A Risk-Neutral Equilibrium Leading to Uncertain Volatility Pricing*

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Abstract

We study the formation of derivative prices in equilibrium between risk-neutral agents with heterogeneous beliefs about the dynamics of the underlying. Under the condition that the derivative cannot be shorted, we prove the existence of a unique equilibrium price and show that it incorporates the speculative value of possibly reselling the derivative. This value typically leads to a bubble; that is, the price exceeds the autonomous valuation of any given agent. Mathematically, the equilibrium price operator is of the same nonlinear form that is obtained in single-agent settings with strong aversion against model uncertainty. Thus, our equilibrium leads to a novel interpretation of this price.

Keywords Heterogeneous Beliefs, Equilibrium, Derivative Price Bubble, Uncertain Volatility Model, Nonlinear Expectation

AMS 2010 Subject Classification 91B51; 91G20; 93E20

1 Introduction

Starting with [2, 28], robust option pricing considers a class of plausible models for the underlying security and seeks strategies that hedge against the model risk. As a result, the associated pricing operator is apparently linked to extreme caution, making it difficult to explain how trades can be initiated at such quotes. In the Uncertain Volatility Model of [2, 28],

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the price corresponds to a model that selects the worst-case volatility from a given range of volatility models at any point in time, thus leading to a Black–Scholes–Barenblatt pricing equation. The non-Markovian version of this operator is known as the G-expectation [36, 37]. More recently, a rich literature considering a variety of hedging instruments and underlying models has emerged; see, among many others, [1, 3, 6, 7, 16, 32] for models in discrete time and [5, 8, 10, 9, 11, 12, 15, 18, 19, 20, 21, 23, 24, 30, 33] for continuous-time models. We refer to [22, 35] for surveys.

In this paper, we show that the same prices also arise as unique equilibria for agents that worry neither about risk nor uncertainty, but instead disagree about the dynamics of the underlying. Thus, in our model, trades occur naturally at prices of the uncertain volatility type. From our point of view, the nonlinearity in the price reflects a speculative component that is added to the fundamental value of the derivative: the agents take into account that they may sell the derivative to an agent with different beliefs at a later point in time. This possibility is know as the "resale option" in the Economics literature.

The basic idea is that if a security exists in finite supply and cannot be shorted, equilibrium prices will reflect the most optimistic belief and therefore have an upward bias. This can be traced back to the static model of [29]. In a dynamic model, the relative optimism or pessimism of the agents changes over time, giving rise to the resale option and causing the agents to trade. This insight is already present in the discrete-time model of [17], where agents disagree about the probability distribution of dividends paid by an asset, and is worked out very elegantly in the continuous-time model of [39] where utility-maximizing agents disagree about the drift rate of an asset; see also [4] for a finite-horizon version of this model. We refer to [40] for a comprehensive survey of this literature, and to [38] for a a review of works on bubbles in Mathematical Finance.

To the best of our knowledge, the present paper is the first to study bubbles in derivatives. In this context, the paradigm of risk-neutral pricing provides a clear definition of fundamental value and therefore of a bubble. Moreover, risk-neutrality results in a great deal of tractability which will allow us to give a simple description of the agents' trading strategies for general models and derivative payoffs. In the remainder of this introduction, we sketch the main ideas of our approach in a simple case with two agents that use local volatility models for the underlying. In the body of the paper, we shall derive our results for n agents with general Markov models.

Our starting point is an underlying security that can be traded without friction at an exogenous martingale price X. The goal is to find an equilibrium price at time $t = 0$ for a derivative written on X, with payoff $f(X(T))$ at maturity T. The derivative is in unit supply and can be traded in continuous time by two agents $i \in \{1, 2\}$. We assume that the derivative cannot be shorted, but refer to Remark 2.6 for a possible relaxation. Each agent is risk-neutral, has their own stochastic model for the dynamics of X, and maximizes the P&L from trading in the derivative. (It is quite natural to allow the agents to trade in the underlying as well, but due to risk-neutrality and the martingale property of X, that would not affect our results.) Specifically, agent i uses a local vol model Q_i for X under which

