t]$$

and the convention that $\infty - \infty = -\infty$. To wit, if $M_T \in L^1(P)$, then $M$ is simply a martingale in the usual sense. We refer to [34] for further background and the proof of the following fact.

**Lemma A.1.** *Let $M$ be an adapted process with $M_0 = 0$. The following are equivalent:*

*(i) $M$ is a local martingale.*

*(ii) $H \bullet M$ is a local martingale whenever $H$ is predictable.*

*(iii) $M$ is a generalized martingale.*

*If $E_P[M_T^-] < \infty$, these conditions are further equivalent to:*

*(iv) $M$ is a martingale.*

Let $S$ be an adapted process with values in $\mathbb{R}^d$. The following is a standard consequence of Lemma A.1.

**Lemma A.2.** *Let $f$ and $g = (g^1, \dots, g^e)$ be $\mathcal{F}$-measurable and let $Q$ be a probability measure such that $E_Q[g^i] = 0$ for all $i$ and $S - S_0$ is a local $Q$-martingale. If there exist $x \in \mathbb{R}$ and $(H, h) \in \mathcal{H} \times \mathbb{R}^e$ such that $x + H \bullet S_T + hg \geq f$ $Q$-a.s., then $E_Q[f] \leq x$.*

*Proof.* If $E_Q[f^-] = \infty$, then $E_Q[f] = -\infty$ by our convention (1.1) and the claim is trivial. Suppose that $E_Q[f^-] < \infty$; then $H \bullet S_T \geq f - x - hg$ implies $(H \bullet S_T)^- \in L^1(Q)$. Therefore, $H \bullet S$ is a $Q$-martingale by Lemma A.1 and we conclude that $x = E_P[x + H \bullet S_T + hg] \geq E_Q[f]$. □

**Lemma A.3.** *Let $Q$ be a probability measure under which $S - S_0$ is a local martingale, let $\varphi \geq 1$ be a random variable, and let $f$ be a random variable satisfying $|f| \leq \varphi$. There exists a probability measure $Q' \sim Q$ under which $S$ is a martingale, $E_{Q'}[\varphi] < \infty$, and $E_{Q'}[f] \geq E_Q[f]$.*

*Proof.* As $Q$ is a local martingale measure, the classical no-arbitrage condition $\mathrm{NA}(\{Q\})$ holds. By Lemma 3.2 there exists a probability $P_* \sim Q$ such that $E_{P_*}[|\varphi|] < \infty$. We shall use $P_*$ as a reference measure; note that $\mathrm{NA}(\{P_*\})$ is equivalent to $\mathrm{NA}(\{Q\})$. It suffices to show that

$$\sup_{Q \in \mathcal{Q}^*_{loc}} E_Q[f] \le \pi(f) \le \sup_{Q \in \mathcal{Q}^*_{\varphi}} E_Q[f],$$

where $\pi(f)$ is the $P_*$-a.s. superhedging price, $\mathcal{Q}^*_{loc}$ is the set of all $Q \sim P_*$ such that $S - S_0$ is a local $Q$-martingale, and $\mathcal{Q}^*_{\varphi}$ is the set of all $Q \sim P_*$ such that $E_Q[\varphi] < \infty$ and $S$ is a $Q$-martingale. The first inequality follows from Lemma A.2. The second inequality corresponds to a version of the classical superhedging theorem with an additional weight function. It can be obtained by following the classical Kreps–Yan separation argument, where the usual space $L^1(P)$ is replaced with the weighted space $L^1_{1/\varphi}(P_*)$ of random variables $X$ such that $E_P[|X/\varphi|] < \infty$. Indeed, the dual $(L^1_{1/\varphi}(P_*))^*$ with respect to the pairing $(X, Z) \mapsto E_P[XZ]$ is given by the space of all random variables $Z$ such that $Z\varphi$ is $P$-a.s. bounded. As a consequence, the separation argument yields a martingale measure $Q \in \mathcal{Q}^*_{\varphi}$, for if $Z \in (L^1_{1/\varphi}(P_*))^*$ is positive and normalized such that $E_P[Z] = 1$, then the measure $Q$ defined by $dQ/dP = Z$ satisfies $E_Q[\varphi] = E_P[Z\varphi] < \infty$. The arguments are known (see, e.g., [51, Theorem 4.1]) and so we shall omit the details. □

# References


[1] B. Acciaio, M. Beiglböck, F. Penkner, and W. Schachermayer. A model-free version of the fundamental theorem of asset pricing and the super-replication theorem. *To appear in Math. Finance*, 2013.