$$dX(t) = \sigma_i(t, X(t))\, dW_i(t), \quad X(0) = x,$$

for some Brownian motion W_i. Given a price process Z for the derivative, agent i chooses a trading strategy H_i to maximize the expected P&L

$$E_i \left[\int_0^T H_i(t)\, dZ(t) \right].$$

The process Z is an equilibrium price if $Z(T) = f(X(T))$ matches the value of the derivative at the maturity and there exist strategies H_1, H_2 which are optimal for the agents and clear the market: $H_1 + H_2 = 1$, since the derivative is in unit supply.

In this setting, each agent's model is complete, so they both have a well-defined notion of a fundamental price. Indeed, agent i's fundamental valuation is the Q_i-expectation $E_i[f(X(T))]$ of the claim which can be found via the solution v_i of the linear PDE

$$\partial_t v(t, x) + \frac{1}{2}\sigma_i^2(t, x)\partial_{xx} v(t, x) = 0, \quad v(T, \cdot) = f.$$

If the derivative can be traded only at the initial time $t = 0$, the equilibrium price is the larger value $\max\{v_1(0, x), v_2(0, x)\}$ of the agents' valuations, since at this price it is optimal for the agent with the higher valuation to hold the derivative and, in view of the no-shorting constraint, holding zero units is optimal for the more pessimistic agent.

In our dynamic model, however, the role of the relative optimist may change depending on the state (t, x), which gives rise to the resale option. We shall show that the equilibrium price is given by the nonlinear PDE

$$\partial_t v(t, x) + \sup_{i=1,2} \frac{1}{2}\sigma_i^2(t, x)\partial_{xx} v(t, x) = 0 \tag{1.1}$$

which corresponds to choosing the more optimistic volatility at any state (t, x). Since this may change between the agents along a trajectory $(t, X(t))$

of the underlying, for instance if the functions σ_i are not ordered or if the function f is not concave or convex, the equilibrium price is typically higher than *both* fundamental valuations—the difference is the value of the resale option or the speculative bubble, since it can be attributed to the possibility of future trading. It is worth noting that the bubble arises in a finite horizon setting where the agents agree about the value $f(X(T))$ at maturity, and despite symmetric information.

We observe that the PDE (1.1) coincides with the Black–Scholes–Barenblatt PDE for an uncertain volatility model with a range $[\underline{\sigma}, \overline{\sigma}]$ of volatilities, where

$$\underline{\sigma}(t,x) = \min\{\sigma_1(t,x), \sigma_2(t,x)\}, \quad \overline{\sigma}(t,x) = \max\{\sigma_1(t,x), \sigma_2(t,x)\},$$

because

$$\sup_{i=1,2} \frac{1}{2}\sigma_i^2(t,x)\partial_{xx}v(t,x) = \sup_{a\in[\underline{\sigma}^2(t,x),\overline{\sigma}^2(t,x)]} \frac{1}{2}a\partial_{xx}v(t,x)$$

are the very same operator. Alternately, this is the G-expectation if σ_i are constant, and the random G-expectation [31, 34] in the general case. In the uncertain volatility setting, the PDE is interpreted as choosing the worst-case volatility within the interval $[\underline{\sigma}, \overline{\sigma}]$ at any state. In our setting, one may think of an imaginary agent that has the more optimistic view among $i \in \{1,2\}$ at any state. Our risk-neutral setting is particularly tractable in that the trades correspond directly to the volatility; indeed, we shall see that the strategies $H_i(t) = h_i(t, X(t))$ are optimal, where

$$h_i(t,x) = \begin{cases} 1, & \text{if } i \text{ is the unique maximizer in (1.1)}, \\ 1/2, & \text{if both } j=1 \text{ and } j=2 \text{ are maximizers}, \\ 0, & \text{else.} \end{cases}$$

(We have chosen a symmetric splitting when both volatilities are maximizers, but any splitting rule will do.) Thus, we see that in any state (t,x), the derivative is held by the more optimistic agent. Or, if we introduced the above imaginary agent in the market, it would be optimal for that agent to hold the derivative at all times. Therefore, it is natural that this agent's valuation becomes the effective pricing mechanism in equilibrium.

The remainder of the article is organized as follows. In Section 2, the above theory is established for n agents using general, multidimensional Markovian models. Theorem 2.3 identifies equilibrium prices with solutions of a PDE, whereas Proposition 2.4 interprets the PDE as a control problem and, in particular, shows uniqueness. Corollary 2.5 presents regularity conditions under which existence and uniqueness can be deduced easily from general PDE results. In Section 3, we present a solvable example with stochastic volatility models of Heston-type where the trading strategies can be described explicitly. The strategies provide some intuition for the agents' resale options and show that trading does indeed occur even for derivatives with convex payoffs, in all but the simplest models.

2 General Model and Main Result

Departing slightly from the above notation, we consider $n \geq 1$ agents and a d-dimensional underlying X. The components of X represent quantities that may or may not be tradable, and thus it is meaningful to allow for non-zero drift. Specifically, let X be the canonical process on $\Omega = C([0,T],\mathbb{R}^d)$ for some time horizon $T > 0$, where Ω is equipped with the canonical filtration and σ-field. For each $1 \leq i \leq n$, we are given a probability Q_i on Ω under which

$$dX(t) = b_i(t, X(t))\,dt + \sigma_i(t, X(t))\,dW_i(t), \quad X(0) = x, \tag{2.1}$$

where W_i is a Brownian motion of (possibly different) dimension d'. We assume that the d-dimensional vector b_i and the $d \times d'$ matrix σ_i are continuous functions of $(t,x) \in [0,T] \times \mathbb{R}^d$ which are Lipschitz continuous (and hence of linear growth) in x, uniformly in t. As a result, the SDE (2.1) has a unique solution and

$$E_i\left[\sup_{t \leq T} |X(t)|^p\right] < \infty, \quad p \geq 0, \tag{2.2}$$

where $E_i[\,\cdot\,]$ denotes the expectation operator under Q_i. See [26, Section 2.5] for these facts.

Definition 2.1. An *admissible* strategy H is a nonnegative, bounded, predictable process, and we write $\mathcal{A}$ for the collection of all these strategies. Given a semimartingale Z under Q_i (to be thought of as the price process of the derivative), a strategy $H_i \in \mathcal{A}$ is *optimal* for agent i if

$$E_i\left[\int_0^T H(t)\,dZ(t)\right] \leq E_i\left[\int_0^T H_i(t)\,dZ(t)\right] < \infty \quad \text{for all} \quad H \in \mathcal{A}.$$

Here and in what follows, we use the convention that $E_i[Y] := -\infty$ whenever $E_i[Y^-] = \infty$, for any random variable Y.

Definition 2.2. Given a function $f : \mathbb{R}^d \to \mathbb{R}$, a process Z is an *equilibrium price* if Z is a semimartingale with $Z(T) = f(X(T))$ a.s. under Q_i for all i and there exist admissible strategies H_i which are optimal and clear the market; i.e.,

$$\sum_{i=1}^{n} H_i(t) = 1, \quad t \in [0,T].$$

To state the main result, let us write

$$C_p^{1,2} := C^{1,2}([0,T) \times \mathbb{R}^d) \cap C_p([0,T] \times \mathbb{R}^d)$$

for the set of continuous functions $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ that satisfy the polynomial growth condition $|u(t,x)| \leq c(1+|x|^p)$ for some $c, p \geq 0$ and admit continuous partial derivatives $\partial_t u$, $\partial_{x_i} u$, $\partial_{x_i x_j} u$ on $[0,T) \times \mathbb{R}^d$. Moreover, we set

$$\mathcal{S} = \bigcap_{i=1}^{d} \mathcal{S}_i, \quad \mathcal{S}_i = \big\{(t,x) \in [0,T) \times \mathbb{R}^d : x \in \operatorname{supp}_{Q_i} X(t)\big\},$$

where $\operatorname{supp}_{Q_i} X(t)$ is the topological support of $X(t)$ under Q_i, and let $\overline{\mathcal{S}}$ denote the closure in $[0,T) \times \mathbb{R}^d$. Similarly, $\mathcal{S}_T = \cap_i \operatorname{supp}_{Q_i} X(T)$; this set is already closed.