[2] C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer, Berlin, 3rd edition, 2006.

[3] M. Beiglböck, P. Henry-Labordère, and F. Penkner. Model-independent bounds for option prices: a mass transport approach. *Finance Stoch.*, 17(3):477–501, 2013.

[4] M. Beiglböck and N. Juillet. On a problem of optimal transport under marginal martingale constraints. *To appear in Ann. Probab.*, 2012.

[5] M. Beiglböck and M. Nutz. Martingale inequalities and deterministic counterparts. *Preprint arXiv:1401.4698v1*, 2014.

[6] D. P. Bertsekas and S. E. Shreve. *Stochastic Optimal Control. The Discrete-Time Case.* Academic Press, New York, 1978.

[7] S. Biagini and R. Cont. Model-free representation of pricing rules as conditional expectations. In *Stochastic processes and applications to mathematical finance*, pages 53–66. World Sci. Publ., Hackensack, NJ, 2007.

[8] B. Bouchard, L. Moreau, and M. Nutz. Stochastic target games with controlled loss. *To appear in Ann. Appl. Probab.*, 2012.

[9] D. T. Breeden and R. H. Litzenberger. Prices of state-contingent claims implicit in option prices. *J. Bus.*, 51(4):621–651, 1978.

[10] H. Brown, D. Hobson, and L. C. G. Rogers. Robust hedging of barrier options. *Math. Finance*, 11(3):285–314, 2001.

[11] H. Buehler. Expensive martingales. *Quant. Finance*, 6(3):207–218, 2006.

[12] P. Carr, K. Ellis, and V. Gupta. Static hedging of exotic options. *J. Finance*, 53:1165–1191, 1998.

[13] P. Carr and D. Madan. A note on sufficient conditions for no arbitrage. *Finance Res. Lett.*, 2:125–130, 2005.

[14] L. Cousot. Conditions on option prices for absence of arbitrage and exact calibration. *J. Bank. Finance*, 31:3377–3397, 2007.

[15] A. M. G. Cox and J. Obłój. Robust pricing and hedging of double no-touch options. *Finance Stoch.*, 15(3):573–605, 2011.

[16] R. C. Dalang, A. Morton, and W. Willinger. Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stochastics Stochastics Rep.*, 29(2):185–201, 1990.

[17] M. H. A. Davis and D. Hobson. The range of traded option prices. *Math. Finance*, 17(1):1–14, 2007.

[18] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 300:463–520, 1994.

[19] F. Delbaen and W. Schachermayer. *The Mathematics of Arbitrage*. Springer, Berlin, 2006.

[20] C. Dellacherie and P. A. Meyer. *Probabilities and Potential B*. North Holland, Amsterdam, 1982.

[21] L. Denis and C. Martini. A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. *Ann. Appl. Probab.*, 16(2):827–852, 2006.

[22] S. Deparis and C. Martini. Superhedging strategies and balayage in discrete time. In *Seminar on Stochastic Analysis, Random Fields and Applications IV*, volume 58 of *Progr. Probab.*, pages 205–219. Birkhäuser, Basel, 2004.

[23] Y. Dolinsky. Hedging of game options under model uncertainty in discrete time. *To appear in Electron. J. Probab.*, 2013.

[24] Y. Dolinsky and H. M. Soner. Martingale optimal transport and robust hedging in continuous time. *To appear in Probab. Theory Related Fields*, 2012.

[25] Y. Dolinsky and H. M. Soner. Robust hedging with proportional transaction costs. *To appear in Finance Stoch.*, 2013.

[26] B. Dupire. Pricing with a smile. *Risk*, 7:18–20, 1994.

[27] D. Fernholz and I. Karatzas. Optimal arbitrage under model uncertainty. *Ann. Appl. Probab.*, 21(6):2191–2225, 2011.

[28] H. Föllmer and A. Schied. *Stochastic Finance: An Introduction in Discrete Time*. W. de Gruyter, Berlin, 2nd edition, 2004.

[29] D. H. Fremlin. *Measure Theory*, volume 4. Torres Fremlin, Colchester, 2003.

[30] A. Galichon, P. Henry-Labordère, and N. Touzi. A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *Ann. Appl. Probab.*, 24(1):312–336, 2014.

[31] P. Henry-Labordère and N. Touzi. An explicit martingale version of Brenier's theorem. *Preprint arXiv:1302.4854v1*, 2013.