We fix a payoff function $f \in C_p(\mathbb{R}^d)$ for the remainder of this section. Our main result identifies equilibrium prices for f with solutions of a PDE; existence and uniqueness will be addressed subsequently.

Theorem 2.3. *(i) Suppose that the PDE*

$$\partial_t v(t,x) + \sup_{i \in \{1,\dots,n\}} \Big\{ b_i \partial_x v(t,x) + \frac{1}{2} \operatorname{Tr}[\sigma_i \sigma_i^\top(t,x) \partial_{xx} v(t,x)] \Big\} = 0 \qquad (2.3)$$

with terminal condition $v(T,\cdot) = f$ has a solution $v \in C_p^{1,2}$. Then, an equilibrium price is given by $Z(t) = v(t,X(t))$. Moreover, the strategies given by $H_i(t) = h_i(t,X(t))$ are optimal, where

$$h_i(t,x) = \begin{cases} 1/m, & \text{if } i \text{ is a maximizer in (2.3)} \\ & \text{and } m \text{ is the total number of maximizers,} \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Conversely, let $v \in C_p^{1,2}$ and suppose that $Z(t) = v(t,X(t))$ is an equilibrium price. Then, v solves the PDE (2.3) on $\overline{\mathcal{S}}$ and satisfies the terminal condition $v(T,\cdot) = f$ on $\mathcal{S}_T$.

Deferring the proof of Theorem 2.3 to the end of this section, we observe that the PDE (2.3) suggests the following control problem. On a given filtered probability space carrying a d'-dimensional Brownian motion W, let Θ be the set of all predictable processes with values in $\{1,\dots,n\}$. For each $\theta\in\Theta$, let $X^{t,x}_\theta(s)$, $s\in[t,T]$ be the solution of the controlled SDE

$$dX(s)=b_{\theta(s)}(s,X(s))\,ds+\sigma_{\theta(s)}(s,X(s))\,dW(s),\quad X(t)=x.$$

It follows from the assumptions on the coefficients b_i,σ_i that this SDE with random coefficients indeed has a unique strong solution which again satisfies (2.2); cf. [26, Section 2.5]. Therefore, we may consider the stochastic control problem

$$V(t,x)=\sup_{\theta\in\Theta}E[f(X^{t,x}_\theta(T))],\quad (t,x)\in[0,T]\times\mathbb{R}^d. \tag{2.4}$$

Standard arguments of stochastic control show that $V\in C_p([0,T]\times\mathbb{R}^d)$ and that V is a viscosity solution of the PDE (2.3) with terminal condition f. However, V need not be smooth in general, and differentiability is relevant in the context of Theorem 2.3 in order to define the agents' strategies and thus, an equilibrium.

Proposition 2.4. *Let $v\in C^{1,2}_p$ be a solution of the PDE* (2.3) *with terminal condition $v(T,\cdot)=f$. Then, v coincides with the value function V of the control problem* (2.4) *and any (measurable) selector*

$$\theta(s,x)\in\operatorname*{arg\,max}_{i\in\{1,\dots,n\}}\Big\{b_i\partial_x v(t,x)+\frac{1}{2}\operatorname{Tr}[\sigma_i\sigma_i^\top(t,x)\partial_{xx}v(t,x)]\Big\}$$

defines an optimal control in feedback form. In particular, uniqueness holds for (2.3) *in the class $C^{1,2}_p$.*

Proof. Since $\{1,\dots,n\}$ is a finite set, the $\arg\max$ is nonempty and we may find a semicontinuous (thus measurable) selector, for instance by choosing the smallest index i in the $\arg\max$. Thus, the claim follows by a standard verification argument; cf. [13, Theorem IV.3.1, p. 157]. □

The proposition gives an interpretation for the equilibrium in Theorem 2.3: the same price would be found by an imaginary agent who prices by taking expectations under a model Q that uses, infinitesimally at any point in time, the drift and volatility coefficients b_i,σ_i that lead to the highest price among the given models $\{1,\dots,n\}$.

Let us now establish existence (and uniqueness) when the inputs are sufficiently smooth. We write $C_b^{1,2}$ for the set of $u \in C^{1,2}([0,T) \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$ such that u, $\partial_t u$, $\partial_x u$, $\partial_{xx} u$ are bounded. Moreover, we recall that a function $y \mapsto A(y)$ with values in the set of $d \times d$ positive symmetric matrices is called uniformly elliptic if there exists a constant $c > 0$ such that $\xi^\top A(y)\xi \geq c|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and all y.

Corollary 2.5. *Suppose that f is bounded, that $b_i, \sigma_i \in C_b^{1,2}$ and that $\sigma_i\sigma_i^\top$ is uniformly elliptic for $1 \leq i \leq n$. Then $\overline{\mathcal{S}} = [0,T) \times \mathbb{R}^d$, $\mathcal{S}_T = \mathbb{R}^d$ and the PDE* (2.3) *has a unique solution $v \in C_p^{1,2}$ with terminal condition f.*

In particular, there exists a unique equilibrium price $Z(t) = v(t, X(t))$ with $v \in C_p^{1,2}$.

Proof. Since b_i is bounded and $\sigma_i\sigma_i^\top$ is uniformly elliptic, the support of Q_i in Ω is the set of all paths $\omega \in C([0,T], \mathbb{R}^d)$ with $\omega(0) = x$; see [41, Theorem 3.1]. The claims regarding $\mathcal{S}$ are a direct consequence.

Turning to the PDE, it follows from [27, Theorem 6.4.3, p. 301] that (2.3) with terminal condition f has a (bounded) solution $v \in C_p^{1,2}$; the conditions in the cited theorem can be verified along the lines of [27, Example 6.1.4, p. 279]. Uniqueness of the solution was already noted in Proposition 2.4, and now the last assertion follows from Theorem 2.3. $\square$

Proof of Theorem 2.3. (i) We have $H_i \in \mathcal{A}$, the market clears and $Z(T) = f(X(T))$. Thus, we fix i and show that H_i is optimal. In view of $v \in C^{1,2}$ and Itô's formula, the process Z admits an Itô decomposition

$$dZ(t) = dA_i(t) + dM_i(t) = \mu_i(t, X(t))\, dt + \partial_x v(t, X_t)\sigma_i(t, X(t))\, dW_i(t) \quad (2.5)$$

for $t \in [0,T)$, where

$$\mu_i(t,x) = \partial_t v(t,x) + b_i \partial_x v(t,x) + \frac{1}{2}\operatorname{Tr}[\sigma_i\sigma_i^\top(t,x)\partial_{xx} v(t,x)]. \quad (2.6)$$

As v solves the PDE (2.3), we deduce that $\mu_i(t,x) \leq 0$. Next, we shall show that $\int H\, dZ$ is a Q_i-supermartingale for every $H \in \mathcal{A}$ and a martingale for $H = H_i$, and as a consequence,

$$E_i\left[\int_0^T H(t)\, dZ(t)\right] \leq 0 = E_i\left[\int_0^T H_i(t)\, dZ(t)\right] < \infty, \quad H \in \mathcal{A},$$

so that H_i is optimal as desired. Indeed, $\mu_i(t,x) \leq 0$ shows that Z is a local supermartingale. As $v \in C_p([0,T] \times \mathbb{R}^d)$, the existence of the moments (2.2) yields that $Z^* := \sup_{t \in [0,T]} |Z(t)| \in L^1(Q_i)$. In particular, Z is of class D and