[32] D. Hobson. Robust hedging of the lookback option. *Finance Stoch.*, 2(4):329–347, 1998.

[33] D. Hobson. The Skorokhod embedding problem and model-independent bounds for option prices. In *Paris-Princeton Lectures on Mathematical Finance 2010*, volume 2003 of *Lecture Notes in Math.*, pages 267–318. Springer, Berlin, 2011.

[34] J. Jacod and A. N. Shiryaev. Local martingales and the fundamental asset pricing theorems in the discrete-time case. *Finance Stoch.*, 2(3):259–273, 1998.

[35] Yu. Kabanov and C. Stricker. A teachers' note on no-arbitrage criteria. In *Séminaire de Probabilités, XXXV*, volume 1755 of *Lecture Notes in Math.*, pages 149–152. Springer, Berlin, 2001.

[36] Yu. Kabanov and Ch. Stricker. The Dalang–Morton–Willinger theorem under delayed and restricted information. In *In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*, volume 1874 of *Lecture Notes in Math.*, pages 209–213. Springer, Berlin, 2006.

[37] N. El Karoui and M.-C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.*, 33(1):29–66, 1995.

[38] D. Kramkov. Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probab. Theory Related Fields*, 105(4):459–479, 1996.

[39] S. J. Leese. Measurable selections and the uniformization of Souslin sets. *Amer. J. Math.*, 100(1):19–41, 1978.

[40] A. Matoussi, L. Piozin, and D. Possamai. Second-order BSDEs with general reflection and Dynkin games under uncertainty. *Preprint arXiv:1212.0476v1*, 2012.

[41] A. Matoussi, D. Possamai, and C. Zhou. Second order reflected backward stochastic differential equations. *Preprint arXiv:1201.0746v1*, 2012.

[42] A. Neufeld and M. Nutz. Superreplication under volatility uncertainty for measurable claims. *Electron. J. Probab.*, 18(48):1–14, 2013.

[43] M. Nutz. Random $G$-expectations. *Ann. Appl. Probab.*, 23(5):1755–1777, 2013.

[44] M. Nutz. Superreplication under model uncertainty in discrete time. *Preprint*

*arXiv:1301.3227v1*, 2013.

[45] M. Nutz and H. M. Soner. Superhedging and dynamic risk measures under volatility uncertainty. *SIAM J. Control Optim.*, 50(4):2065–2089, 2012.

[46] M. Nutz and R. van Handel. Constructing sublinear expectations on path space. *Stochastic Process. Appl.*, 123(8):3100–3121, 2013.

[47] M. Nutz and J. Zhang. Optimal stopping under adverse nonlinear expectation and related games. *Preprint arXiv:1212.2140v1*, 2012.

[48] S. Peng. Nonlinear expectations and stochastic calculus under uncertainty. *Preprint arXiv:1002.4546v1*, 2010.

[49] D. Possamai, G. Royer, and N. Touzi. On the robust superhedging of measurable claims. *Preprint arXiv:1302.1850v1*, 2013.

[50] R. T. Rockafellar. Integral functionals, normal integrands and measurable selections. In *Nonlinear Operators and the Calculus of Variations*, volume 543 of *Lecture Notes in Math.*, pages 157–207, Springer, Berlin, 1976.

[51] W. Schachermayer. The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. *Math. Finance*, 14(1):19–48, 2004.

[52] H. M. Soner, N. Touzi, and J. Zhang. Martingale representation theorem for the $G$-expectation. *Stochastic Process. Appl.*, 121(2):265–287, 2011.

[53] H. M. Soner, N. Touzi, and J. Zhang. Dual formulation of second order target problems. *Ann. Appl. Probab.*, 23(1):308–347, 2013.

[54] Y. Song. Some properties on $G$-evaluation and its applications to $G$-martingale decomposition. *Sci. China Math.*, 54(2):287–300, 2011.

[55] D. Stroock and S. R. S. Varadhan. *Multidimensional Diffusion Processes.* Springer, New York, 1979.

[56] X. Tan and N. Touzi. Optimal transportation under controlled stochastic dynamics. *Ann. Probab*, 41(5):3201–3240, 2013.

[57] D. H. Wagner. Survey of measurable selection theorems. *SIAM J. Control Optimization*, 15(5):859–903, 1977.

[58] D. H. Wagner. Survey of measurable selection theorems: an update. In *Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979)*, volume 794 of *Lecture Notes in Math.*, pages 176–219. Springer, Berlin, 1980.