and thus its (Doob–Meyer) decomposition satisfies $|A_i(T)| = A_i^* \in L^1(Q_i)$; cf. [25, Theorem 1.4.10, p. 24]. As a consequence, dropping the index i for brevity,

$$E[M^*] \le |Z(0)| + E[Z^*] + E[A^*] < \infty.$$

The BDG inequalities [25, Theorem 3.3.28, p. 166] now show that for any bounded predictable process H,

$$E[(H \bullet M)^*] \preceq E[(H^2 \bullet \langle M \rangle)(T)^{1/2}] \preceq E[\langle M \rangle (T)^{1/2}] \preceq E[M^*] < \infty,$$

where $\preceq$ denotes inequality up to a constant and $\bullet$ denotes integration. As a result, $H \bullet M$ is a true martingale. By (2.6), the definition of h_i entails that $h_i \mu_i = 0$, and we conclude that $\int H_i \, dZ = \int H_i \, dM$ is a Q_i-martingale, whereas for a general $H \in \mathcal{A}$, nonnegativity of H yields $H(t)\mu_i(t, X(t)) \le 0$ and thus $\int H \, dZ = \int H \, dA + \int H \, dM$ is a supermartingale. This completes the proof of (i).

(ii) Let $v \in C_p^{1,2}$ and suppose that $Z(t) = v(t, X(t))$ is an equilibrium price. Then, as $Z(T) = f(T, X(T))$ Q_i-a.s. for all i, the terminal condition $v(T, \cdot) = f$ holds on $\mathcal{S}_T$.

Under Q_i, Z again admits a decomposition (2.5)–(2.6), and our first goal is to show that $\beta_i(t) := \mu_i(t, X(t)) \le 0$ Q_i-a.s. Suppose for contradiction that $Q_i\{\beta_i(t) > 0\} > 0$ for some t. Then, we can find stopping times $\tau_1 \le \tau_2$ such that $\beta_i > 0$ on $[\tau_1, \tau_2]$ and $Q_i\{\tau_1 < \tau_2\} > 0$, for instance by setting

$$\tau_1 = \inf\{t \ge 0 : \beta_i(t) \ge \varepsilon\} \wedge T, \quad \tau_2 = \inf\{t \ge \tau_1 : \beta_i(t) \le \varepsilon/2\} \wedge T$$

for small enough $\varepsilon > 0$ and noting that β_i has continuous paths. Moreover, if (τ^k) is a localizing sequence for the local martingale M, the stopping times $\tau_1 \wedge \tau^k$ and $\tau_1 \wedge \tau^k$ still have the desired properties for large enough k, so we may assume that the stopped process $M(\cdot \wedge \tau_2)$ is a true martingale. As a result, the strategy defined by $H^\lambda(t) = \lambda \mathbf{1}_{]\tau_1, \tau_2]}$ for $\lambda > 0$ is admissible for agent i and satisfies

$$E_i \left[\int_0^T H^\lambda(t) \, dZ(t) \right] = \lambda E_i \left[\int_0^T H^1(t) \beta_i(t) \, dt \right] > 0.$$

The left-hand side can be made arbitrarily large by increasing λ, a contradiction to our assumption that Z is an equilibrium price. We have therefore shown that $\beta_i(t) \le 0$ Q_i-a.s. for all $t < T$, and hence $\mu_i \le 0$ on $\mathcal{S}$, by continuity.

In particular, as in (i), $\int H \, dZ$ is a supermartingale for any $H \in \mathcal{A}$. On the other hand, as the strategy $H \equiv 0$ is an admissible choice, the optimal strategy H_i of agent i must satisfy $E_i[\int_0^T H_i\,dZ] \geq 0$, so that the process $\int H_i\,dZ$ is a martingale. In particular,

$$H_i(t)\mu_i(t, X(t)) = 0 \quad (dt \times Q_i)\text{-a.e.} \quad \text{for all } i.$$

Since $H_i \geq 0$ and $\sum_i H_i = 1$, it follows that

$$\sup_i \mu_i = 0 \quad \text{on} \quad \mathcal{S}$$

as the functions μ_i are continuous. In view of (2.6), this is precisely the claimed PDE on $\mathcal{S}$, and it extends to $\overline{\mathcal{S}}$ by continuity. $\square$

Remark 2.6. While we have assumed that the derivative cannot be shorted, the same equilibrium price arises if shorting is possible but incurs a sufficiently large cost; for instance, a quadratic instantaneous cost. In this case, the price is still given by the same PDE (2.3), although the agents' optimal strategies change. We have chosen the more stringent setting for this paper because its optimal strategies create a clearer analogy to the Uncertain Volatility Model.

3 Example with Stochastic Volatility

In this section, we solve an example where two agents use stochastic volatility models of Heston-type and disagree about the speed of mean reversion in the volatility process. Classical rational expectations models with homogeneous beliefs typically lead to no-trade equilibria, as surveyed in [40, Section 4]. In the present context, the simplest example where each agent believes in a different Bachelier (or Black–Scholes) model with constant volatility, also leads to a no-trade equilibrium for a convex option payoff f, because the agent expecting the highest volatility will then hold the derivative at all times. The example presented here illustrates that this pathology typically disappears in more complex models. Indeed, we shall see that, with heterogeneous beliefs about the mean-reversion speed of the volatility, a derivative with convex payoff is traded whenever the volatility process crosses the mean reversion level—which happens with positive probability on any time interval.

Using the customary notation (S, Y) instead of $X = (X^1, X^2)$, we consider the two-dimensional SDE

$$\begin{aligned} dS(t) &= \alpha(Y(t))\,dW(t), \quad S(0) = s, \\ dY(t) &= \lambda_i(\bar{Y} - Y(t))\,dt + \beta(Y(t))\,dZ(t), \quad Y(0) = y, \end{aligned}$$

where S represents the spot price of the underlying and Y is the non-tradable process driving the volatility of S. Here, W and Z are independent Brownian motions and the positive functions α, β are such that α^2, β^2 are Lipschitz-continuous and uniformly bounded away from zero; moreover, α is increasing. The mean-reversion level $\bar{Y} \in \mathbb{R}$ is common to both agents, whereas the speed of mean reversion $\lambda_i > 0$ depends on the agent $i \in \{1, 2\}$; for concreteness, we suppose that $\lambda_1 > \lambda_2$. Finally, the option is given by $f(S(T))$ for a convex payoff function $s \mapsto f(s)$ of polynomial growth; a typical example is a call or put option.

Proposition 3.1. *In the stated model, there exists a unique equilibrium price* $Z(t) = v(t, S(t), Y(t))$ *with* $v \in C_p^{1,2,2}$*, and the strategies given by*

$$H_1(t) = \begin{cases} 1, & Y(t) < \bar{Y}, \\ 1/2, & Y(t) = \bar{Y}, \\ 0, & Y(t) > \bar{Y} \end{cases}$$

and $H_2(t) = 1 - H_1(t)$ *are optimal. That is, the agent with faster (slower) mean reversion holds the option whenever* Y *is below (above) the level of mean reversion.*

This result confirms the intuition that at any given time, the agent expecting a higher future volatility will hold the derivative: when $Y(t) < \bar{Y}$, a faster mean reversion indeed corresponds to a higher expectation about the future volatility, and vice versa. As a result, the derivative is traded whenever Y crosses the level $\bar{Y}$.

Proof of Proposition 3.1. The PDE (2.3) for this example reads

$$\partial_t v + \frac{\alpha^2}{2}\partial_{ss} v + \frac{\beta^2}{2}\partial_{yy} v + \sup_{\lambda \in \{\lambda_1, \lambda_2\}} \big\{\lambda(\bar{Y} - y)\partial_y v\big\} = 0. \tag{3.1}$$

We show in Lemma 3.2 below that this equation has a solution $v \in C_p^{1,2,2}$ with $\partial_y v \geq 0$. Then, it follows from Theorem 2.3 (i) that $Z(t) = v(t, S(t), Y(t))$ is an equilibrium price and that the indicated strategies are optimal. Moreover, Proposition 2.4 shows that v is the unique solution in $C_p^{1,2,2}$. As in the proof of Corollary 2.5, uniform ellipticity implies that $\overline{\mathcal{S}} = [0, T) \times \mathbb{R}^2$ and $\mathcal{S}_T = \mathbb{R}^2$, and now Theorem 2.3 (ii) implies the uniqueness of the equilibrium. □

The following result was used in the preceding proof.

Lemma 3.2. *The PDE* (3.1) *with terminal condition* f *admits a solution* $v \in C_p^{1,2,2}$ *with* $\partial_y v \geq 0$.

Proof. We first consider the linear equation

$$\partial_t v + \frac{\alpha^2}{2}\partial_{ss} v + \frac{\beta^2}{2}\partial_{yy} v + \gamma \partial_y v = 0, \quad v(T,\cdot) = f, \tag{3.2}$$

where the coefficient γ is given by

$$\gamma(y) = \begin{cases} \lambda_1(\bar{Y} - y), & y \le \bar{Y}, \\ \lambda_2(\bar{Y} - y), & y > \bar{Y}. \end{cases}$$

This equation can be obtained formally from (3.1) under the ansatz that the "vega" $\partial_y v$ is nonnegative, which can be expected due to the convexity of f. Reversing the formal derivation, we shall prove below that (3.2) has a solution $v \in C_p^{1,2,2}$ with $\partial_y v \ge 0$. It then follows that v is also a solution of (3.1), as desired. To this end, define a function v by

$$v(t,s,y) = E[f(S'(T))], \tag{3.3}$$

where S' is the first component of the solution to the SDE

$$\begin{aligned} dS'(r) &= \alpha(Y'(r))\, dW(r), \\ dY'(r) &= \gamma(Y'(r))\, dr + \beta(Y'(r))\, dZ(r) \end{aligned}$$

with initial value (s,y) at time $t \le T$. Since $f \in C_p(\mathbb{R})$, we have that $v \in C_p([0,T] \times \mathbb{R}^2)$; cf. [26, Theorem 3.1.5, p. 132].

To see that $v \in C^{1,2,2}$, consider the PDE (3.2) on the bounded domain $D = [0,T) \times (-N,N)^2$ for $N > 0$ and use the function v as boundary condition on the parabolic boundary of D. This initial-boundary value problem has a unique solution $\tilde{v} \in C^{1,2,2}(D) \cap C_0(\overline{D})$; cf. [14, Theorem 6.3.6, p. 138]. Moreover, by the Markov property, the Feynman–Kac representation of $\tilde{v}$ on D shows that $\tilde{v} = v$ on D, and in particular that v is differentiable as desired.

It remains to show that $\partial_y v \ge 0$. By the independence of the Brownian motions W and Z, the expectation (3.3) can be computed by first integrating the payoff f against the conditional distribution of $S'(T)$ given the path $(Y'(r))_{r \in [t,T]}$ of the volatility process and then integrating with respect to the law of Y'. This conditional distribution is Gaussian; more precisely, the conditional expectation given $(Y'(r))_{r \in [t,T]}$ and initial the conditions $S'(t) = s, Y'(t) = y$ is

$$\int_{-\infty}^{\infty} f\left(s + z\sqrt{\int_t^T \alpha^2(Y'(r))\, dr}\right) \phi(z)\, dz, \tag{3.4}$$

where ϕ denotes the density function of the standard normal distribution. Since f is convex, this quantity is increasing with respect to the variance parameter $\int_t^T \alpha^2(Y'(r))\,dr$. This parameter, in turn, is increasing with respect to y because α is an increasing function and $Y'(r)$ is a.s. increasing in the initial value y of Y' by the comparison theorem for SDEs; cf. [25, Proposition 5.2.18]. As a result, the conditional option price (3.4) is increasing in y, and then by monotonicity of the expectation operator, the same holds for the unconditional option price $v(t, s, y)$, so that $\partial_y v \geq 0$ as posited. This completes the proof. $\square$

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